Diamond Subgraphs in the Reduction Graph of a One-Rule String Rewriting System

Arthur Adinayev *

✉ arthuad@ac.sce.ac.il

Itamar Stein†

✉ Steinita@gmail.com

Abstract

In this paper, we study a certain case of a subgraph isomorphism problem. We consider the Hasse diagram of the lattice M_k (the unique lattice with $k+2$ elements and one anti-chain of length k) and find the maximal k for which it is isomorphic to a subgraph of the reduction graph of a given one-rule string rewriting system. We obtain a complete characterization for this problem and show that there is a dichotomy. There are one-rule string rewriting systems for which the maximal such k is 2 and there are cases where there is no maximum. No other intermediate option is possible.

Mathematics Subject Classification. 68Q42, 68R15

Keywords: one-rule string rewriting systems, reduction graph, subgraph isomorphism problem

1 Introduction

The (directed) reduction graph of a string rewriting system (SRS) S is the graph whose vertices are words, and whose edges are the one-step reductions. In this paper we study the reduction graph of one-rule SRSs. Despite their simple appearance, there are many open problems regarding one-rule SRSs. For instance, the well-known word problem asks whether two given words are in the same connected component of the reduction graph and it is a long standing open problem whether it is decidable for one-rule SRSs (see [6, Section 2] for

*Software Engineering Department, Shamoon College of Engineering, Israel
†Mathematics Unit, Shamoon College of Engineering, Israel (Corresponding Author)

a survey). Another example is the termination problem which asks if there is an infinite path in the reduction graph and it is also not known if this problem is decidable for one-rule SRSs (see [5, 7] and [4, Problem 21b]). The reduction graph has a central role in the treatment of each of these problems and many other questions regarding SRSs or monoids presented by SRSs. Therefore, any progress in understanding its structure is of value.

One way to have a better understanding of a graph G is by finding basic graphs that are or aren't isomorphic to a subgraph of G. We consider subgraphs to be more related to standard concepts of SRSs rather than other types of embeddings. For instance, an equivalent formulation of the termination problem is whether the reduction graph has a subgraph which is a homomorphic image of the infinite path graph. In this paper, we take a basic finite graph denoted M_k (to be defined shortly) and consider the question of whether M_k is isomorphic to a subgraph of the reduction graph of a given one-rule SRS S. Except from being an interesting question on its own, this kind of a problem can lead to new properties and characterizations of one-rule SRSs that might be of use for other types of questions. Indeed, some of the properties and notions that appear in this research (for instance, left\right cancellativity) have been used in the study of the word problem and other similar combinatorial questions (see [1, Chapter II] and [2]).

The reduction graph of a one-rule SRS $\langle A \mid u \to v \rangle$ is always a graded graph (in the sense that for any two vertices x, y any two paths from x to y has the same length), so clearly it has only graded subgraphs. The graph we consider in this paper is the Hasse diagram of the lattice M_k, where M_k is the lattice with $k + 2$ elements $\{x, y, z_1, \dots, z_n\}$ such that $x \le z_i \le y$ for $1 \le i \le n$ and $\{z_1, \dots, z_n\}$ are pairwise incomparable. It is clearly a graded graph, and for the sake of simplicity, we denote it also by M_k. As already mentioned, given a one-rule SRS $S = \langle A \mid u \to v \rangle$ whose reduction graph is denoted G_S, our goal is to determine whether M_k is embeddable in G_S, or in other words, what is the maximal value of k for which M_k is embeddable in G_S.

The paper is organized as follows. Neglecting few trivial cases ($u = v$ or $|A| = 1$) and assuming without loss of generality that $|u| \le |v|$, we divide the problem into several cases. In Section 3.1 we prove that if S is left (right) cancellative (i.e., u and v has different first (respectively, last) letters) then M_3 is not embeddable in G_S, hence $k = 2$ is maximal. In Section 3.2 we generalize this to any system where v is not bordered with u, i.e., u is not a prefix or not a suffix of v. In Section 3.3 we discuss systems where $u = 1$ and prove that if $v \ne b^n$ for every $b \in A$ then M_k is embeddable in G_S for any natural k. On the other hand, if $v = b^n$ for some $b \in A$ then $k = 2$ is again the maximum. In Section 3.4 we deal with the remaining case where $u \ne 1$ and v is bordered with u. We use Adyan reduction [2] to reduce this case to a system of the form $\langle \tilde{A} \mid 1 \to \tilde{v} \rangle$ which is the case solved in Section 3.3.

In conclusion, we have obtained a dichotomy between cases where M_k is embeddable in G_S for every natural k and cases where $k = 2$ is the maximal value for which M_k is embeddable in G_S.

2 Preliminaries

A *directed graph* is a tuple $(V, E, \mathbf{d}, \mathbf{r})$ consists of a set (of vertices) V, a set (of edges) E and two functions $d, r : E \to V$ associating each edge $e \in E$ with a domain vertex $\mathbf{d}(e)$ and a range vertex $\mathbf{r}(e)$. A *subgraph* $G' = (V', E', \mathbf{d}', \mathbf{r}')$ of G is a graph such that $V' \subseteq V$, $E' \subseteq E$ and $\mathbf{d}', \mathbf{r}' : E' \to V'$ are the corresponding restrictions of $\mathbf{d}$ and $\mathbf{r}$ (in particular, this requires that $\mathbf{d}(E') \subseteq V'$ and $\mathbf{r}(E') \subseteq V'$). Let $G_1 = (V_1, E_1, \mathbf{d}_1, \mathbf{r}_1)$ and $G_2 = (V_2, E_2, \mathbf{d}_2, \mathbf{r}_2)$ be two graphs. A *graph homomorphism* $f : G_1 \to G_2$ consists of two functions $f_V : V_1 \to V_2$ and $f_E : E_1 \to E_2$ such that

$$\mathbf{d}_2(f_E(e)) = f_V(\mathbf{d}_1(e)), \quad \mathbf{r}_2(f_E(e)) = f_V(\mathbf{r}_1(e))$$

for every $e \in E_1$. We say that f is an embedding (so G_1 is embedded in G_2) if f_E and f_V are injective functions.

The set of all words over an alphabet A is denoted by A^*. We denote the empty word by 1 and the set of all non-empty words by A^+. Let $u, v \in A^*$ be some words. We say that u is a prefix (suffix) of v if there exists $x \in A^*$ such that $v = ux$ (respectively, $v = xu$). Also, u is called a factor of v if there exist $x, y \in A^*$ such that $v = xuy$. We say that v is bordered with u if u is both a prefix and a suffix of v. Recall that the length of a word $u \in A^*$ is the number of letters in u and it is denoted $|u|$. We say that the letter $a \in A$ is *at position* i of u if $u = xay$ for some $x, y \in A^*$ and $|x| = i$.

Let A be some set and let R be a relation on A^*. A tuple $S = \langle A \mid R \rangle$ is called a string rewriting system (SRS). Elements of R are usually written in the form $u_i \to v_i$ instead of (u_i, v_i). Let $S = \langle A \mid R \rangle$ be an SRS. The *single-step reduction relation* induced by R is a relation on A^* denoted $\to_R$ which is defined by $w \to_R w'$ if $w = xuy$ and $w' = xvy$ for some $x, y \in A^*$ and $u \to v \in R$. If $|x| = i$ we say that the rule $u \to v$ is being used at position i in the reduction $w \to_R w'$. We denote by G_S the *reduction graph* of S. It is the (directed) graph defined as follows. The set of vertices of G_S is the set A^* of all words over A. Given $w, w' \in A^*$, edges $w \to w'$ correspond to tuples $(i, u \to v)$ where $u \to v$ is a rule in R and $w \to_R w'$ is a one-step reduction where $u \to v$ is being used at position i. If S has only one rule, namely when R is a singleton, we can identify an edge only with the position i where the unique rewrite rule is being used. A path in the reduction graph is called a *reduction* of S.

3 The embeddability of M_k in the reduction graph of a one-rule SRS

Definition 3.1. Denote by M_k the directed graph whose set of vertices is $\{x, y, z_1, \ldots, z_k\}$ and for every $1 \le i \le k$ there are two edges $x \to z_i$ and $z_i \to y$. Note that M_k is "diamond shaped", for instance, M_3 is the Hasse diagram of the diamond lattice:

$$\begin{tikzcd}
 & z_1 \arrow[dr] & \\
x \arrow[ur] \arrow[r] \arrow[dr] & z_2 \arrow[r] & y \\
 & z_3 \arrow[ur] &
\end{tikzcd}$$

We want to consider the following question. Given a one-rule SRS $S = \langle A \mid u \to v \rangle$, what is the maximal k for which M_k is isomorphic to a subgraph of the reduction graph G_S?

We start with some simple observations. If $u = v$ then the reduction graph contains only loops and even M_1 is not a embeddable in G_S so from now on we assume $u \neq v$. If $|A| = 1$ then every connected component of G_S with more than one vertex is just an (infinite) path graph. Therefore only M_1 is embeddable in G_S and we can assume from now on that $|A| > 1$. Another simple observation is that M_k is embeddable in G_S for $S = \langle A \mid u \to v \rangle$ if and only if it is embeddable in $G_{S^{-1}}$ where S^{-1} is the converse system $S^{-1} = \langle A \mid v \to u \rangle$. Therefore, without loss of generality we can assume that $|u| \leq |v|$.

If an SRS $S = \langle A \mid u \to v \rangle$ satisfies both $|A| > 1$ and $u \neq v$, it is easy to see that M_2 is embeddable in G_S. Indeed, choose a word $w \in A^*$ such that $uwv \neq vwu$ (for instance, if $\max\{|u|, |v|\} < l$ we can choose $w = a^l b^l$). The reduction graph of S contains the subgraph

$$\begin{tikzcd}
 & uwv \arrow[dr] & \\
uwu \arrow[ur] \arrow[dr] & & vwv \\
 & vwu \arrow[ur] &
\end{tikzcd}$$

which is isomorphic to M_2. The question left is whether there are other values of k for which M_k is embeddable in G_S? We split this question into several cases.

3.1 Left (right) cancellative SRSs

Let $S = \langle A \mid u \to v \rangle$ be a one-rule SRS such that $u, v \neq 1$. We say that S is *left cancellative* if the first letter of u and v are different.

Remark 3.2. The term "left cancellative" comes from the well-known fact that the first letter of u and v are different if and only if the semigroup presented by S is left cancellative, i.e., $ax = ay$ implies $x = y$ (see [1, Chapter II Theorem 2], also stated clearly in [6, Theorem 16]).

In this section we will prove that M_3 is not embeddable in G_S if S is a left cancellative SRS.

Given a reduction of some SRS

$$x_1 \to x_2 \to \ldots \to x_n$$

we want a way to mark letters that are involved in the rewriting. For this we introduce a technical tool. Given a set of letters $A = \{a_1, \ldots, a_n\}$ we define a set of "decorated" copies $A^\bullet = \{a_1^\bullet, \ldots, a_n^\bullet\}$. Let $u \in A^*$ and assume $u = u_1 \ldots u_k$ where every u_i is a letter of A. We denote by $u^\bullet = u_1^\bullet \ldots u_k^\bullet$ a decorated copy of the word u. Denote by $\pi : A \cup A^\bullet \to A$ a function defined $\pi(a_i) = \pi(a_i^\bullet) = a_i$ which clearly extends to a projection $\pi : (A \cup A^\bullet)^* \to A^*$. Now we can define:

Definition 3.3. Let $S = \langle A \mid R \rangle$ be an SRS. Define a new SRS, denoted $\overline{S} = \langle \overline{A}, \overline{R} \rangle$, in the following way. The set of letters of $\overline{S}$ is $\overline{A} = A \cup A^\bullet$. For every rule $u \to v$ in R and for every word $\overline{u} \in (A \cup A^\bullet)^*$ such that $\pi(\overline{u}) = u$ the relation $\overline{R}$ will have the rule $\overline{u} \to v^\bullet$.

Example 3.4. If $S = \langle a, b \mid ab \to bba \rangle$ then the SRS $\overline{S}$ is

$$\overline{S} = \langle a, a^\bullet, b, b^\bullet \mid ab \to b^\bullet b^\bullet a^\bullet, \quad a^\bullet b \to b^\bullet b^\bullet a^\bullet, \quad ab^\bullet \to b^\bullet b^\bullet a^\bullet, \quad a^\bullet b^\bullet \to b^\bullet b^\bullet a^\bullet \rangle$$

It is obvious that every reduction

$$\overline{x}_1 \to \ldots \to \overline{x}_n$$

of $\overline{S}$ can be projected into a reduction of S

$$\pi(\overline{x}_1) \to \ldots \to \pi(\overline{x}_n)$$

by deleting all the "decorations". Moreover, it is easy to see that every reduction of S

$$x_1 \to x_2 \to \ldots \to x_n$$

can be "lifted" into a reduction of $\overline{S}$

$$\overline{x}_1 \to \overline{x}_2 \ldots \to \overline{x}_n$$

such that $\pi(\overline{x_i}) = x_i$ and $\overline{x_1} = x_1$. The decorated letters in this reduction will be the letters that are "involved" in the reduction or "affected" by it.

Example 3.5. Consider the SRS S in example 3.4 and the reduction

$$abaabb \overset{(3)}{\to} ababbab \overset{(2)}{\to} abbbabab \overset{(4)}{\to} abbbbbaab$$

where the numbers over the arrows are the positions in which the rewrite is being done. This reduction can be lifted to the reduction

$$abaabb \overset{(3, ab \to b^\bullet b^\bullet a^\bullet)}{\to} abab^\bullet b^\bullet a^\bullet b \overset{(2, ab^\bullet \to b^\bullet b^\bullet a^\bullet)}{\to} abb^\bullet b^\bullet a^\bullet b^\bullet a^\bullet b \overset{(4, a^\bullet b^\bullet \to b^\bullet b^\bullet a^\bullet)}{\to} abb^\bullet b^\bullet b^\bullet b^\bullet a^\bullet a^\bullet b$$

of the SRS $\overline{S}$.

The following observation about reductions in $\overline{S}$ will be useful.

Lemma 3.6. *Let $S = \langle A \mid u \to v \rangle$ be a one-rule SRS and consider a reduction*

$$x_1 \to x_2 \to \ldots \to x_n$$

of S and its lifting

$$\overline{x}_1 \to \overline{x}_2 \to \ldots \to \overline{x}_n$$

to a reduction of $\overline{S}$. Assume that the first decorated letter of $\overline{x}_n$ is at position i then

1. *No step in the reduction is carried out at position j for $j < i$.*
2. *There is a step in the reduction carried out at position i.*
3. *If S is left cancellative then the letter at position i of x_1 is the first letter of u and the letter at position i of x_n is the first letter of v.*

Proof. Statements (1) and (2) are clear so we will prove (3). Denote by a the first letter of u and by b the first letter of v. Assume that in the step $x_k \to x_{k+1}$ the rewrite rule is carried out at position i (such step exists by (2)). Therefore, the letter at position i of x_k is a and the letter at position i of x_{k+1} is b. Since no step is carried out at position j for $j < i$, the first letter of x_1 is also a. In addition, the first letter of u and v are different so we can not carry out any step at position i in the reduction $x_{k+1} \to \ldots \to x_n$. Therefore, the letter at position i of x_n is b as required. □

Lemma 3.7. *Let $S = \langle A \mid u \to v \rangle$ be a left cancellative SRS and let $x \to z_1 \to y$ and $x \to z_2 \to y$ be two reductions in S. Denote the corresponding "lifted" reductions in $\bar{S}$ by*

$$\overline{x} \to \overline{z_1} \to \overline{y_1}, \quad \overline{x} \to \overline{z_2} \to \overline{y_2}$$

(a priori, $\overline{y_1} \neq \overline{y_2}$ because they might have different decorations). Then, the first decorated positions of $\overline{y_1}$ and $\overline{y_2}$ are equal.

Proof. Denote by i_1 (i_2) the first decorated position of $\overline{y_1}$ (respectively, $\overline{y_2}$). We continue to use a for the first letter of u and b for the first letter of v. Assume without loss of generality that $i_1 < i_2$. Applying part (3) of Lemma 3.6 on the reduction $x \to z_1 \to y$, we obtain that b is the letter at position i_1 of y and a is the letter at position i_1 of x. Applying part (1) of Lemma 3.6 on $x \to z_2 \to y$, we obtain that a is the letter at position i_1 of y (since there are no steps carried out in this reduction at position j for $j < i_2$). This is a contradiction so $i_1 = i_2$ as required. □

Proposition 3.8. *Let $S = \langle A \mid u \to v \rangle$ be a left cancellative SRS. Then M_3 is not isomorphic to a subgraph of G_S.*

Proof. Consider three reductions

$$x \to z_1 \to y, \quad x \to z_2 \to y, \quad x \to z_3 \to y$$

such that z_1, z_2, z_3 are all distinct and lift them into three reductions in $\overline{S}$

$$\overline{x} \to \overline{z_1} \to \overline{y_1}, \quad \overline{x} \to \overline{z_2} \to \overline{y_2}, \quad \overline{x} \to \overline{z_3} \to \overline{y_3}.$$

According to Lemma 3.7 the first decorated positions of $\overline{y_1}$, $\overline{y_2}$ and $\overline{y_3}$ are identical. Denote this position by i. Part (2) of Lemma 3.6 implies that in each one of the three reduction there is a rewrite step carried out at position i. Without loss of generality we assume that in the first reduction this is the first step

$$x \overset{(i)}{\to} z_1.$$

In the second reduction this cannot be the first step

$$x \overset{(i)}{\to} z_2$$

because this will imply $z_1 = z_2$ in contradiction to our assumption. Therefore, this must be the second step

$$z_2 \overset{(i)}{\to} y.$$

For the third reduction we cannot have

$$x \overset{(i)}{\to} z_3$$

as this implies $z_1 = z_3$ and we cannot have

$$z_3 \overset{(i)}{\to} y$$

as this implies $z_2 = z_3$. This is a contradiction which finishes the proof. □

Remark 3.9. Clearly, a dual result holds for right cancellative SRSs.

3.2 SRSs where v is not bordered with u

In this section we generalize the results of Section 3.1 to a wider class of SRSs.

Proposition 3.10. *Let $S = \langle A \mid u \to v \rangle$ be an SRS such that u is not a prefix of v, then M_3 is not embeddable in G_S.*

Proof. Denote by p the maximal prefix of u which is also a prefix of v. Therefore, we can write $u = pu'$ and $v = pv'$ for some words u', v'. It might be the case that $p = 1$ (if S is left cancellative) but note that $u' \neq 1$ since u is not a prefix of v and $v' \neq 1$ since we are assuming $|u| \leq |v|$. The maximality of p implies that the SRS defined by $S' = \langle A \mid u' \to v' \rangle$ is left cancellative. Now, note that any reduction $x \to y$ which is carried out using the rule $pu' \to pv'$ can be carried out using the rule $u' \to v'$. Therefore, G_S is a subgraph of $G_{S'}$. Since M_3 is not embeddable in $G_{S'}$ by Proposition 3.8 it is not embeddable in G_S as well. □

Clearly, a dual result holds for SRSs where u is not a suffix of v so we can conclude:

Proposition 3.11. *Let $S = \langle A \mid u \to v \rangle$ be an SRS. If v is not bordered with u (i.e., u is not a prefix of v or not a suffix of v) then M_3 is not embeddable in G_S.*

3.3 Special one-rule SRSs

In this section we deal with SRSs of the form $S = \langle A \mid 1 \to v \rangle$. We remark that SRSs of the form $\langle A \mid v_i \to 1 \rangle$ are called *special* (see [3, Definition 3.4.1]). We have already mentioned that M_k is embeddable in G_S if and only if it is embeddable in $G_{S^{-1}}$ where S^{-1} is the converse system. So we can say that in this section we consider special one-rule SRSs. There are few subcases.

Lemma 3.12. *If $v = b^n$ for some letter $b \in A$ then M_3 is not embeddable in G_S.*

Proof. Any word $x \in A^*$ can be uniquely decomposed into

$$x = b^{m_0} a_{i_1} b^{m_1} a_{i_2} b^{m_2} \cdots b^{m_{l-1}} a_{i_l} b^{m_l}$$

where $a_{i_1}, \ldots, a_{i_l} \in A$ are letters distinct from b and $m_0, \ldots, m_l$ are non-negative integers. If $x \to z$ is a one-step reduction then

$$z = b^{m'_0} a_{i_1} b^{m'_1} a_{i_2} b^{m'_2} \cdots b^{m'_{l-1}} a_{i_l} b^{m'_l}$$

such that $m'_i = m_i + n$ for some $i \in \{0, \ldots, l\}$ and $m'_j = m_j$ if $j \neq i$. It is clear that we can identify x with the tuple $(m_0, \ldots, m_l)$ and a one-step reduction is equivalent to adding n to one of the entries. Therefore, a two step reduction $x \to z_1 \to y$ is equivalent to adding n to two of the entries (or twice to the same one). Now, it is clear that there could be at most one additional reduction $x \to z_2 \to y$ from x to y with $z_1 \neq z_2$. This finishes the proof. □

Lemma 3.13. *For any $k \in \mathbb{N}$, the graph M_k is embeddable in G_S for $S = \langle A \mid 1 \to ab \rangle$.*

Proof. Choose $k \in \mathbb{N}$ and take $x = (aabb)^{k-1}$. For $0 \le i \le k-1$ define $z_i = (aabb)^i ab(aabb)^{k-i-1}$. It is clear that z_i is obtained from x by applying the rewrite rule at position $4i$. Moreover, it is clear that $z_i \neq z_j$ for $i \neq j$. Now, applying the rewrite rule at position $4i+1$ we obtain a reduction $z_i \to y$ where $y = (aabb)^k$. This yields a subgraph isomorphic to M_k as required. □

Lemma 3.14. *Let $S = \langle A \mid 1 \to v \rangle$ be an SRS such that $v \neq b^n$ for every $b \in A$. Then, M_k is embeddable in G_S for every k.*

Proof. Assume that the first letter of v is a so $v = av'$ where v' contains at least one letter distinct from a. Define a monoid homomorphism $f : \{a, b\}^* \to A^*$ which is the extension of

$$f(a) = a, \quad f(b) = v'.$$

It is easy to see that f is injective and that $f(ab) = av' = v$. Therefore, it induces a graph embedding

$$\hat{f} : G_T \to G_S$$

where $T = \langle a, b \mid 1 \to ab \rangle$. In particular, it embeds the subgraph of G_T isomorphic to M_k (which exists by Lemma 3.13) onto an isomorphic subgraph of G_S.

Combining Lemma 3.12 and Lemma 3.14 we conclude this section. □

Proposition 3.15. *Let $S = \langle A \mid 1 \to v \rangle$ be an SRS. If $v = b^n$ for some $b \in A$ then $k = 2$ is the maximal value such that M_k is embeddable in G_S. Otherwise, M_k is embeddable in G_S for every natural k.*

3.4 SRSs where v is bordered with u

In this section we will show that any system $S = \langle A \mid u \to v \rangle$ where v is bordered with u can be reduced using Adyan reduction [2] into an SRS of the form $\tilde{S} = \langle \tilde{A} \mid 1 \to \tilde{v} \rangle$ such that M_k is embeddable in G_S if and only if it is a embeddable in $G_{\tilde{S}}$. Therefore, we can use Proposition 3.15 in order to determine whether M_k is a subgraph of S. We remark that a similar approach of using Adyan reductions for other one-rule problems was used in [7] and [8, Section 6].

We start with some basic definitions required for the reduction.

Definition 3.16. Let $u \in A^*$ be some word. Its set of *self-overlaps* is defined by

$$\mathrm{OVL}(u) = \{w \in A^+ \mid \exists x, y \in A^+ \quad u = xw = wy\}.$$

The word u is called *self-overlap-free if* $\mathrm{OVL}(u) = \varnothing$.

Let T be a self-overlap-free word over some alphabet A. Enumerate all words in A^* without T as a factor by

$$R_1, R_2, \ldots$$

and let B be an infinite set of new letters

$$B = \{b_1, b_2, \ldots\} \quad (B \cap A = \varnothing).$$

Denote the set of words bordered with T by Bord_T and note that every word $x \in \mathrm{Bord}_T$ can be decomposed uniquely into

$$x = TR_{i_1}TR_{i_2} \cdots TR_{i_k}T.$$

Adyan and Oganesyan define a bijection $\varphi_T : \mathrm{Bord}_T \to B^*$ inductively by

$$\varphi_T(x) = \begin{cases} 1 & x = T \\ \varphi_T(x_1)b_i & x = x_1 R_i T, \quad x_1 \in \mathrm{Bord}_T\,. \end{cases}$$

It is important to observe some properties of φ_T.

Lemma 3.17. *For every $x \in \mathrm{Bord}_T$ we have that $|\varphi_T(x)| < |x|$.*

Proof. This can easily be proved by induction since $0 = |\varphi_T(T)| < |T|$ and $1 = |b_i| \le |R_i T|$ even if R_i is the empty word. □

Lemma 3.18. *Let $u, v \in \mathrm{Bord}_T$ such that u is a prefix of v, then $\varphi_T(u)$ is a prefix of $\varphi_T(v)$.*

Proof. It is clear from the definition of φ_T that

$$\varphi_T(Tx_1Tx_2T) = \varphi_T(Tx_1T)\varphi_T(Tx_2T).$$

Therefore, if $u = T\overline{u}T$ and $v = T\overline{u}TwT$ then

$$\begin{aligned}\varphi_T(v) &= \varphi_T(T\overline{u}TwT) = \varphi_T(T\overline{u}T)\varphi_T(TwT)\\ &= \varphi_T(u)\varphi_T(TwT)\end{aligned}$$

so $\varphi_T(u)$ is indeed a prefix of $\varphi_T(v)$. □

A dual argument shows that if u is a suffix of v then $\varphi_T(u)$ is a suffix of $\varphi_T(v)$. Therefore, we obtain:

Lemma 3.19. *Let $u, v \in \mathrm{Bord}_T$ be distinct words such that v is bordered with u then $\varphi_T(v)$ is bordered with $\varphi_T(u)$.*

From now on we consider an SRS $S = \langle A \mid u \to v\rangle$ such that ($u \neq v$ and) v is bordered with u. This implies that $u \in \mathrm{OVL}(v)$. Denote by T the shortest element of $\mathrm{OVL}(u)$ or $T = u$ if $\mathrm{OVL}(u) = \varnothing$. Clearly, T is self-overlap-free and $T \in \mathrm{OVL}(v)$ so both u and v are bordered with T. (A system $S = \langle A \mid u \to v\rangle$ with this property is called *reducible* in [2]). We make some observations on the existence of a subgraph of G_S isomorphic to M_k.

Lemma 3.20. *If M_k is embeddable in G_S then it is also isomorphic to a subgraph of G_S whose vertices are in Bord_T.*

Proof. Assume

$$x \to z \to y$$

is a reduction in G_S. Note that any word $x \in A^*$ which contains T as a factor can be written uniquely as $x = x'\overline{x}x''$ where $\overline{x} \in \mathrm{Bord}_T$ and x', x'' do not contain T as a factor. Therefore, we can write the above reduction as

$$x'\overline{x}x'' \to z'\overline{z}z'' \to y'\overline{y}y''.$$

Since u and v are bordered with T, it is clear that

$$x' = z' = y', \quad x'' = z'' = y''$$

and

$$\overline{x} \to \overline{z} \to \overline{y}$$

is also a reduction. Therefore, if we have k different reductions

$$x \to z_1 \to y, \ldots, x \to z_k \to y$$

there are k corresponding reductions

$$\overline{x} \to \overline{z_1} \to \overline{y}, \ldots, \overline{x} \to \overline{z_k} \to \overline{y}$$

such that $\overline{x}, \overline{y}, \overline{z_1}, \ldots \overline{z_k} \in \mathrm{Bord}_T$. Since the steps $x \to z_i$ and $x \to z_j$ are carried out at different positions for $i \neq j$ we know that $\overline{x} \to \overline{z_i}$ and $\overline{x} \to \overline{z_j}$ are carried out in different positions and hence $\overline{z_i} \neq \overline{z_j}$. Therefore, we have a subgraph isomorphic to M_k such that all the vertices are bordered with T as required. $\square$

Lemma 3.21. *Let $S = \langle A \mid u \to v \rangle$ be an SRS such that v is bordered with u and let T be defined as above. Then M_k is embeddable in G_S if and only if it is embeddable in $G_{\hat{S}}$ for $\hat{S} = \langle B \mid \varphi_T(u) \to \varphi_T(v) \rangle$.*

Proof. Recall that φ_T is a bijection $\varphi_T : \mathrm{Bord}_T \to B^*$. It is clear that φ_T^{-1} maps any subgraph of $G_{\hat{S}}$ onto an isomorphic subgraph of G_S. On the other direction, if G_S has a subgraph isomorphic to M_k, then by Lemma 3.20 it has such subgraph whose vertices are elements of Bord_T. Therefore, φ_T maps it onto a subgraph of $G_{\hat{S}}$ isomorphic to M_k as required. $\square$

Lemma 3.22. *Let B be an alphabet (perhaps infinite) and let $S = \langle B \mid u \to v \rangle$ be an SRS. Let $B' \subseteq B$ be the (finite) set of letters from B that occur in u and v and define $S' = \langle B' \mid u \to v \rangle$. Then, M_k is embeddable in G_S if and only if it is embeddable in $G_{S'}$.*

Proof. It is clear that $G_{S'}$ is a subgraph of G_S by inclusion so any subgraph of $G_{S'}$ is a subgraph of G_S. On the other direction denote by π the standard projection $\pi : B^* \to (B')^*$ defined by

$$\pi(b) = \begin{cases} b & b \in B' \\ 1 & b \notin B.' \end{cases}$$

It is clear that if

$$x \to y$$

is a reduction of G_S carried out at position i then

$$\pi(x) \to \pi(y)$$

is also a reduction of $G_{S'}$. Moreover, the letter at position i of x is a letter of B' (it is the first letter of u). Therefore, if

$$x \to z_1 \to y, \ldots, x \to z_k \to y$$

are k reductions in G_S such that $z_i \neq z_j$ for $i \neq j$ then

$$\pi(x) \to \pi(z_1) \to \pi(y), \ldots, \pi(x) \to \pi(z_k) \to \pi(y)$$

are k reductions in $G_{S'}$ such that $\pi(z_i) \neq \pi(z_j)$ for $i \neq j$. This finishes the proof. $\square$

We can now state the main result of this section.

Proposition 3.23. *Let $S = \langle A \mid u \to v\rangle$ be an SRS such that v is bordered with u then we can construct another SRS $\tilde{S} = \langle \tilde{A} \mid 1 \to \tilde{v}\rangle$ such that M_k is embeddable in G_S if and only if it is embeddable in $G_{\tilde{S}}$.*

Proof. Choose T to be the shortest element of $\mathrm{OVL}(u)$ (or $T = u$ if $\mathrm{OVL}(u) = \varnothing$). Take B' to be the set of letters from B that occur in $\varphi_T(u)$ and $\varphi_T(v)$. Denote

$$A_1 = B', \quad u_1 = \varphi_T(u), \quad v_1 = \varphi_T(v)$$

and $S_1 = \langle A_1 \mid u_1 \to v_1\rangle$. By Lemma 3.21 and Lemma 3.22, M_k is embeddable in G_S if and only if it is embeddable in G_{S_1}. There is no reason to expect that $u_1 = 1$. However, by Lemma 3.19 v_1 is still bordered with u_1 so we choose T_1 to be the shortest element of $\mathrm{OVL}(u_1)$ or $T_1 = u_1$ if $\mathrm{OVL}(u_1) = \varnothing$. Now we can continue this process and construct $S_2 = \langle A_2 \mid u_2 \to v_2\rangle$ with $u_2 = \varphi_{T_1}(u_1)$, $v_2 = \varphi_{T_1}(v_1)$ and so on. Since $|\varphi_T(x)| < |x|$ this process must terminate. It will terminate when $u_k = \varphi_{T_{k-1}}(u_{k-1}) = 1$. Then we can define $\tilde{A} = A_k$ and $\tilde{v} = v_k$ and obtain a system $\tilde{S} = \langle \tilde{A} \mid 1 \to \tilde{v}\rangle$ which satisfy the desired result. $\square$

Remark 3.24. Note that T can be easily obtained from u and v and it is also a routine procedure to calculate $\langle B' \mid \varphi_T(u) \to \varphi_T(v)\rangle$ (note that B' is a finite set). Therefore, the process described in Proposition 3.23 can be effectively computed.

Proposition 3.23 is enough in order to solve the case of this section. Given an SRS $S = \langle A \mid u \to v\rangle$ such that v is bordered with u we can carry out the procedure described in Proposition 3.23 and obtain an SRS $\tilde{S} = \langle \tilde{A} \mid 1 \to \tilde{v}\rangle$ which is the case dealt with in Proposition 3.15.

4 Conclusion

Combining the results of Section 3 we obtain the following theorem which gives a complete answer to the question of whether M_k is embeddable in the reduction graph of a one-rule SRS.

Theorem 4.1. *Let $S = \langle A \mid u \to v\rangle$ be a one-rule SRS such that $u \neq v$, $|u| \leq |v|$ and $|A| > 1$. Then:*

1. *If v is not bordered with u then $k = 2$ is the maximal value such that M_k is embeddable in G_S.*

2. *If v is bordered with u then we can use Adyan reductions as described in Proposition 3.23 and obtain an SRS $\tilde{S} = \langle \tilde{A} \mid 1 \to \tilde{v} \rangle$. In this case:*

   (a) *If $\tilde{v} = b^n$ for some $b \in \tilde{A}$ then $k = 2$ is the maximal value such that M_k is embeddable in G_S.*

   (b) *If $\tilde{v} \neq b^n$ for every $b \in \tilde{A}$ then M_k is embeddable in G_S for every k.*

Remark 4.2. Since the procedure described in Proposition 3.23 is effective, Theorem 4.1 implies that the question of whether M_k is embeddable in G_S for a given SRS $S = \langle A \mid u \to v \rangle$ is decidable.

If v is bordered with u we can consider the SRS $\langle A \mid u \to v \rangle$ as equivalent in some sense to an SRS $\langle \tilde{A} \mid 1 \to \tilde{v} \rangle$ so it can be considered as a very specific case. Therefore, one way to interpret Theorem 4.1 is that M_3 is not embeddable in the reduction graph of a "standard" one rule SRS. This gives some restriction on the possible structure of the reduction graph of a "typical" case. It is an interesting question whether other similar restrictions can be found.

References

[1] S. I. Adjan. *Defining relations and algorithmic problems for groups and semigroups.* Proceedings of the Steklov Institute of Mathematics, No. 85 (1966). Translated from the Russian by M. Greendlinger. American Mathematical Society, Providence, R.I., 1966.

[2] SI Adjan and GU Oganesjan. On the word and divisibility problems in semigroups with a single defining relation. *Mathematics of the USSR-Izvestiya*, 12(2):207, 1978.

[3] Ronald V. Book and Friedrich Otto. *String-rewriting systems.* Texts and Monographs in Computer Science. Springer-Verlag, New York, 1993.

[4] Nachum Dershowitz. Open. Closed. Open. In *Term rewriting and applications*, volume 3467 of *Lecture Notes in Comput. Sci.*, pages 376–393. Springer, Berlin, 2005.

[5] Alfons Geser. Termination of string rewriting rules that have one pair of overlaps. In *Rewriting techniques and applications*, volume 2706 of *Lecture Notes in Comput. Sci.*, pages 410–423. Springer, Berlin, 2003.

[6] Gerard Lallement. The word problem for Thue rewriting systems. In *Term rewriting (Font Romeux, 1993)*, volume 909 of *Lecture Notes in Comput. Sci.*, pages 27–38. Springer, Berlin, 1995.

[7] Kayoko Shikishima-Tsuji, Masashi Katsura, and Yuji Kobayashi. On termination of confluent one-rule string-rewriting systems. *Inform. Process. Lett.*, 61(2):91–96, 1997.

[8] Itamar Stein. Reducing the gradedness problem of string rewriting systems to a termination problem. *RAIRO Theor. Inform. Appl.*, 49(3):233–254, 2015.