Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and exceptional groups of Lie type

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Abstract

This paper determined all pairs $(\mathcal{D}, G)$ where $\mathcal{D}$ is a non-symmetric 2-(v, k, λ) design with $(r, \lambda) = 1$ and G is the almost simple flag-transitive automorphism group of $\mathcal{D}$ with an exceptional socle of Lie type. We prove that if $T \trianglelefteq G \leq Aut(T)$ where T is an exceptional group of Lie type, then T must be the Ree group or Suzuki group, and there just five non-isomorphic designs $\mathcal{D}$.

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1 Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a set of k-subsets of $\mathcal{P}$ called blocks, such that any 2 points are contained in exactly λ blocks. A flag is an incident point-block pair (α, B). An automorphism of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The design is non-trivial if $2 < k < v - 1$ and non-symmetric if $v < b$. All automorphisms of the design $\mathcal{D}$ form a group called the full automorphism group of $\mathcal{D}$, denoted by $Aut(\mathcal{D})$. Let $G \leq Aut(\mathcal{D})$, the design $\mathcal{D}$ is called point (block, flag)-transitive if G acts transitively on the set of points (blocks, flags), and point-primitive if G acts primitively on $\mathcal{P}$. Note that a finite primitive group is almost simple if it is isomorphic to a group G for which $T \cong Inn(T) \leq G \leq Aut(T)$ for some non-abelian simple group T.

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Let $G \leq Aut(\mathcal{D})$, and r be the number of blocks incident with a given point. In [6], P. Dembowski proved that if G is a flag-transitive automorphism group of a 2-design $\mathcal{D}$ with $(r, \lambda) = 1$, then G is point-primitive. In 1988, P. H. Zieschang [32] proved that if $\mathcal{D}$ is a 2-design with $(r, \lambda) = 1$ and $G \leq Aut(\mathcal{D})$ is flag transitive, then G must be of almost simple or affine type. Such 2-designs have been studied in [1, 2, 29, 31], where the socle of G is a sporadic, an alternating group or elementary abelian p-group, respectively. In this paper, we continue to study the case that the socle of G is an exceptional simple group of Lie type. We get the following:

Theorem 1 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a non-symmetric 2-(v, k, λ) design with $(r, \lambda) = 1$ and G an almost simple flag-transitive automorphism group of $\mathcal{D}$ with the exceptional socle T of Lie type in characteristic p and $q = p^e$. Let B be a block of $\mathcal{D}$. Then one of the following holds:*

(1) *$T = {}^2G_2(q)$ with $q = 3^{2n+1} \geq 27$, and $\mathcal{D}$ is one of the following:*

  (i) *a Ree unital with $G_B = \mathbb{Z}_2 \times L_2(q)$;*

  (ii) *a 2-$(q^3 + 1, q, q - 1)$ design with $G_B = Q_1 : K$;*

  (iii) *a 2-$(q^3 + 1, q, q - 1)$ design with $G_B = Q_2 : K$;*

  (iv) *a 2-$(q^3 + 1, q^2, q^2 - 1)$ design with $G_B = Q' : K$,*

  where $Q \in Syl_3(T)$, and the definitions of Q_1, Q_2 and K refer to Section 3.

(2) *$T = {}^2B_2(q)$ with $q = 2^{2n+1} \geq 8$, and $\mathcal{D}$ is a 2-$(q^2 + 1, q, q - 1)$ design with $G_B = Z(Q) : K$, where $Q \in Syl_2(T)$ and $K = \mathbb{Z}_{q-1} \cong \mathbb{F}_q^*$.*

2 Preliminary results

We first give some preliminary results about designs and almost simple groups.

Lemma 2.1 ([29, Lemma 2.2]) *For a 2-(v, k, λ) design $\mathcal{D}$, it is well known that*

(1) *$bk = vr$;*

(2) *$\lambda(v - 1) = r(k - 1)$;*

(3) *$v \le \lambda v < r^2$;*

(4) *if $G \le Aut(\mathcal{D})$ is flag-transitive and $(r, \lambda) = 1$, then $r \mid (|G_\alpha|, v-1)$ and $r \mid d$, for any non-trivial subdegree d of G.*

Lemma 2.2 *Assume that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. Let $\alpha \in \mathcal{P}$ and $B \in \mathcal{B}$. Then*

(1) *$G = TG_\alpha$ and $|G| = f|T|$ where f is a divisor of $|Out(T)|$;*

(2) *$|G : T| = |G_\alpha : T_\alpha| = f$;*

(3) *$|G_B|$ divides $f|T_B|$, and $|G_{\alpha B}|$ divides $f|T_{\alpha B}|$ for any flag (α, B).*

Proof. Note that G is an almost simple primitive group by [5]. So (1) holds and (2) follows from (1). Since $T \trianglelefteq G$, then $|B^T|$ divides $|B^G|$ and $|(\alpha, B)^T|$ divides $|(\alpha, B)^G|$, hence $|G_B : T_B|$ divides f, and $|G_{\alpha B} : T_{\alpha B}|$ divides f, (3) holds. □

Lemma 2.3 ([6, 2.2.5]) *Let $\mathcal{D}$ be a 2-(v, k, λ) design. If $\mathcal{D}$ satisfies $r = k + \lambda$ and $\lambda \le 2$, then $\mathcal{D}$ is embedded in a symmetric 2-$(v + k + \lambda, k + \lambda, \lambda)$ design.*

Lemma 2.4 ([6, 2.3.8]) *Let $\mathcal{D}$ be a 2-(v, k, λ) design and $G \le Aut(\mathcal{D})$. If G is 2-transitive on points and $(r, \lambda) = 1$, then G is flag transitive.*

Lemma 2.5 *Let A, B, C be subgroups of group G. If $B \le A$, then*

$$|A : B| \ge |(A \cap C) : (B \cap C)|.$$

Lemma 2.6 ([17]) *Suppose that T is a simple group of Lie type in characteristic p and acts on the set of cosets of a maximal parabolic subgroup. Then T has a unique subdegree which is a power of p except T is $L_d(q)$, $\Omega^+_{2m}(q)$ (m is odd) or $E_6(q)$.*

Lemma 2.7 [26, 1.6](Tits Lemma) *If T is a simple group of Lie type in characteristic p, then any proper subgroup of index prime to p is contained in a parabolic subgroup of T.*

In the following, for a positive integer n, n_p denotes the p-part of n and $n_{p'}$ denotes the p'-part of n, i.e., $n_p = p^t$ where $p^t \mid n$ but $p^{t+1} \nmid n$, and $n_{p'} = n/n_p$.

Lemma 2.8 *Assume that* G *and* $\mathcal{D}$ *satisfy the hypothesis of Theorem 1. Then* $|G| < |G_\alpha|^3$ *and if* G_α *is a non-parabolic maximal subgroup of* G*, then* $|G| < |G_\alpha||G_\alpha|_{p'}^2$ *and* $|T| < |Out(T)|^2|T_\alpha||T_\alpha|_{p'}^2$.

Proof. From Lemma 2.1, since r divides every non-trivial subdegree of G, then r divides $|G_\alpha|$, and so $|G| < |G_\alpha|^3$. If G_α is not parabolic, then p divides $v = |G : G_\alpha|$ by Lemma 2.7. Since r divides $v - 1$, $(r, p) = 1$ and so r divides $|G_\alpha|_{p'}$. It follows that $r < |G_\alpha|_{p'}$, and hence $|G| < |G_\alpha||G_\alpha|_{p'}^2$ by Lemma 2.1. Now by Lemma 2.2(2), we have that $|T| < |Out(T)|^2|T_\alpha||T_\alpha|_{p'}^2$. □

Lemma 2.9 ([20, Theorem 2, Table III]) *If* T *is a finite simple exceptional group of Lie type such that* $T \leq G \leq Aut(T)$*, and* G_α *is a maximal subgroup of* G *such that* $T_0 = Soc(G_\alpha)$ *is not simple, then one of the following holds:*

(1) G_α *is parabolic;*

(2) G_α *is of maximal rank;*

(3) $G_\alpha = N_G(E)$*, where* E *is an elementary abelian group given in [4, Theorem 1 (II)];*

(4) $T = E_8(q)$ *with* $p > 5$*, and* T_0 *is either* $A_5 \times A_6$ *or* $A_5 \times L_2(q)$*;*

(5) T_0 *is as in Table 1.*

Table 1

T	T_0
$F_4(q)$	$L_2(q) \times G_2(q)(p > 2, q > 3)$
$E_6^\epsilon(q)$	$L_3(q) \times G_2(q), U_3(q) \times G_2(q)(q > 2)$
$E_7(q)$	$L_2(q) \times L_2(q)(p > 3), L_2(q) \times G_2(q)(p > 2, q > 3),$
	$L_2(q) \times F_4(q)(q > 3), G_2(q) \times Sp_6(q)$
$E_8(q)$	$L_2(q) \times L_3^\epsilon(q)(p > 3), L_2(q) \times G_2(q) \times G_2(q)(p > 2, q > 3),$
	$G_2(q) \times F_4(q), L_2(q) \times G_2(q^2)(p > 2, q > 3)$

Lemma 2.10 ([19, Theorem 3]) *Let* T *be a finite simple exceptional group of Lie type, with* $T \leq G \leq Aut(T)$*. Assume* G_α *is a maximal subgroup of* G *and* $Soc(G_\alpha) = T_0(q)$ *is a simple group of Lie type over* $\mathbb{F}_q(q > 2)$ *such that* $\frac{1}{2}rank(T) < rank(T_0)$*; assume also that* (T, T_0) *is not* $(E_8, {}^2A_5(5))$ *or* $(E_8, {}^2D_5(3))$*. Then one of the following holds:*

(1) *G_α is a subgroup of maximal rank;*

(2) *T_0 is a subfield or twisted subgroup;*

(3) *$T = E_6(q)$ and $T_0 = C_4(q)(q$ odd$)$ or $F_4(q)$.*

Lemma 2.11 ([22, Theorem 1.2]) *Let T be a finite simple exceptional group of Lie type such that $T \le G \le Aut(T)$, and G_α a maximal subgroup of G with socle $T_0 = T_0(q)$ a simple group of Lie type in characteristic p. Then if $rank(T_0) \le \frac{1}{2} rank(T)$, we have the following bounds:*

(1) *if $T = F_4(q)$, then $|G_\alpha| < 4q^{20} \log_p q$;*

(2) *if $T = E_6^\epsilon(q)$, then $|G_\alpha| < 4q^{28} \log_p q$;*

(3) *if $T = E_7(q)$, then $|G_\alpha| < 4q^{30} \log_p q$;*

(4) *if $T = E_8(q)$, then $|G_\alpha| < 12q^{56} \log_p q$.*

In all cases, $|G_\alpha| < 12|G|^{\frac{5}{13}} \log_p q$.

The following lemma gives a method to check the existence of the design with the possible parameters.

Lemma 2.12 *For the given parameters (v, b, r, k, λ) and the group G, the conditions that there exists a design $\mathcal{D}$ with such parameters satisfying G which is flag-transitive and point primitive is equivalent to the following four steps holding for some subgroup H of G with index b and its orbit of size k:*

(1) *G has at least one subgroup H of order $|G|/b$;*

(2) *H has at least one orbit O of length k;*

(3) *the size of O^G is b;*

(4) *the number of blocks incident with any two points is a constant.*

When we run through all possibilities of H and its orbits with size k, then we found all designs with such parameters and admitting $G \leq Aut(\mathcal{D})$ is flag-transitive and point primitive. This is the essentially strategy adopted in [29].

We now give some information about the Ree group ${}^2G_2(q)$ with $q = 3^{2n+1}$ and its subgroups, which from [8, 11, 15] and would be used later.

Set $m = 3^{n+1}$, and so $m^2 = 3q$. The Ree group ${}^2G_2(q)$ is generated by Q, K and τ, where Q is Sylow 3-subgroup of ${}^2G_2(q)$, $K = \{diag(t^m, t^{1-m}, t^{2m-1}, 1, t^{1-2m}, t^{m-1}, t^{-m}) \mid t \in \mathbb{F}_q^*\} \cong \mathbb{Z}_{q-1}$ and $\tau^2 = 1$ such that τ inverts K, and $|{}^2G_2(q)| = (q^3+1)q^3(q-1)$.

Lemma 2.13 (1) ([15]) *${}^2G_2(q)$ is 2-transitive of degree q^3+1.*

(2) ([7, p.252]) *The stabilizer of one point is $Q : K$, and $N_{{}^2G_2(q)}(Q) = Q : K$.*

(3) ([11, p.292]) *The stabilizer K of two points is cyclic of order $q-1$ and the stabilizer of three points is of order2.*

(4) ([11, p.292]) *The Sylow 2-subgroup of ${}^2G_2(q)$ is elementary abelian with order 8.*

Lemma 2.14 ([8, Lemma 3.3]) *Let $M \leq {}^2G_2(q)$ and M be maximal in ${}^2G_2(q)$. Then either M is conjugate to $M_6 := {}^2G_2(3^\ell)$ for some divisor ℓ of $2n+1$, or M is conjugate to one of the subgroups M_i in the following table:*

Table 2: The maximal subgroups of ${}^2G_2(q)$

Group	Structure	Remarks
M_1	$Q : K$	the normalizer of Q in ${}^2G_2(q)$
M_2	$\mathbb{Z}_2 \times L_2(q)$	the centralizer of an involution in ${}^2G_2(q)$
M_3	$(\mathbb{Z}_2^2 \times D_{(q+1)/2}) : \mathbb{Z}_3$	the normalizer of a four-subgroup
M_4	$\mathbb{Z}_{q+m+1} : \mathbb{Z}_6$	the normalizer of $\mathbb{Z}_{q+m+1}$
M_5	$\mathbb{Z}_{q-m+1} : \mathbb{Z}_6$	the normalizer of $\mathbb{Z}_{q-m+1}$

Moreover, we see that from [8], the Sylow 3-subgroup Q can be identified with the group consisting of all triples (α, β, γ) from $\mathbb{F}_q$ with multiplication:

$$(\alpha_1, \beta_1, \gamma_1)(\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2 - \alpha_1\alpha_2^m, \gamma_1 + \gamma_2 - \alpha_1^m\alpha_2^m - \alpha_2\beta_1 + \alpha_1\alpha_2^{m+1}).$$

It is easy to check that $(0,0,\gamma)(0,\beta,0) = (0,\beta,\gamma)$. Set $Q_1 = \{(0,0,\gamma)|\gamma \in \mathbb{F}_q\}$ and $Q_2 = \{(0,\beta,0)|\beta \in \mathbb{F}_q\}$, then $Q_1 \cong Q_2 \cong \mathbb{Z}_3^{2n+1}$.

For a group Q, $Z(Q)$, $\Phi(Q)$, Q' denote the center, Frattini subgroup, and the derived subgroup of Q, respectively. Then $Q' = \Phi(Q) = Q_1 \times Q_2$, $Z(Q) = Q_1$, and Q' is an elementary abelian 3-group. For any $(\alpha,\beta,\gamma) \in Q$ and $k \in K$,

$$(\alpha,\beta,\gamma)^k = (k\alpha,\ k^{1+m}\beta,\ k^{2+m}\gamma).$$

Lemma 2.15 ([8, 15]) *Let Q, M, Q_2, M_2 and K as above, then*

(1) *the normalizer of any subgroup of Q is contained in M_1;*

(2) *for any $g \in {}^2G_2(q)$, either $Q^g = Q$ or $Q^g \cap Q = 1$;*

(3) *Q_2 is a Sylow 3-subgroup of M_2 and $N_{M_2}(Q_2) = 2 \times (Q_2 : \langle k^2 \rangle)$ with $\langle k \rangle = K$.*

Lemma 2.16 ([8, Lemma 3.2]) *The following hold for the cyclic subgroup K:*

(1) *K is transitive on $Q_1 \setminus \{1\}$ acting by conjugation;*

(2) *K has two orbits $(0,1,0)^K$, $(0,-1,0)^K$ on $Q_2 \setminus \{1\}$ acting by conjugation.*

From above lemmas, we have the following properties of the subgroups of ${}^2G_2(q)$.

Lemma 2.17 *If $H \leq M_1$ and $(q-1) \mid |H|$, then $K \leq H$.*

Proof. Let p be a prime divisor of $q-1$. If $P \in Syl_p(M_1)$, then since $(p,3) = 1$ and $Q \cap K = 1$, we have $P \in Syl_p(K)$. Note that K is cyclic, the Sylow p-subgroup of K is unique, and so the Sylow p-subgroup of M_1 is unique. On the other hand, if $P_0 \in Syl_p(H)$, since $H \leq M_1$, then $P_0 = P \cap H$. Moreover, $|P_0| = |P|$ implies that $P = P_0 \leq H$. Since p is arbitrary, all Sylow subgroups of K are contained in H, and so $K \leq H$. □

Corollary 1 *Let $H \leq M_1$ and $|H| = q(q-1)$. Then $H = A : K$ where A is the Sylow 3-subgroup of H.*

Proof. Since $Q \trianglelefteq M_1$, we have $A = H \cap Q$ and $A \trianglelefteq H$. By Lemma 2.17, $K \leq H$. Now $A \cap K = 1$, and so $H = A : K$. □

Lemma 2.18 *Let Q_2 be a Sylow 3-subgroup of M_2 and $H_2 := N_{M_2}(Q_2)$. If $Q_2 \leq Q$ and $M_1 = Q : K$, then the following hold:*

(1) *$H_2 = Q_2 : K$ and $H_2 \leq M_1$;*

(2) *for any $H \leq M_2$ satisfying $|H| = q(q-1)$, there exists $c \in M_2$ such that $H = H_2^c$ and $H \leq M_1^c$.*

Proof. Clearly, (1) holds by Lemma 2.15(1) and Corollary 1. Let $H \leq M_2$ and $|H| = q(q-1)$. Note that $M_2 \cong \mathbb{Z}_2 \times L_2(q)$. Since $H \lesssim \mathbb{Z}_2 \times L_2(q)$ and $H_2 \lesssim \mathbb{Z}_2 \times L_2(q)$, then by the list of maximal subgroups of $L_2(q)$, we know that $H \cong H_2 \cong \mathbb{Z}_2 \times ([q] : Z_{\frac{q-1}{2}})$. Let σ be an automorphism from H_2 to H. Then $Q_2^\sigma \trianglelefteq H$ since $Q_2 \trianglelefteq H_2$. Moreover, since $q \mid |H|$, the Sylow 3-subgroup of H is conjugate to Q_2 in M_2 and so $Q_2^\sigma = Q_2^c \trianglelefteq H$ for $c \in M_2$. It follows that

$$H \leq N_{M_2}(Q_2^c) = N_{M_2}(Q_2)^c = H_2^c.$$

Therefore $H = H_2^c$.

Note that if $Q^c \neq Q$, then from $Q_2^c \leq Q^c$ and Lemma 2.15(1), we get $H = N_{M_2}(Q_2^c) \leq M_1^c$, and so (2) holds.

Now, we prove that $Q^c \neq Q$. If $Q^c = Q$, then $Q_2^c \leq Q$, and so $H \leq M_1$. By Corollary 1, we have $H = Q_2^c : K$ and $H_2 = Q_2 : K$. Since $Q_2 \trianglelefteq Q'$, $Q_2^c \trianglelefteq Q'$. Recall that $Q' = Q_1 \times Q_2$ is an elementary abelian 3-group, so $Q_2^c \cap Q_1 \neq 1$ or $Q_2^c \cap Q_2 \neq 1$. Now suppose that $(0, \beta, 0) \in Q_2^c \cap Q_2$, since $Q_2^c \cap Q_2 \leq Q_2$, we have $(0, \beta, 0)^{-1} = (0, -\beta, 0) \in Q_2^c \cap Q_2$. This, together with $K \leq H$ and $K \leq H_2$, implies $(0, \beta, 0)^K \cup (0, -\beta, 0)^K = Q_2 \setminus \{1\} = Q_2^c \setminus \{1\}$. Hence $Q_2^c = Q_2$, a contradiction. Similarly, if $Q_2^c \cap Q_1 \neq 1$, we have $Q_2^c = Q_1$, a contradiction. □

Lemma 2.19 *Suppose that $H \leq {}^2G_2(q)$ and $|H| = q(q-1)$. Then H is conjugate to $H_1 = Q_1 : K$ or $H_2 = Q_2 : K$, and there are only two conjugacy classes of subgroups of order $q(q-1)$ in ${}^2G_2(q)$.*

Proof. Let $H \leq {}^2G_2(q)$ and $|H| = q(q-1)$. By Lemma 2.14, H must be contained in a conjugacy of M_1 or M_2. Firstly, if $H^{g^{-1}} \leq M_1$, then by Corollary 1, $H^{g^{-1}} = A : K$ where A is a Sylow 3-subgroup of $H^{g^{-1}}$. We now show that $A \leq Q'$. Assume that F is a maximal subgroup of Q such that $A \leq F$. If $A \cap Q' = 1$, then by Lemma 2.5 and the fact $Q' \leq F$, we have $|F : A| \geq |F \cap Q' : A \cap Q'| = q^2$, and so $|F| \geq q^3$, a contradiction. Therefore, there exists an element $(0,\beta,\gamma) \in A \cap Q'$, which implies that $A \setminus \{1\} = (0,\beta,\gamma)^K \subseteq Q' \setminus \{1\}$ and hence $A \leq Q'$. It follows that $A \cap Q_1 \neq 1$ or $A \cap Q_2 \neq 1$. Similar to the proof of Lemma 2.18, if $A \cap Q_1 \neq 1$, then $A = Q_1$ and so $H^{g^{-1}} = H_1$, and if $A \cap Q_2 \neq 1$, then $A = Q_2$ and so $H^{g^{-1}} = H_2$. Secondly, if H contained in a conjugacy of M_2, then H is conjugate to H_2 by Lemma 2.18(2). □

Lemma 2.20 *Let $H \leq {}^2G_2(q)$ and $|H| = q^2(q-1)$. Then H is conjugate to $Q' : K$, and there are only one conjugacy class of subgroups of order $q^2(q-1)$ in ${}^2G_2(q)$.*

Proof. Since Q' *char* $Q \trianglelefteq M_1$, so $Q' : K$ is a subgroup of M_1 with order $q^2(q-1)$. Suppose that $H \leq {}^2G_2(q)$ and $|H| = q^2(q-1)$. By Lemma 2.14, we have $H^{g^{-1}} \leq M_1$. Similarly as the proof of Corollary 1, we get that $H^{g^{-1}}$ has the structure $A : K$ where A is the Sylow 3-subgroup of $H^{g^{-1}}$. Let F be a maximal subgroup of Q satisfying $A \leq F$. Since $|F : A| \geq |F \cap Q_i : A \cap Q_i|$, we have $|A \cap Q_i| > 1$, which implies $Q_i = Q_i^K \leq A^K = A$ for $i = 1, 2$. So $Q' \leq A$, and it follows that $Q' = A$ and $H^{g^{-1}} = Q' : K$ in M_1. □

Similarly, we have the following result on the Suzuki group ${}^2B_2(q)$ by [9] and [7, p.250].

Lemma 2.21 *Suppose that Q is the Sylow 2-subgroup of ${}^2B_2(q)$ and $M_1 = Q : K$ is the normalizer of Q. Let $H \leq {}^2B_2(q)$ and $|H| = q(q-1)$. Then H is conjugate to $Z(Q) : K$. There exists a unique conjugacy class of subgroups of order $q(q-1)$ in ${}^2B_2(q)$.*

3 Proof of Theorem 1

3.1 T is the Ree group

Proposition 3.1 *Suppose that G and $\mathcal{D}$ satisfy hypothesis of Theorem 1. Let B be a block. If $T = {}^2G_2(q)$ with $q = 3^{2n+1}$, then $\mathcal{D}$ is the Ree unital or one of the following:*

(1) *$\mathcal{D}$ is a 2-$(q^3+1, q, q-1)$ design with $G_B = Q_1 : K$ or $Q_2 : K$;*

(2) *$\mathcal{D}$ is a 2-(q^3+1, q^2, q^2-1) with $G_B = Q' : K$.*

This proposition will be proved into two steps. We first assume that there exists a design satisfying the assumptions and obtain the possible parameters (v, b, r, k, λ) in Lemma 3.1, then prove the existence of the designs using Lemma 2.12.

Lemma 3.1 *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T = {}^2G_2(q)$ with $q = 3^{2n+1}$, then $(v, b, r, k, \lambda) = (q^3+1, q^2(q^3+1), q^3, q, q-1)$ or $(q^3+1, q(q^3+1), q^3, q^2, q^2-1)$ or $\mathcal{D}$ is the Ree unital.*

Proof. Let $T_\alpha := G_\alpha \cap T$. Since G is primitive on $\mathcal{P}$, then T_α is one of the cases in Lemma 2.14 by [13]. First, the cases that $T_\alpha = \mathbb{Z}_2^2 \times D_{(q+1)/2}$ and $\mathbb{Z}_{q\pm m+1} : \mathbb{Z}_6$ are impossible by Lemma 2.8. If $T_\alpha = \mathbb{Z}_2 \times L_2(q)$, then $v = q^2(q^2 - q + 1)$ and $(|G_\alpha \cap T|, v - 1) = (q(q^2 - 1), q^4 - q^3 + q^2 - 1) = q - 1$. But since r divides $f(|G_\alpha \cap T|, v - 1)$, which is too small to satisfy $v < r^2$. Similarly, T_α cannot be ${}^2G_2(3^\ell)$.

We next assume that $T_\alpha = Q : K$, and so $v = q^3 + 1$. Moreover, from [7, p.252], T is 2-transitive on $\mathcal{P}$, so T is flag-transitive by Lemma 2.4. Hence we may assume that $G = T = {}^2G_2(q)$. The equations in Lemma 2.1 show

$$b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda q^3(q^3+1)}{k(k-1)},$$

then by the flag-transitivity of T, we have

$$|T_B| = \frac{|T|}{b} = \frac{(q-1)k(k-1)}{\lambda}.$$

Let M be a maximal subgroup of T such that $T_B \leq M$. Then since $|T_B| \mid |M|$ and $q \geq 27$, M must be M_1 or M_2 shown in Lemma 2.14.

If $T_B \leq M_1$, then $k(k-1) \mid \lambda q^3$. Furthermore, since $(r, \lambda) = 1$ and so $\lambda \mid (k-1)$ by Lemma 2.1(2). Therefore $\lambda = k - 1$, and it follows that $r = v - 1 = q^3$ and $k \mid q^3$. Note that M_1 is point stabilizer of T in this action. So there exists α such that $M_1 = T_\alpha$ and $T_B \leq T_\alpha$. However, the flag-transitivity of T implies $\alpha \notin B$. For any point $\gamma \in B$, $T_{\gamma B} \leq T_{\alpha\gamma}$. By Lemma 2.13, $|T_{\alpha\gamma}| = q - 1$, and so $|T_{\gamma B}| \mid (q-1)$. On the other hand, from

$$|B^{T_\gamma}| = |T_\gamma : T_{\gamma B}| \leq |B^{G_\gamma}| = |G_\gamma : G_{\gamma B}| = r = q^3,$$

we have $T_{\gamma B} = T_{\alpha\gamma}$ and so $B^{T_{\alpha\gamma}} = B$. Since the stabilizer of three points is of order 2 by Lemma 2.13, so the size of $T_{\alpha\gamma}$-orbits acting on $\mathcal{P} \setminus \{\alpha, \gamma\}$ is $q - 1$ or $\frac{1}{2}(q-1)$. This , together with $B^{T_{\alpha\gamma}} = B$ and $\alpha \notin B$, implies that $k - 1 = a\frac{(q-1)}{2}$ for an integer a. Recall that $k \mid q^3$ and $k < r$, we get $k = q$ or $k = q^2$. If $k = q$, then

$$b = q^2(q^3+1), r = q^3, \lambda = q - 1.$$

If $k = q^2$, we have

$$b = q(q^3+1), r = q^3, \lambda = q^2 - 1.$$

Now we deal with the case that $T_B \leq M_2$ by the similar method in [12, Theorem 3.2].

If T_B is a solvable subgroup of $M_2 \cong \mathbb{Z}_2 \times L_2(q)$, then T_B must map into either $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_2 \times D_{q\pm1}$ or $\mathbb{Z}_2 \times ([q] : \mathbb{Z}_{\frac{q-1}{2}})$. Obviously, the former two cases are impossible. For the last case, $T_B \lesssim \mathbb{Z}_2 \times ([q] : \mathbb{Z}_{\frac{q-1}{2}})$. Since $T_B \leq M_2$, by Lemma 2.18, this can be reduced to the case $T_B \leq M_1$.

If T_B is non-solvable, then it embeds in $\mathbb{Z}_2 \times L_2(q_0)$ with $q_0^\ell = q = 3^{2n+1}$. The condition that $|T_B|$ divides $|\mathbb{Z}_2 \times L_2(q_0)|$ forces $q_0 = q$ and so T_B is isomorphic to $\mathbb{Z}_2 \times L_2(q)$ or $L_2(q)$.

If $T_B \cong \mathbb{Z}_2 \times L_2(q)$, then $T_B = M_2$ and so $b = q^2(q^2-q+1)$. Hence, from Lemma 2.1, we have $k \mid q(q+1)$, $q^2 \mid r$ and $r \mid q^3$. Since $k \geq 3$, then the fact that the stabilizer of three points is of order 2 implies that T_B cannot acting trivially on the block B. Moreover, since $q+1$ is the smallest degree of any non-trivial action of $L_2(q)$, we have $k = \frac{\lambda(v-1)}{r} + 1 \geq q+1$.

If the design $\mathcal{D}$ is a linear space, then $\mathcal{D}$ is the Ree unital (see [12]) with parameters

$$(v, b, r, k, \lambda) = (q^3+1, q^2(q^2-q+1), q^2, q+1, 1)$$

and T is flag-transitive with the block stabilizer M_2.

If $\lambda > 1$, we claim that $\lambda = k-1$. Clearly, $\lambda \mid (k-1)$ as $(r, \lambda) = 1$ by Lemma 2.1(2). If $3 \mid (k-1)$ and $(k, 3) = 1$, then since $k \mid q(q+1)$ and $k \geq q+1$, we have $k = q+1$ and so $\lambda \mid q$, which contradicts $(r, \lambda) = 1$ as $q^2 \mid r$. Hence $(k-1, 3) = 1$. Moreover, $(k-1) \mid \lambda q^3$ implies that $(k-1) \mid \lambda$. So we have $\lambda = k-1$.

Let Δ_1, Δ_2,..., Δ_t be the orbits of M_2. Since M_2 is the block stabilizer of the Ree unital, it has an orbit of size $q+1$. Without loss of generality, suppose that $|\Delta_1| = q+1$. On the one hand, recall that $k \mid q(q+1)$ and T is flag transitive, $T_B = M_2$ has at least one orbit with size less than $q(q+1)$. On the other hand, we show that $|\Delta_i| > q(q+1)$ for $i \neq 1$ in the following and we obtain the desired contradiction. Assume that $\delta \in \mathcal{P} \setminus \Delta_1$, we claim that $(M_2)_\delta$ is a 2-group. Let p be a prime divisor of $|(M_2)_\delta|$ and P be a Sylow p-subgroup of $(M_2)_\delta$. If $p \neq 2$ and $p \neq 3$, then since $(M_2)_\delta \leq T_\delta$, we have $p \mid (q-1)$. Obviously, since Δ_1 is an orbit of M_2 and $P \leq (M_2)_\delta$, and so P acts invariantly on Δ_1 and $\mathcal{P} \setminus \Delta_1$. Note that the length of a P-orbit is either 1 or divided by p, so P fixes at least two points in Δ_1. Moreover, P also fixes δ. Therefore P fixes at least three points of $\mathcal{P}$, which is impossible as the order of the stabilizer of three points is 2 by Lemma 2.13(3). If $p = 3$, since P fixes the point $\delta \in \mathcal{P}\setminus\Delta_1$ and $|\mathcal{P}\setminus\Delta_1| = q^3-q$, then P fixes at least three points in $\mathcal{P}\setminus\Delta_1$, which is also impossible. As a result, $(M_2)_\delta$ is a 2-group. The fact that the Sylow 2-subgroup of T is of order 8 implies that the sizes of the M_2-orbits Δ_i $(i \neq 1)$ are at least $\frac{q(q^2-1)}{8}$ and hence larger than $q(q+1)$, which contradicts the fact $k \mid q(q+1)$. Therefore, $T_B \ncong \mathbb{Z}_2 \times L_2(q)$. Similarly, $T_B \ncong L_2(q)$. Thus T_B is not a non-solvable subgroup in M_2. □

Proof of Proposition 3.1. We use Lemma 2.12 to prove the existence of the design with parameters listed in Lemma 3.1.

Assume that $(v,b,r,k,\lambda) = (q^3+1, q^2(q^3+1), q^3, q, q-1)$. Then from Lemma 2.19 we known that there are only two conjugacy classes of subgroups of order $q(q-1)$ in T and $H_1 = Q_1 : K \leq T_\alpha$ and $H_2 = Q_2 : K \leq T_\alpha$ as representatives, respectively.

First, we consider the orbits of H_1. Let $\gamma \neq \alpha$ be the point fixed by K. Since $K \leq H_1$, then $K_\gamma = K \leq (H_1)_\gamma \leq T_{\alpha\gamma} = K$, which implies $(H_1)_\gamma = T_{\alpha\gamma}$ and so $|H_1 : (H_1)_\gamma| = |\gamma^{H_1}| = q$. It is easy to see that $|\delta^{H_1}| \neq q$ for any point $\delta \neq \alpha, \gamma$. Therefore, H_1 has only one orbit of size q. Let $B_1 = \gamma^{H_1}$.

Now we show that $H_1 = T_{B_1}$, which implies $|B_1^T| = b$. Since $H_1 \leq T_{B_1}$ and $B_1 = \gamma^{H_1} = \gamma^{T_{B_1}}$, then $|H_1 : (H_1)_\gamma| = |T_B : T_{\gamma B_1}| = q$. If $K = (H_1)_\gamma < T_{\gamma B_1}$, then 3 divides $|T_{\gamma B_1} : T_{\delta\gamma B_1}|$ for any $\delta \in B_1 \setminus \{\gamma\}$ by Lemma 2.13(3). It follows that $3 \mid (q-1)$, a contradiction. As a result, $K = (H_1)_\gamma = T_{\gamma B_1}$ and so $H_1 = T_{B_1}$. Let $\mathcal{B}_1 := B_1^T$. Therefore $|\mathcal{B}_1| = |T : H_1| = b$. Let $\mathcal{B}_1$ be the set of blocks.

Finally, since T is 2-transitive on $\mathcal{P}$, the number of blocks which incident with two points is a constant. Hence $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}_1)$ is a 2-$(q^3+1, q, q-1)$ design admitting T as a flag transitive automorphism group by Lemma 2.12.

In a similar way, we get the design $\mathcal{D}_2$ satisfying all hypothesis when the subgroup is $H_2 = Q_2 : K$. Furthermore, since H_1 is not isomorphic to H_2, so $\mathcal{D}_1$ is not isomorphic to $\mathcal{D}_2$ by [6, 1.2.17].

Similarly , if $(v,b,r,k,\lambda) = (q^3+1, q(q^3+1), q^3, q^2, q^2-1)$, we can construct the design with these parameters. □

3.2 T is the Suzuki group

Proposition 3.2 *Suppose that G and $\mathcal{D}$ satisfy hypothesis of Theorem 1. If $T = {}^2B_2(q)$ with $q = 2^{2n+1}$, then $\mathcal{D}$ is a 2-$(q^2+1, q, q-1)$ design with $G_B = Z(Q) : K$ where $Q \in Syl_2(T)$ and $K = Z_{q-1}$.*

Proof. Suppose that $T = {}^2B_2(q)$ with order $(q^2+1)q^2(q-1)$. Then $|G| = f(q^2+1)q^2(q-1)$ where f divides $|Out(T)|$. By [9] or [27], the order of G_α is one of the following:

(1) $fq^2(q-1)$;

(2) $2f(q-1)$;

(3) $4f(q \pm \sqrt{2q}+1)$;

(4) $f(q_0^2+1)q_0^2(q_0-1)$ with $q_0^\ell = q$.

Since $|G| < |G_\alpha|^3$, we first have that $|G_\alpha| \neq 2f(q-1)$. If $|G_\alpha| = 4f(q \pm \sqrt{2q}+1)$, from the inequality $|G| < |G_\alpha|^3$, we get $f(q^2+1)q^2(q-1) < (4f)^3(2q)^3$, and so $q^2+q+1 \leq 4^3f^22^3$. Since $f \leq |Out(T)| = e$ and $q = p^e$, hence $q+1 < 4^32^3$ and $q = 2^7$, 2^5 or 2^3. If $q = 2^7$, then $|G| = f2^{14}(2^{14}-1)(2^7-1) > f^34^3(2^7+2^4+1)^3 = |G_\alpha|^3$ where $f = 7$ or 1, a contradiction. If $q = 2^5$, then $v = 198400$ or 325376 for $|G_\alpha| = 4f(q+\sqrt{2q}+1)$ or $4f(q-\sqrt{2q}+1)$ respectively. By calculating $(|G_\alpha|, v-1)$, since r divides $(|G_\alpha|, v-1)$, we know that r is too small. Similarly, we get $q \neq 2^3$.

If $|G_\alpha| = f(q_0^2+1)q_0^2(q_0-1)$ with $q_0^\ell = q$, then the inequality $|G| < |G_\alpha||G_\alpha|_{p'}^2$ forces $m = 3$. So $v = (q_0^4-q_0^2+1)q_0^4(q_0^2+q_0+1)$. Since r divides $(|G_\alpha|_{p'}, v-1)$, then $r \leq |G_\alpha|_{p'} \leq fq_0^3 < q_0^{9/2}$. From $v < r^2$, we get $(q_0^4-q_0^2+1)q_0^4(q_0^2+q_0+1) < r^2 < q_0^9$, which is impossible.

Now assume that $|G_\alpha| = fq^2(q-1)$. Then $v = q^2+1$ and T is 2-transitive by [7, p.250]. Hence, T is flag-transitive by Lemma 2.4. Similarly, we have $|T_B| = \frac{|T|}{b} = \frac{k(k-1)(q-1)}{\lambda}$. Let M be the maximal subgroup of T such that $T_B \leq M$ as in Lemma 3.1. The fact that $|T_B|$ divides $|M|$ implies that $|M| = q^2(q-1)$ and $k(k-1)$ divides λq^2. Similar to the proof of Lemma 3.1, we have $T_{\gamma B} = T_{\alpha\gamma}$ with the order $q-1$. Furthermore, we get

$$(v, b, r, k, \lambda) = (q^2+1, q(q^2+1), q^2, q, q-1).$$

Next we prove the existence of the design with above parameters by Lemma 2.12. Firstly, from Lemma 2.21 we know that the Suzuki group has a unique conjugacy class of subgroups of order $q(q-1)$, let $H := Z(Q) : K \leq T_\alpha$ as the representative.

Note that K is the stabilizers of two points in ${}^2B_2(q)$ by [11, p.187]. Let $\gamma \neq \alpha$ be the point fixed by K and $B = \gamma^H$. Then similar as the proof of Proposition 3.1 we get that B is the only H-orbit of length q and $H = T_B$. Let $\mathcal{B} = B^T$ be the set of blocks. Finally, since T is 2-transitive on $\mathcal{P}$, the number of blocks which incident with two points is a constant. Hence $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-$(q^2+1, q, q-1)$ design admitting T be a flag transitive automorphism group by Lemma 2.12. □

3.3 T is one of the remaining families

In this subsection, let

$$\mathcal{T} = \{{}^2F_4(q), {}^3D_4(q), G_2(q), F_4(q), E_6^{\epsilon}(q), E_7(q), E_8(q)\},$$

we will prove that there are no new design arise when $T \in \mathcal{T}$.

First, we show that G_α cannot be a parabolic subgroup of G for any $T \in \mathcal{T}$.

Lemma 3.2 *Suppose that G and $\mathcal{D}$ satisfy hypothesis of Theorem 1. If $T \in \mathcal{T}$, then G_α cannot be a parabolic subgroup of G.*

Proof. By Lemma 2.6, for all cases that $T \in \mathcal{T} \setminus E_6(q)$, there is a unique subdegree which is a power of p, so r is a power of p by Lemma 2.1(4). We can easily check that r is too small and the condition $r^2 > v$ cannot be satisfied. Now, assume that $T = E_6(q)$. If G contains a graph automorphism or $G_\alpha \cap T$ is P_2 or P_4, then there is also a unique subdegree which is a power of p and so r is too small again. If $G_\alpha \cap T$ is P_3 with type A_1A_4, then

$$v = \frac{(q^3+1)(q^4+1)(q^9-1)(q^6+1)(q^4+q^2+1)}{(q-1)}.$$

Since r divides $(|G_\alpha|, v-1)$, we have $r \mid eq(q-1)^5(q^5-1)$ and so r is too small to satisfy $r^2 > v$. If $G_\alpha \cap T$ is P_1 with type D_5, then

$$v = \frac{(q^8+q^4+1)(q^9-1)}{q-1}.$$

From [16], we know that there exists two non-trivial subdegrees:

$$d = \frac{q(q^3+1)(q^8-1)}{(q-1)} \quad \text{and} \quad d' = \frac{q^8(q^4+1)(q^5-1)}{(q-1)}.$$

Since $(d, d') = q(q^4+1)$, we have $r \mid q(q^4+1)$ by Lemma 2.1(4), which contradicts with $r^2 > v$. □

Let $\mathcal{T}_1 = \{F_4(q), E_6^{\epsilon}(q), E_7(q), E_8(q)\}$.

Lemma 3.3 *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T \in \mathcal{T}_1$ and G_α is non-parabolic, then G_α cannot be a maximal subgroup of maximal rank.*

Proof. If G_α is non-parabolic and of maximal rank, then for any $T \in \mathcal{T}_1$, we have a complete list of $T_\alpha := G_\alpha \cap T$ in [18, Tables 5.1-5.2]. All subgroups in [18, Table 5.2] and some cases in [18, Table 5.1] can be ruled out by the inequality $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$. Since r divides $(|G_\alpha|, v-1)$, for the remaining cases we have that $r^2 < v$, a contradiction.

For example, if $T = F_4(q)$ with order $q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$. Then T_α is one of the following: (1) $2.(L_2(q) \times PSp_6(q)).2$ (q odd); (2) $d.\Omega_9(q)$; (3) $d^2.P\Omega_8^+(q).S_3$; (4) ${}^3D_4(q).3$; (5) $Sp_4(q^2).2$ (q even); (6) $(Sp_4(q) \times Sp_4(q)).2$($q$ even); (7) $h.(L_3^\epsilon(q) \times L_3^\epsilon(q)).h.2$, with $d = (2, q-1)$ and $h = (3, q-\epsilon)$.

If $T_\alpha = 2.(L_2(q) \times PSp_6(q)).2$ with q odd, then

$$|T_\alpha| = q^{10}(q^2-1)^2(q^4-1)(q^6-1) \ \text{ and } \ v = q^{14}(q^4+1)(q^4+q^2+1)(q^6+1).$$

Since $(q^2+1) \mid v$ and $(q^4+q^2+1) \mid v$, then $(|G_\alpha|, v-1) \mid |Out(T)|(q^2-1)^4$ and so $r^2 < q^9 < v$, a contradiction.

If $T_\alpha = 2.P\Omega_9(q)$ with q odd, then

$$|T_\alpha| = q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1) \ \text{ and } \ v = q^8(q^8+q^4+1).$$

Since $q \mid v$, $(q^4+q^2+1) \mid v$, $v-1 \equiv 2 \pmod{q^4-1}$, we get r divides $2^4|Out(T)|(q^4+1)$ and so $r^2 < v$, a contradiction.

Cases (3)-(6) can be ruled out similarly, and Case (7) cannot occur because of $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$. $\square$

Lemma 3.4 *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T \in \mathcal{T}_1$ and G_α is non-parabolic, then $T_0 = Soc(G_\alpha \cap T)$ is simple and $T_0 = T_0(q_0) \in Lie(p)$.*

Proof. Assume that $T_0 = Soc(G_\alpha \cap T)$ is not simple. Then by Lemma 2.9 and Lemma 3.3, one of the following holds:

(1) $G_\alpha = N_G(E)$, where E is an elementary abelian group given in [4, Theorem 1(II)];

(2) $T = E_8(q)$ with $p > 5$, and T_0 is either $A_5 \times A_6$ or $A_5 \times L_2(q)$;

(3) T_0 is as in Table 1.

From [4, Theorem 1(II)], we check that all subgroups in Case (1) are local and too small to satisfy $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$.

The order of subgroup in Case (2) is too small.

For Case (3), since G_α is not simple and not local by [4, Theorem 1], G_α is of maximal rank by [25, p.346], which has already been ruled out in Case (1). Therefore, T_0 is simple.

Now assume that $T_0 = T_0(q_0) \notin Lie(p)$. Then for all T, we find the possibilities of T_0 in [21, Table 1]. Some cases can be ruled out by the inequality $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$. In each of the remaining cases, since r must divides $(|G_\alpha|, v-1)$, r is too small to satisfy $v < r^2$. For example, assume that $T = F_4(q)$. If $T_0 \notin Lie(p)$, then according to [21, Table 1], it is one of the following: A_{5-10}, $L_2(7)$, $L_2(8)$, $L_2(13)$, $L_2(17)$, $L_2(25)$, $L_2(27)$, $L_3(3)$, $U_3(3)$, $U_4(2)$, $Sp_6(2)$, $\Omega_8^+(2)$, ${}^3D_4(2)$, J_2, J_2, $A_{11}(p = 11)$, $L_3(4)(p = 3)$, $L_4(3)(p = 2)$, ${}^2B_2(8)(p = 5)$, $M_{11}(p = 11)$. The possibilities of T_0 such that $|G| < |G_\alpha|^3$ are $A_9(q = 2)$, $A_{10}(q = 2)$, $Sp_6(2)(q = 2)$, $\Omega_8^+(2)(q = 2, 3)$, ${}^3D_4(2)(q = 2, 3)$, $J_2(q = 2)$, $L_4(3)(q = 2)$. However, since $r \mid (|G_\alpha|, v-1)$, we have $r^2 < v$ for all these cases, which is a contradiction.□

Lemma 3.5 *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T_0 = T_0(q_0)$ is a simple group of Lie type and G_α is non-parabolic, then $T \notin \mathcal{T}_1$.*

Proof. First assume that $T = F_4(q)$. If $\text{rank}(T_0) > \frac{1}{2}\text{rank}(T)$, then by Lemma 2.10 and Lemma 3.3, the only possible cases of $G_\alpha \cap T$ satisfying $|G| < |G_\alpha|^3$ are $F_4(q^{\frac{1}{2}})$ and $F_4(q^{\frac{1}{3}})$ when $q_0 > 2$. If $G_\alpha \cap T = F_4(q^{\frac{1}{2}})$, then $v = q^{12}(q^6+1)(q^4+1)(q^3+1)(q+1) > q^{26}$. Since q, $q+1$, q^2+1 and q^3+1 are factors of v, then $r \mid 2e(q-1)^2(q^3-1)^2$ by $r \mid (|G_\alpha|, v-1)$, which implies that $r^2 < v$, a contradiction. If $G_\alpha \cap T = F_4(q^{\frac{1}{3}})$, since $p \mid v$, then r divides $|G_\alpha|_{p'}$, which also implies $r^2 < v$. When $q_0 = 2$, the subgroups $T_0(2)$ with $\text{rank}(T_0) > \frac{1}{2}\text{rank}(T)$ that satisfy $|G| < |G_\alpha|^3$ are $A_4^\epsilon(2)$, $B_3(2)$, $B_4(2)$, $C_3(2)$, $C_4(2)$ or $D_4^\epsilon(2)$. But in each case, $r \mid (|G_\alpha|, v-1)$ forces $r^2 < v$, a contradiction. If $\text{rank}(T_0) \le \frac{1}{2}\text{rank}(T)$, then from Lemma 2.11, we have $|G_\alpha| < 4q^{20}\log_p q$. Looking at the orders of groups of Lie type, we see that if $|G_\alpha| < 4q^{20}\log_p q$, then $|G_\alpha|_{p'} < q^{12}$, and so $|G_\alpha||G_\alpha|^2_{p'} < |G|$, contrary to Lemma 2.8.

For $T = E_6^\epsilon(q)$, if $\text{rank}(T_0) > \frac{1}{2}\text{rank}(T)$, then when $q_0 > 2$, by Lemma 2.10 the only possibilities are $E_6^\epsilon(q^{\frac{1}{2}})$, $E_6^\epsilon(q^{\frac{1}{3}})$, $C_4(q)$ and $F_4(q)$. In all these cases r are too small. When $q_0 = 2$, the possibilities $T_0(2)$ satisfying $|G| < |G_\alpha|^3$ with order dividing $|E_6^\epsilon(2)|$ are $A_5^\epsilon(2)$, $B_4(2)$, $C_4(2)$, $D_4^\epsilon(2)$ and $D_5^\epsilon(2)$. However, since $r \mid (|G_\alpha|, v-1)$, for all these cases we obtain $r^2 < v$, a contradiction. If $\text{rank}(T_0) \le \frac{1}{2}\text{rank}(T)$, then from Lemma 2.11, we have $|G_\alpha| < 4q^{28}\log_p q$. By further check the orders of groups of Lie type, we see that $|G_\alpha|_{p'} < q^{17}$, and so $|G_\alpha||G_\alpha|^2_{p'} < |G|$, a contradiction.

Assume that $T = E_7(q)$. If $rank(T_0) \leq \frac{1}{2}rank(T)$, then by Lemma 2.11 $|G_\alpha|^3 \leq |G|$, a contradiction. If $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$, then when $q_0 > 2$, B by Lemma 2.10, the only cases $T \cap G_\alpha$ satisfying $|G| < |G_\alpha|^3$ are $G_\alpha \cap T = E_7(q^{\frac{1}{s}})$, where $s = 2$ or 3. But in all cases we have $r^2 < v$. If $q_0 = 2$, then the possible subgroups such that $|G| < |G_\alpha|^3$ with order dividing $|E_7(2)|$ are $A_6^\epsilon(2)$, $A_7^\epsilon(2)$, $B_5(2)$, $C_5(2)$, $D_5^\epsilon(2)$ and $D_6^\epsilon(2)$. However in all of these cases, since $r \mid (|G_\alpha|, v-1)$ we have $r^2 < v$, a contradiction.

Assume that $T = E_8(q)$. If $\mathrm{rank}(T_0) \leq \frac{1}{2}\mathrm{rank}(T)$, then by Lemma 2.11 we get $|G_\alpha|^3 < |G|$, a contradiction. Therefore, $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$. If $q_0 > 2$, then Lemma 2.10 implies $G_\alpha \cap T = E_8(q^{\frac{1}{s}})$, with $s = 2$ or 3. However in both cases we get a small r with $r^2 < v$, a contradiction. If $q_0 = 2$, then $\mathrm{rank}(X_0) \geq 5$. All subgroups satisfying $|G_\alpha|^3 > |G|$ are $A_8^\epsilon(2)$, $B_7(2)$, $B_8(2)$, $C_7(2)$, $C_8(2)$, $D_8^\epsilon(2)$ and $D_7^\epsilon(2)$. But for all these cases we have $r^2 < v$. □

Lemma 3.6 *If $T = G_2(q)$ with $q = p^e > 2$, then G_α cannot be a non-parabolic maximal subgroup of G.*

Proof. Suppose that $T = G_2(q)$ with $q > 2$ since $G_2(2)' = PSU_3(3)$. All maximal subgroups of G can be found in [13] for odd q and in [3] for even q.

Assume that G_α be a non-parabolic maximal subgroup of G. First we deal with the case where $G_\alpha \cap T = SL_3^\epsilon(q).2$ with $\epsilon = \pm$. Then we have $v = \frac{1}{2}q^3(q^3 + \epsilon 1)$. By Lemma 2.1 and [25, Section 8] we conclude that r divides $\frac{(q^3-\epsilon 1)}{2}$ for odd q (cf. [25, Section 4, Case 1, $i = 1$]) and r divides $(q^3 - \epsilon 1)$ for even q (cf. [25, Section 3, Case 8]). The case that q odd is ruled out by $v < r^2$. If q is even, then $r = q^3 - \epsilon 1$. This, together with $k < r$, implies $k - 1 = \lambda\frac{q^3+\epsilon 2}{2}$, and so $\lambda = 1$ or $\lambda = 2$. From the result of [25] we known that $\lambda \neq 1$. If $\lambda = 2$, then since $k < r$, we have $\epsilon = -$. It follows that $k = q^3 - 1$ and $r = q^3 + 1$. This is impossible by Lemma 2.3 and [24, Theorem 1].

Now, if $G_\alpha \cap T = {}^2G_2(q)$ with $q = 3^{2n+1} \geq 27$, then $v = q^3(q+1)(q^3-1)$. Note that $q \mid v$ and $(q^2-1, v-1) = 1$, we have $(|G_\alpha|, v-1) \mid e(q^2-q+1)$, and it follows that $r^2 < v$, a contradiction.

The cases that $G_\alpha \cap T$ is $G_2(q_0)$ or $(SL_2(q) \circ SL_2(q)) \cdot 2$ can be ruled out similarly.

Using the inequality $|G| < |G_\alpha|^3$ and the fact that r divides $(|G_\alpha|, v-1)$, we find r too small to satisfy $r^2 > v$ for every other maximal subgroup. □

Lemma 3.7 *If $T = {}^2F_4(q)$, then G_α cannot be a non-parabolic maximal subgroup.*

Proof. Let $T = {}^2F_4(q)$ and G_α be a non-parabolic maximal subgroup of G. Then from the list of the maximal subgroups of G in [23], there are no subgroups G_α satisfying $|G| < |G_\alpha||G_\alpha|^2_{p'}$, except for the case $q = 2$. For the case $q = 2$, $G_\alpha \cap T$ is $L_3(3).2$ or $L_2(25)$. However in each case, since r divides $(|G_\alpha|, v-1)$, and so r is too small. □

Lemma 3.8 *If $T = {}^3D_4(q)$, then G_α cannot be a non-parabolic maximal subgroup.*

Proof. If $T = {}^3D_4(q)$ and G_α is a non-parabolic maximal subgroup of G, then all possibilities of $G_\alpha \cap T$ are listed in [14]. However, for all cases, the fact that r divide $(|G_\alpha|, v-1)$ give a small r which cannot satisfy the condition $v < r^2$. For example, if $G_\alpha \cap T$ is $G_2(q)$ of order $q^6(q^2-1)(q^6-1)$, then $v = q^6(q^8+q^4+1)$. Since $q \mid v$ and $(q^4+q^2+1) \mid v$, then $r \mid 3e(q^2-1)^2$, which contradicts with $v < r^2$. □

Lemma 3.9 *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If the socle $T \in \mathcal{T}$, then G_α cannot be a non-parabolic maximal subgroup.*

Proof. It is follows from Lemmas 3.3−3.8. □

Proof of Theorem 1. Now Theorem 1 is an immediate consequence of Propositions 3.1-3.2 and of Lemmas 3.2 and 3.9.

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