

GROUP CONVOLUTIONAL CODES

S. Estrada

Dept. de Matemática Aplicada.

Universidad de Murcia. Campus de Espinardo

Espinardo, Murcia, 30100, Spain

J. R. García Rozas, J. Peralta, E. Sánchez García

Dept. Álgebra y Análisis Matemático

Universidad de Almería

04120 Almería, Spain

e-mail: jrgrozas@ual.es

Abstract

In this note we introduce the concept of group convolutional code. We make a complete classification of the minimal S_3 -convolutional codes over the field of five elements by means of Jategaonkar's theorems.

Mathematics subject classification (2000): 16S36, 94B10.

Keywords: skew polynomial rings, Jategaonkar's theorem, convolutional codes, group codes.

1 Introduction

Block codes as left ideals in group algebras were introduced by S. D. Bermann in [1]. After that, several papers of MacWilliams, Landrock, Damgard, Lieber, Ward, Zimmermman and others gave more credit to this theory ([3], [10], [11],[12],[13],[18]). In the context of convolutional codes, P. Piret [15],

studied the H -codes, which can be seen as a generalized version of the group block codes in the convolutional case.

On the other hand, the concept of cyclic convolutional codes and their first properties were proposed by P. Piret and C. Roos in [14] and [16], respectively. More recently, H. Gluesing-Luerssen et al. ([5], [6]) continue the study of cyclic convolutional codes. In the present paper, we give a definition of group convolutional code, which is a generalization of cyclic convolutional code and group block code. We introduce some important techniques in non-commutative algebra, concretely, the structure theorems of skew polynomials rings given in [9] by Jategaonkar.

The paper is organized as follows. In Section 2 we make the necessities definitions related with convolutional codes we will use throughout the paper. Then we introduce the concept of group convolutional code and minimal one, this last will be the main object of our study since they are the building blocks for the rest of the codes. Next we summarize Jategaonkar's result on the structure of skew polynomial rings over semisimple rings, that we will use in the last section. Finally, Section 3 deals with the classification of the minimal S_3 -convolutional codes over the field of five elements. The isomorphism established between the skew polynomial ring and certain direct sums of rings of matrices over simplest skew polynomial rings will be crucial. Note that these codes are the smallest non-commutative group convolutional codes to consider. This result opens the way to consider more complicated examples.

2 Preliminaries and first results

Throughout this paper, \mathbb{F} denotes a finite field and n a positive integer such that the characteristic of \mathbb{F} , $\text{char}(\mathbb{F})$, does not divide n . This assumption guarantees that for any group G of order n , the group algebra $\mathbb{F}[G]$ is semi-simple.

This paper deals with convolutional codes with additional algebraic structure. We adopt the following definition of convolutional code from [6].

Definition 1 *A convolutional code of length n and dimension k is a direct summand \mathcal{C} of $\mathbb{F}[z]^n$ of rank k as $\mathbb{F}[z]$ -module.*

Let r be a positive integer. Any matrix $M \in M_{r \times n}(\mathbb{F}[z])$ with rows given by a generating set of \mathcal{C} as $\mathbb{F}[z]$ -module is called generating matrix of the code \mathcal{C} . If $r = k$, then M is called generator matrix or encoder of \mathcal{C} .

The maximal degree of the k -minors of an encoder M is called the *complexity* of the code. A code of complexity zero is said to be a block code.

The *free distance* of a convolutional code is defined as follows. First, given $v = \sum_{i=0}^m v_i z^i \in \mathbb{F}[z]^n$ where $v_j \in \mathbb{F}^n$, we define its weight as $wt(v) = \sum_{i=0}^m wt(v_i)$, where $wt(v_i)$ is the usual Hamming weight of the vector $v_i \in \mathbb{F}^n$. Then, the free distance of a convolutional code $\mathcal{C} \subseteq \mathbb{F}[z]^n$ is defined as, $dist(\mathcal{C}) = \min\{wt(v) \mid v \in \mathcal{C} - \{0\}\}$.

We call (n, k, δ) -convolutional code a code with length n , dimension (or rank) k and complexity δ . We say that a (n, k, δ) -convolutional code with free distance d , \mathcal{C} , is a MDS code (maximal distance separable) if $d = S(n, k, \delta)$, where $S(n, k, \delta)$ is the generalized Singleton bound, $S(n, k, \delta) = (n - k)(\lfloor \frac{\delta}{k} \rfloor + 1) + \delta + 1$. For a given size field q , we have the so called *Griesmer bound* for convolutional codes over the field of q elements. It is defined as $G(n, k, \delta; m)_q = \max\{d' \in \{1, \dots, S(n, k, \delta)\} \mid \sum_{l=0}^{k(m+i)-\delta-1} \lceil \frac{d'}{q^l} \rceil \leq n(m+i)\}$ for all $i \in \widehat{N}\}$.

Here m denotes the maximum taken over the Forney indices of a (n, k, δ) -convolutional code, and it is called the *memory* of the code. Also, \widehat{N} denotes $\{1, 2, \dots\}$ if $km = \delta$ or $\{0, 1, 2, \dots\}$ if $km > \delta$. A convolutional code over a field of q elements is said to be optimal if it reaches the Griesmer bound (see [7]).

Let $G = \{g_1, \dots, g_n\}$ be a finite group of order n . We consider the group \mathbb{F} -algebra $A = \mathbb{F}[G]$ and the \mathbb{F} -isomorphism $\beta : \mathbb{F}^n \rightarrow A$ given by $\beta(v_1, \dots, v_n) := \sum_{i=1}^n v_i g_i$. On the other hand, we have the canonical isomorphism $\psi : \mathbb{F}[z]^n \rightarrow \mathbb{F}^n[z]$. Given $y \in \mathbb{F}[z]^n$, let $\psi(y) = \sum_{j \geq 0} z^j w_j \in \mathbb{F}^n[z]$. Then we define $\rho : \mathbb{F}[z]^n \rightarrow A[z]$ by $\rho(y) = \sum_{j \geq 0} z^j \beta(w_j)$. It is clear that ρ is a $\mathbb{F}[z]$ -isomorphism. We identify the $\mathbb{F}[z]$ -submodules of $\mathbb{F}[z]^n$ with the $\mathbb{F}[z]$ -submodules of $A[z]$ via ρ .

Now, let σ be an \mathbb{F} -automorphism of A and $\mathcal{R} = A[z; \sigma]$ be the skew polynomial ring. The multiplication rule in \mathcal{R} is given by $az = z\sigma(a)$ for all $a \in A$. The map $\rho_\sigma : \mathbb{F}[z]^n \rightarrow A[z; \sigma]$ defined just like ρ is the key for the next definitions (in [5] essentially appears the respective definitions in the particular case of a cyclic group). Note that ρ_σ is an isomorphism of left $\mathbb{F}[z]$ -modules.

Definition 2 Let $\mathcal{C} \subseteq \mathbb{F}[z]^n$ be a convolutional code. We say that \mathcal{C} is a (G, σ) -convolutional code if $\rho_\sigma(\mathcal{C})$ is a direct summand left ideal of \mathcal{R} .

We will see that this definition coincides with the usual one where only is required that $\rho_\sigma(\mathcal{C})$ is a direct summand as $\mathbb{F}[z]$ -module.

Proposition 1 Let $\mathcal{C} \subseteq \mathbb{F}[z]^n$ be a convolutional code. The following conditions are equivalent.

- a) \mathcal{C} is a (G, σ) -convolutional code.
- b) $\rho_\sigma(\mathcal{C})$ is a left ideal of \mathcal{R} and there is an $\mathbb{F}[z]$ -submodule K of \mathcal{R} such that $\rho_\sigma(\mathcal{C}) \oplus K = \mathcal{R}$.

Proof. a) \Rightarrow b) is obvious since any left ideal of \mathcal{R} is, in particular, a $\mathbb{F}[z]$ -submodule.

b) \Rightarrow a) Suppose $\rho_\sigma(\mathcal{C}) \oplus K = \mathcal{R}$ as $\mathbb{F}[z]$ -modules. Then there is an $\mathbb{F}[z]$ -linear map $\pi : \mathcal{R} \rightarrow \rho_\sigma(\mathcal{C})$ such that $\pi(x) = x$ for all $x \in \rho_\sigma(\mathcal{C})$. Define $\bar{\pi} : \mathcal{R} \rightarrow \rho_\sigma(\mathcal{C})$ by $\bar{\pi}(a) = \frac{1}{m}(\sum_{d \in U} d \pi(d^{-1}a))$, where $U = U(\mathbb{F}[G])$ is the group of units of $\mathbb{F}[G]$ and m is its order. It is clear that $\bar{\pi}(x) = x$ for all $x \in \rho_\sigma(\mathcal{C})$. We will show that $\bar{\pi}$ is \mathcal{R} -linear and so $\rho_\sigma(\mathcal{C})$ would be a direct summand of \mathcal{R} as left \mathcal{R} -modules. It is enough to prove that $\bar{\pi}(ha) = h\bar{\pi}(a)$ and $\bar{\pi}(za) = z\bar{\pi}(a)$ for all $h \in G, a \in \mathcal{R}$. Now, $\bar{\pi}(ha) = \frac{1}{m}(\sum_{d \in U} d \pi(d^{-1}ha)) = \frac{1}{m}(\sum_{d \in U} hh^{-1}d \pi(d^{-1}ha)) = h(\frac{1}{m}(\sum_{d \in U} h^{-1}d \pi(d^{-1}ha))) = h\bar{\pi}(a)$.

Also, $\bar{\pi}(za) = \frac{1}{m}(\sum_{d \in U} d \pi(d^{-1}za)) = z(\frac{1}{m}(\sum_{d \in U} \sigma(d) \pi(\sigma(d)^{-1}a))) = z\bar{\pi}(a)$, (the last equality holds because σ produces a permutation on the elements in U). \square

Definition 3 We say that a (G, σ) -convolutional code \mathcal{C} is minimal if $\rho_\sigma(\mathcal{C})$ is indecomposable as left $A[z; \sigma]$ -module.

Proposition 2 a) Any minimal (G, σ) -convolutional code does not contain any other proper (G, σ) -convolutional code.

b) Any (G, σ) -convolutional code is a direct sum of minimal (G, σ) -convolutional codes.

Proof. a) Let \mathcal{C} be a minimal (G, σ) -convolutional code and $\mathcal{L} \subseteq \mathcal{C}$ a (G, σ) -convolutional code different from \mathcal{C} . Then $\mathcal{R} = \rho_\sigma(\mathcal{L}) \oplus K$ as left \mathcal{R} -modules for some $K \leq \mathcal{R}$. This implies that $\rho_\sigma(\mathcal{C}) = \rho_\sigma(\mathcal{L}) \oplus (K \cap \rho_\sigma(\mathcal{C}))$ which is a contradiction with the minimality of \mathcal{C} .

b) Let \mathcal{C} be a (G, σ) -convolutional code. Then $I = \rho_\sigma(\mathcal{C})$ is a direct summand left ideal of \mathcal{R} . If I is indecomposable then it is done. In the contrary case, $I = I_1 \oplus I_2$ where I_i is a nonzero left ideal of \mathcal{R} for $i = 1, 2$. Again if both I_i are indecomposable it is done. This procedure can be repeated and must stop since the ideal I has finite rank as $\mathbb{F}[z]$ -module and the I_i 's are free $\mathbb{F}[z]$ -modules.

It is standard that any minimal (G, σ) -convolutional code is generated as left $A[z; \sigma]$ -module by a primitive idempotent element of $A[z; \sigma]$. This paper mainly deals with the problem of finding these primitive idempotents. We are interested in the matrix approach of $A[z; \sigma]$. Next, we make an account of results on the interpretation of the elements of $A[z; \sigma]$ as matrices in some matrix ring. We use Jategaonkar's results (cf. [9]) in order to give an explicit isomorphism of rings between $A[z; \sigma]$ and the rings constructed via matrix rings.

For the rest of this section, let A be a finite ring (non necessarily commutative), $\sigma : A \rightarrow A$ be an automorphism and z an indeterminate. The skew polynomial ring $\mathcal{R} = A[z; \sigma]$ admits a variable change in z such that \mathcal{R} is again a skew polynomial ring: let u be a unit in A and \bar{u} the inner automorphism of A defined by $\bar{u}(a) = u^{-1}au$, $a \in A$. It is easy to check that $A[z; \sigma] = A[zu; \bar{u}\sigma]$.

The following rings are intimately related to the skew polynomial rings. Let K be a ring and $\rho : K \rightarrow K$ an automorphism. Let $D = K[x; \rho]$, $m > 0$ and P the subring of $M_m(D)$ consisting of all the matrices (d_{ij}) satisfying the next two conditions: (1) $d_{ij} \in D \forall i, j$; (2) $d_{ij} \in xD$ if $i > j$. We denote the subring P by $\{K, m, \rho, x\}$. We also denote by I_n the set $\{1, \dots, n\}$.

We recall the concept of *set of matrix units* that appears in [8, P. 52]. Let A be a ring. A finite subset $\{e_{ij} : i, j \in I_n\}$ in A is called set of matrix units in A if verifies the following two conditions:

$$\sum_{i=1}^n e_{ii} = 1 \quad \text{and} \quad e_{ij}e_{kl} = \delta_{jk}e_{il}$$

where δ_{jk} is the Kronecker delta. In particular, $e_{ij} \neq 0$ for all $i, j \in I_n$.

A central idempotent element f in A is called *semiprimitive* if f is primitive in the center of A .

The following fact will be used frequently in the next section. Let A be a semisimple finite ring and $\{f_1, \dots, f_m\}$ a complete set of semiprimitive

idempotent elements in A . Assume that $\sigma : A \rightarrow A$ is an automorphism such that $\sigma(f_i) = f_{\pi(i)}$ where π is the cycle over I_m given by $\pi = (1\ 2\dots m)$. Let $\mathcal{R} = A[z; \sigma]$. Then, by [9, Lemma 3.1], there exists a finite field K , an automorphism $\rho : K \rightarrow K$ and a positive integer n such that $\mathcal{R} \cong M_n(\{K, m, \rho, x\})$ for some indeterminate x .

Note that the above positive integer n is the cardinality of a complete set of matrix units in Af_1 .

3 S_3 -convolutional codes

In this section we are going to determinate the minimal S_3 -convolutional codes over the field with five elements via Jategaonkar's theorems [9]. We fix the field with 5 elements \mathbb{F}_5 and let $A = \mathbb{F}_5[S_3]$. The ring A is semisimple by Maschke Theorem. First, we calculate a complete set of primitive orthogonal idempotents elements of A by means of theory of Young diagrams (see [2, pg. 190]). The list of the four idempotent is the following:

$$\varepsilon_1 = I + (1\ 2) + (1\ 3) + (2\ 3) + (1\ 2\ 3) + (1\ 3\ 2),$$

$$\varepsilon_2 = I + 4(1\ 2) + 4(1\ 3) + 4(2\ 3) + (1\ 2\ 3) + (1\ 3\ 2),$$

$$\varepsilon_3 = 2I + 3(1\ 2) + 2(2\ 3) + 3(1\ 2\ 3),$$

$$\varepsilon_4 = 2I + 2(1\ 2) + 3(2\ 3) + 3(1\ 3\ 2).$$

Then $A = \varepsilon_1 A \oplus \varepsilon_2 A \oplus \varepsilon_3 A \oplus \varepsilon_4 A$, where $\varepsilon_1 A \cong \varepsilon_2 A \cong \mathbb{F}_5$ and $\varepsilon_3 A \oplus \varepsilon_4 A \cong M_2(\mathbb{F}_5)$ as rings. The corresponding semiprimitive idempotents are $f_1 = \varepsilon_1$, $f_2 = \varepsilon_2$ and $f_3 = \varepsilon_3 + \varepsilon_4$.

We consider two classes of \mathbb{F}_5 -automorphism of A attending to the feasible permutation that produces over the set $\{f_1, f_2, f_3\}$. One class will be represented by the identity permutation and the other by the permutation $(1\ 2)$. By [9, Theorem 3.3], two automorphisms that produce the same permutation also produce isomorphic skew polynomial rings. Moreover, we will prove later that they are *isometric*, in the sense that there is ring isomorphisms between them that preserve the weight of the elements. So we only take in our study the identity automorphism (for the identity permutation) and any automorphism $\sigma \in \text{Aut}_{\mathbb{F}_5}(A)$ such that $\sigma(f_1) = f_2$, $\sigma(f_2) = f_1$ and $\sigma(f_3) = f_3$ (note that any automorphism maps f_1 to f_1 or f_2 , f_2 to f_2 or f_1 and f_3 to f_3).

3.1 The case of the permutation (1 2)

We begin with the second type of automorphism. We take the automorphism σ such that $\sigma(I) = I$, $\sigma(1\ 2) = 4(1\ 2)$, $\sigma(1\ 3) = 4(1\ 3)$, $\sigma(2\ 3) = 4(2\ 3)$, $\sigma(1\ 2\ 3) = (1\ 2\ 3)$, $\sigma(1\ 3\ 2) = (1\ 3\ 2)$. It can be checked that σ verifies the above conditions over $\{f_1, f_2, f_3\}$.

By [9, Lemma 3.2], $A[z; \sigma] = Ag_1[zg_1; \sigma_1] \oplus Ag_2[zg_2; \sigma_2]$, where $g_1 = f_1 + f_2$, $g_2 = f_3$, $\sigma_1 = \sigma|_{Ag_1}$ and $\sigma_2 = \sigma|_{Ag_2}$.

Let $b_1 = I$, $b_2 = (1\ 2)$, $b_3 = (1\ 3)$, $b_4 = (2\ 3)$, $b_5 = (1\ 2\ 3)$ and $b_6 = (1\ 3\ 2)$. Given $h \in A[z; \sigma]$, we have

$$h = \sum_{i=0}^m z^i \left(\sum_{j=1}^6 a_{ij} b_j \right) = \sum_{j=1}^6 \left(\sum_{i=0}^m z^i a_{ij} \right) b_j$$

with $a_{ij} \in \mathbb{F}_5$. Then,

$$h = hg_1 + hg_2 = \sum_{j=1}^6 \left(\sum_{i=0}^m (zg_1)^i a_{ij} \right) b_j g_1 + \sum_{j=1}^6 \left(\sum_{i=0}^m (zg_2)^i a_{ij} \right) b_j g_2.$$

We study separately hg_1 and hg_2 .

By [9, Theorem 2.1], there exists an isomorphism $\phi_1 : Ag_1[zg_1; \sigma_1] \rightarrow \mathcal{S}$, where $\mathcal{S} = \{\mathbb{F}_5, 2, \rho, x\} \subseteq M_2(\mathbb{F}_5[x; \rho])$, $\varepsilon_2 A \cong \mathbb{F}_5$, $x = (zg_1)^2$ and $\rho = \sigma_1^2 = id_{\varepsilon_2 A} : \varepsilon_2 A \rightarrow \varepsilon_2 A$. Hence, the ring \mathcal{S} is simply the subring of $M_2(\mathbb{F}_5[x])$ given by $\mathcal{S} = \left\{ \begin{pmatrix} p_{11} & p_{12} \\ xp_{21} & p_{22} \end{pmatrix} \mid p_{ij} \in \mathbb{F}_5[x] \right\}$. To understand $\phi_1(hg_1)$ is enough to calculate $\phi_1((zg_1))$ and $\phi_1(b_j g_1)$, for all $j \in I_6$. It is easy to see that $b_j g_1$ is equal to $2(I + (1\ 2\ 3) + (1\ 3\ 2))$ or $2((1\ 2) + (1\ 3) + (2\ 3))$ for all $j \in I_6$. Then, by the proof of [9, Theorem 2.1], we have $\phi_1(zg_1) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$, $\phi_1(((1\ 2) + (1\ 3) + (2\ 3))g_1) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $\phi_1((I + (1\ 2\ 3) + (1\ 3\ 2))g_1) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Note that $\phi_1(\varepsilon_1 g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\phi_1(\varepsilon_2 g_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and by applying ϕ_1 to the sum $((1\ 2) + (1\ 3) + (2\ 3))g_1 + (I + (1\ 2\ 3) + (1\ 3\ 2))g_1 = \varepsilon_1 g_1$ we get precisely $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Now we focus our attention on the direct summand $Ag_2[zg_2; \sigma_2]$. By [9, Lemma 3.1], there exists an isomorphism $\psi : Ag_2[zg_2; \sigma_2] \rightarrow M_2(\mathbb{F}_5[zg_2u; \bar{u}\sigma_2])$, where u is a unit in Ag_2 . We will make effective this isomorphism.

First we find an isomorphism $\delta : Ag_2 \rightarrow M_2(\mathbb{F}_5)$. Let $\varepsilon_{33} = \varepsilon_3$, $\varepsilon_{44} = \varepsilon_4$, $\varepsilon_{34} = (1\ 3)\varepsilon_4$ and $\varepsilon_{43} = (1\ 3)\varepsilon_3$. Then, by the theory of Young diagrams, the set $\{\varepsilon_{33}, \varepsilon_{34}, \varepsilon_{43}, \varepsilon_{44}\}$ is a set of matrix units for Ag_2 (see [8]). Hence the assignation $\varepsilon_{33} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\varepsilon_{34} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\varepsilon_{43} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\varepsilon_{44} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, will produce the isomorphism δ . Concretely, given $ag_2 \in Ag_2$, we define $a_{ij} = \sum_{k=3}^4 \varepsilon_{ki} ag_2 \varepsilon_{jk}$ with $j, i \in \{3, 4\}$. Then a_{ij} belongs to the center of Ag_2 ([8]), $Cent(Ag_2) \cong \mathbb{F}_5$, and $\delta(ag_2) = (a_{ij})$ verifies the above.

Now we need to know how $\sigma_2 : Ag_2 \rightarrow Ag_2$ is induced in $M_2(\mathbb{F}_5)$, i.e., we must find an automorphism $\hat{\sigma}_2 : M_2(\mathbb{F}_5) \rightarrow M_2(\mathbb{F}_5)$ such that the diagram

$$\begin{array}{ccc} Ag_2 & \xrightarrow{\sigma_2} & Ag_2 \\ \delta \downarrow & & \downarrow \delta \\ M_2(\mathbb{F}_5) & \xrightarrow{\hat{\sigma}_2} & M_2(\mathbb{F}_5) \end{array}$$

is commutative. It is clear that $\hat{\sigma}_2 = \delta \sigma_2 \delta^{-1}$. Then, easy calculations show that $\hat{\sigma}_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B \begin{pmatrix} a & b \\ c & d \end{pmatrix} B^{-1}$, where $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$.

Hence we have the isomorphism induced by δ in the obvious manner: $\bar{\delta} : Ag_2[zg_2; \sigma_2] \rightarrow M_2(\mathbb{F}_5)[y; \hat{\sigma}_2]$, $zg_2 \mapsto y$, $\bar{\delta}|_{Ag_2} = \delta$. On the other hand, $M_2(\mathbb{F}_5)[y; \hat{\sigma}_2] = M_2(\mathbb{F}_5)[yB; \bar{B}\hat{\sigma}_2] = M_2(\mathbb{F}_5)[yB]$. Taking $yB = x$, we finally get

$$M_2(\mathbb{F}_5)[y; \hat{\sigma}_2] = M_2(\mathbb{F}_5)[x] \cong M_2(\mathbb{F}_5[x]),$$

with the last isomorphism the canonical one. Let

$$\phi_2 : Ag_2[zg_2; \sigma_2] \rightarrow M_2(\mathbb{F}_5[x])$$

be the composition of $\bar{\delta}$ with the canonical isomorphism. Then, it is clear that $\phi_2(zg_2) = xB^{-1}$ and given $\alpha = a_{33}\varepsilon_{33} + a_{44}\varepsilon_{44} + a_{34}\varepsilon_{34} + a_{43}\varepsilon_{43} \in Ag_2$, $\phi_2(\alpha) = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$.

Once we have completely described the isomorphisms ϕ_1 and ϕ_2 , we have the ring isomorphism $\phi = \phi_1 \oplus \phi_2 : A[z; \sigma] = Ag_1[zg_1; \sigma_1] \oplus Ag_2[zg_2; \sigma_2] \longrightarrow \mathcal{S} \oplus M_2(\mathbb{F}_5[x])$. This isomorphism will allow us to make calculations in $\mathcal{S} \oplus M_2(\mathbb{F}_5[x])$ and then to reflect them in $A[z; \sigma]$. We are interested in the S_3 -convolutional codes, these are obtained by means of the direct summands left ideals of $A[z; \sigma]$. Hence, we get the primitive idempotents of \mathcal{S} and $M_2(\mathbb{F}_5[x])$, and then we apply ϕ^{-1} to them. Note that it is easy to see that any idempotent in \mathcal{S} or $M_2(\mathbb{F}_5[x])$ is primitive.

The idempotent matrices of \mathcal{S} are of the form $A = \begin{pmatrix} r & s \\ xt & 1-r \end{pmatrix}$ with $r(1-r) = xts$. First we suppose that r is different from 0 and 1. We have two possibilities: $x|r$ or $x|(1-r)$. If $x|r$, we call $r = kxd$, $s = dq$, $xt = kxp$, $1-r = pq$. Then $\begin{pmatrix} p & -d \\ xk & q \end{pmatrix} \cdot A = C$, where $C = \begin{pmatrix} 0 & 0 \\ kx & q \end{pmatrix}$, and $\begin{pmatrix} p & -d \\ xk & q \end{pmatrix}$ has $\begin{pmatrix} q & d \\ -xk & p \end{pmatrix}$ as inverse in \mathcal{S} . Hence $\bullet < A > = \bullet < C >$. If $xk = \sum_{i=0}^n \alpha_i x^{i+1}$ and $q = \sum_{i=0}^m \beta_i x^i$, then $\phi_1^{-1}(C) = \varepsilon_2(\delta + \gamma) = u$ where $\delta = \sum_{i=0}^n \alpha_i z^{2i+1}$, $\gamma = \sum_{i=0}^m \beta_i z^{2i}$. Since $b_i \varepsilon_2 = \varepsilon_2$ or $4\varepsilon_2$, we get a convolutional code of rank 1, with the $\mathbb{F}_5[z]$ -basis $\{(\delta + \gamma, \delta + 4\gamma, \delta + 4\gamma, \delta + 4\gamma, \delta + \gamma, \delta + \gamma)\}$, and complexity $\max\{2deg(k) + 1, 2deg(q)\}$.

In the second case, that is, when $x|(1-r)$, we have $\bullet < A > = \bullet < C >$, where C is now $C = \begin{pmatrix} k & q \\ 0 & 0 \end{pmatrix}$. In the same way as above, we get a convolutional code of rank 1, with basis $\{(\delta + \gamma, \delta + 4\gamma, \delta + 4\gamma, \delta + 4\gamma, \delta + \gamma, \delta + \gamma)\}$, and complexity $\max\{2deg(k), 2deg(q) + 1\}$, where $\delta = \sum_{i=0}^n \alpha_i z^{2i}$, $\gamma = \sum_{i=0}^m \beta_i z^{2i+1}$.

Finally, we compute rank, basis, and complexity of the codes that we get when $r = 0, 1$:

$\begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix}$: rank 1, with basis $\{(1, 4, 4, 4, 1, 1)\}$, and complexity zero.

$\begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix}$: rank 1, with basis $\{(1 + \gamma, 4\gamma + 1, 4\gamma + 1, 4\gamma + 1, 1 + \gamma, 1 + \gamma)\}$,

and complexity $2deg(s) + 1$. (If $s = \sum_{i=0}^n \alpha_i x^i$, then $\gamma = \sum_{i=0}^n \alpha_i z^{2i+1}$).

$\begin{pmatrix} 0 & 0 \\ xt & 1 \end{pmatrix}$: rank 1, with basis $\{(1 + \gamma, 4 + \gamma, 4 + \gamma, 4 + \gamma, 1 + \gamma, 1 + \gamma)\}$,

and complexity $2deg(t) + 1$. (If $t = \sum_{i=0}^n \alpha_i x^i$, then $\gamma = \sum_{i=0}^n \alpha_i z^{2i+1}$).

$\begin{pmatrix} 1 & 0 \\ xt & 0 \end{pmatrix}$: rank 1, with basis $\{(1, 1, 1, 1, 1, 1)\}$, and complexity zero.

We resume all the above by stating that any minimal S_3 -convolutional code corresponding to an idempotent of \mathcal{S} has the basis

$\{(f(z), f(-z), f(-z), f(-z), f(z), f(z))\}$ or $\{(f(z), -f(-z), -f(-z), -f(-z), f(z), f(z))\}$, where $f(z)$ is a polynomial in $\mathbb{F}_5[z]$, $f(z) = \sum_{i=0}^n a_i z^i$, with $\sum a_{2i} z^{2i}$ and $\sum a_{2i+1} z^{2i+1}$ coprime (or, equivalently, $f(z)$ and $f(-z)$ coprime), or $\sum a_{2i} z^{2i} = 1$ and $\sum a_{2i+1} z^{2i+1} = 0$. Hence the complexity is always $deg(f)$. In both cases, these codes can be seen as codes of length 2 by concatenation.

For several small $deg(f)$ we can compute the free distance of some of these codes. For example, if $f(z) = bz + a$ with $a, b \neq 0$ the code generated by $(f(z), f(-z), f(-z), f(-z), f(z), f(z))$ has free distance 12 and so is a MDS code. It is also easy to see that if $f(z) = a + bz + cz^2$ with $a, b, c \neq 0$, then the code generated by $(f(z), f(-z), f(-z), f(-z), f(z), f(z))$ has free distance 18 and so is a MDS code too.

Now we focus our attention into the idempotents of $M_2(\mathbb{F}_5[x])$. Set $d = \phi_2^{-1}(B) = 4(1\ 2\ 3) + (1\ 3\ 2) \in Ag_2$. Note that $d^2 = 2g_2$, hence $d^{2t} = 2^t g_2$ and $d^{2t+1} = 2^t d g_2$.

We consider an idempotent matrix in $M_2(\mathbb{F}_5[x])$: $\begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ with $r \neq 0, 1$. Since $r(1-r) = ts$, we call $r = ah$, $s = hc$, $t = ab$, $1-r = bc$. Then we have the following equalities:

$$\begin{pmatrix} a & c \\ -b & h \end{pmatrix} \cdot \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix} \cdot \begin{pmatrix} h & -c \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} a & c \\ -b & h \end{pmatrix}^{-1} = \begin{pmatrix} h & -c \\ b & a \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & c \\ -b & h \end{pmatrix} \cdot \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}.$$

Hence the left ideals generated by $\begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ and $\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$ are the same.

So we only have to transform $\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$ into an element of $A[z; \sigma]$ and then calculate the associated convolutional code.

Let $a = \sum_{i=0}^n \alpha_i x^i, c = \sum_{i=0}^m \beta_i x^i \in \mathbb{F}_5[x]$. Then,

$$\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = \\ \sum_{i=0}^n \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \alpha_i + \sum_{i=0}^m \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \beta_i.$$

Hence,

$$\phi_2^{-1} \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} = \sum_{i=0}^n (zg_2)^i d^i \varepsilon_3 \alpha_i + \sum_{i=0}^m (zg_2)^i d^i \varepsilon_{34} \beta_i$$

and so

$$\phi^{-1} \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} = \sum_{i=0}^n z^i d^i \varepsilon_3 \alpha_i + \sum_{i=0}^m z^i d^i \varepsilon_{34} \beta_i = u.$$

(Note that g_2 is the identity in Ag_2).

Set $a' = \sum_{i=0}^n z^i \alpha_i d^i, c' = \sum_{i=0}^m z^i \beta_i d^i$. Breaking a' and c' according to the parity of the z -degree of the monomials we write: $a_1 = \sum z^{2i} \alpha_{2i} 2^i, a_2 = \sum z^{2i+1} \alpha_{2i+1} 2^i, c_1 = \sum z^{2i} \beta_{2i} 2^i, c_2 = \sum z^{2i+1} \beta_{2i+1} 2^i$. Then $u = (a_1 + da_2)\varepsilon_3 + (c_1 + dc_2)\varepsilon_{34} \in A[z; \sigma]$.

In order to determinate the associated S_3 -convolutional code, we must calculate $b_i \varepsilon_3, b_i \varepsilon_{34}, b_i \varepsilon_3 d$ and $b_i \varepsilon_{34} d$, and then calculate $b_i u$. The final expression of each $b_i u$ will be of the form $b_i u = a_1 u_{i1} + a_2 u_{i2} + c_1 u_{i3} + c_2 u_{i4}$, with $u_{ij} \in A$. This happens since $\sigma_2^2 = I$. Taking this into account, with the help of GAP software [19], we get the generating matrix whose files are the following:

$$b_1 u \mapsto w_1 = (2a_1 + 3a_2 + 4c_2, 3a_1 + 3a_2 + 4c_2, 4a_2 + 2c_1 + 3c_2, 2a_1 + 3a_2 + 3c_1 + 3c_2, 3a_1 + 3a_2 + 2c_1 + 3c_2, 4a_2 + 3c_1 + 3c_2),$$

$$b_2 u \mapsto w_2 = (3a_1 + 2a_2 + c_2, 2a_1 + 2a_2 + c_2, a_2 + 3c_1 + 2c_2, 3a_1 + 2a_2 + 2c_1 + 2c_2, 2a_1 + 2a_2 + 3c_1 + 2c_2, a_2 + 2c_1 + 2c_2),$$

$$b_3u \mapsto w_3 = (a_2 + 2c_1 + 2c_2, 3a_1 + 2a_2 + 2c_1 + 2c_2, 2a_1 + 2a_2 + c_2, a_2 + 3c_1 + 2c_2, 3a_1 + 2a_2 + c_2, 2a_1 + 2a_2 + 3c_1 + 2c_2),$$

$$b_4u \mapsto w_4 = (2a_1 + 2a_2 + 3c_1 + 2c_2, a_2 + 3c_1 + 2c_2, 3a_1 + 2a_2 + 2c_1 + 2c_2, 2a_1 + 2a_2 + c_2, a_2 + 2c_1 + 2c_2, 3a_1 + 2a_2 + c_2),$$

$$b_5u \mapsto w_5 = (4a_2 + 3c_1 + 3c_2, 2a_1 + 3a_2 + 3c_1 + 3c_2, 3a_1 + 3a_2 + 4c_2, 4a_2 + 2c_1 + 3c_2, 2a_1 + 3a_2 + 4c_2, 3a_1 + 3a_2 + 2c_1 + 3c_2),$$

$$b_6u \mapsto w_6 = (3a_1 + 3a_2 + 2c_1 + 3c_2, 4a_2 + 2c_1 + 3c_2, 2a_1 + 3a_2 + 3c_1 + 3c_2, 3a_1 + 3a_2 + 4c_2, 4a_2 + 3c_1 + 3c_2, 2a_1 + 3a_2 + 4c_2).$$

It is easy to see that $w_1 = -w_2$, $w_4 = -w_2 - w_3$, $w_5 = -w_3$ and $w_6 = w_2 + w_3$. Therefore the code has rank 2, $\{w_2, w_3\}$ is a basis and the complexity is $\max\{2\deg(a), 2\deg(c)\}$.

When the idempotent matrix of $M_2(\mathbb{F}_5[x])$ has $r = 0$ or $r = 1$, we can reduce its study to the above case. Concretely, we have

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -1 & s \end{pmatrix} \cdot \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ t & 1 \end{pmatrix} &= \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where the left side matrices of the product are invertible in $M_2(\mathbb{F}_5[x])$, (the matrix $\begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix}$ is not necessary to be reduced).

Therefore, all the minimal S_3 -convolutional codes corresponding to idempotents in the component $M_2(\mathbb{F}_5[x])$ have the basis

$$\{(3a_1 + 2a_2 + c_2, 2a_1 + 2a_2 + c_2, a_2 + 3c_1 + 2c_2, 3a_1 + 2a_2 + 2c_1 + 2c_2,$$

$$\begin{aligned}
& (2a_1 + 2a_2 + 3c_1 + 2c_2, a_2 + 2c_1 + 2c_2), \\
& (a_2 + 2c_1 + 2c_2, 3a_1 + 2a_2 + 2c_1 + 2c_2, 2a_1 + 2a_2 + c_2, a_2 + 3c_1 + 2c_2, \\
& 3a_1 + 2a_2 + c_2, 2a_1 + 2a_2 + 3c_1 + 2c_2) \}
\end{aligned}$$

where $a_1 = \sum z^{2i} \alpha_{2i} 2^i$, $a_2 = \sum z^{2i+1} 2^i \alpha_{2i+1}$, $c_1 = \sum z^{2i} \beta_{2i} 2^i$, $c_2 = \sum z^{2i+1} 2^i \beta_{2i+1}$, and $a = \sum \alpha_i z^i$, $c = \sum \beta_i z^i$ are any coprime polynomials in $\mathbb{F}_5[z]$, or $a = 0$, $c = 1$, or $a = 1$, $c = 0$. The rank is always 2 and the complexity is always $\max\{2\deg(a), 2\deg(c)\}$. Note that, in the above basis, the second vector is obtained from the first one by permuting the components with $(1\ 5\ 6)(2\ 3\ 4)$.

3.2 The case of the identity permutation

Now we study the S_3 -convolutional codes that are obtained when the automorphism maps f_1 to f_1 . We can take, without loss of generality, $\sigma = id_A$. Then

$$A[z; \sigma] = A[z] = Ag_1[z] \oplus Ag_2[z] = (A\varepsilon_1[z] \oplus A\varepsilon_2[z]) \oplus Ag_2[z] \cong (\mathbb{F}_5[z] \oplus \mathbb{F}_5[z]) \oplus M_2(\mathbb{F}_5)[z] \cong (\mathbb{F}_5[z] \oplus \mathbb{F}_5[z]) \oplus M_2(\mathbb{F}_5[z]).$$

Hence $Ag_1[z]$ has only two idempotents different from 0 and 1, concretely, ε_1 and ε_2 , which generate two direct summand left ideals of $Ag_1[z]$. The S_3 -convolutional code associated to ε_1 has rank 1, a basis is $\{(1, 1, 1, 1, 1, 1)\}$, that is, it is a block code. The S_3 -convolutional code associated to ε_2 has also rank 1, a basis is $\{(1, 4, 4, 4, 1, 1)\}$, i.e., it is a block code too. These are the only minimal codes to consider in the component $Ag_1[z]$.

Next, we study the component $Ag_2[z]$. In the same way that in the case $\sigma \neq id_A$ above, we find idempotent elements in $Ag_2[z]$ corresponding to the respective idempotent matrices in $M_2(\mathbb{F}_5[z])$.

We start with the same situation that in the case $\sigma \neq id$. We consider an arbitrary idempotent matrix $\begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ with $r(1-r) = ts$ and $r \neq 0, 1$. We will reach to the same conclusion that in the case $\sigma \neq id$: it is enough to work with the matrix $\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$. Then, this matrix is performed into the element $a\varepsilon_3 + c\varepsilon_{34}$ of $Ag_2[z]$. The associated generating matrices of the minimal S_3 -convolutional codes are obtained in a similar way to the case

$\sigma \neq id_A$: we only have to put in those matrices $a_2 = c_2 = 0$ and consider $a_1 = a$, $c_1 = c$ as arbitrary coprime polynomials in $\mathbb{F}_5[z]$. The generating matrix of the code has the following rows:

$$\begin{aligned} w_1 &= (2a, 3a, 2c, 2a + 3c, 3a + 2c, 3c), \\ w_2 &= (3a, 2a, 3c, 3a + 2c, 2a + 3c, 2c), \\ w_3 &= (2c, 3a + 2c, 2a, 3c, 3a, 2a + 3c), \\ w_4 &= (2a + 3c, 3c, 3a + 2c, 2a, 2c, 3a) \\ w_5 &= (3c, 2a + 3c, 3a, 2c, 2a, 3a + 2c) \\ w_6 &= (3a + 2c, 2c, 2a + 3c, 3a, 3c, 2a). \end{aligned}$$

Then

$$w_1 = -w_2, \quad w_4 = -w_2 - w_3, \quad w_5 = -w_3, \quad w_6 = w_2 + w_3$$

Therefore the code has rank 2, $\{w_2, w_3\}$ is a basis and the complexity is $\max\{2\deg(a), 2\deg(c)\}$.

When $r = 0$ or $r = 1$, we can also reduce the matrices to reach out the above case and then we get some particular cases.

We can compute, by comparing column and row distances of the generator matrices and using GAP software [19], all the optimal minimal $(6, 2, 2)$ S_3 -convolutional codes which are obtained by means of the identity permutation. Note that the Griesmer bound for the field \mathbb{F}_5 and memory $m = 1$ is $G_5(6, 2, 2; 1) = 10$ which is less than the Singleton bound (which is 11). In the following table appears all the possible values for a and c that produce non equivalent optimal codes in this situation.

a	$z + 1$	$z + 1$	$z + 1$	$z + 1$	$z + 1$	$z + 1$	$z + 1$	$z + 2$	$z + 2$
c	$2z + 1$	$3z + 1$	$4z + 1$	$z + 2$	$z + 3$	$z + 4$	$z + 1$	$2z + 2$	
a	$z + 2$	$z + 2$	$z + 2$	$z + 2$	$z + 3$	$z + 3$	$z + 3$	$z + 3$	
c	$3z + 2$	$4z + 2$	$z + 3$	$z + 4$	$z + 1$	$z + 2$	$2z + 3$	$3z + 3$	
a	$z + 3$	$z + 3$	$z + 4$	$z + 4$	$z + 4$	$z + 4$	$z + 4$	$z + 4$	
c	$4z + 3$	$z + 4$	$z + 1$	$z + 2$	$z + 3$	$2z + 4$	$3z + 4$	$4z + 4$	

Table 1: Optimal minimal $(6, 2, 2; 1)_5$ S_3 -convolutional codes for the identity permutation

3.3 Weight-preserving ring automorphisms

We will show that two different \mathbb{F}_5 -automorphisms of A that produce the same permutation on the set $\{f_1, f_2, f_3\}$ also produce **isometric** skew polynomial rings. This is a very important issue for guaranteeing a complete classification of S_3 -convolutional codes with controlled free distances into a concrete skew polynomial ring.

Any \mathbb{F}_5 -automorphism σ of A verifies $\sigma(g) = kh$ with $k \in \mathbb{F}_5 - \{0\}$ and $g, h \in S_3$. More precisely, $\sigma((i j)) = u \cdot (l m)$ with $u \in \{1, -1\}$ and $\sigma((i j m)) = (j i m)$. Hence we get six automorphisms for the case $\sigma(f_1) = f_2$ and six for the case $\sigma(f_1) = f_1$.

Let σ, τ two \mathbb{F}_5 -automorphisms verifying $\sigma(f_1) = f_2$ $\tau(f_1) = f_2$. We will define a ring isometry $\chi : A[z; \sigma] \rightarrow A[z; \tau]$. We have

$$A[z; \sigma] = (Ag_1)[zg_1; \sigma_1] \oplus (Ag_2)[zg_2; \sigma_2]$$

and $A[z; \tau] = (Ag_1)[zg_1; \tau_1] \oplus (Ag_2)[zg_2; \tau_2]$, where σ_i, τ_i are the restriction automorphisms to $Ag_i, i = 1, 2$.

The ring Ag_1 is generated as, \mathbb{F}_5 -vector space, by $c_1 = I + (1\ 2\ 3) + (1\ 3\ 2)$ and $c_2 = (1\ 2) + (1\ 3) + (2\ 3)$. Therefore $\sigma_1(c_1) = c_1 = \tau_1(c_1)$ and $\sigma_1(c_2) = -c_2 = \tau_1(c_2)$. Let $\chi_1 : (Ag_1)[zg_1; \sigma_1] \rightarrow (Ag_1)[zg_1; \tau_1]$ be simply the identity map.

Now we will define a ring isometry $\chi_2 : (Ag_2)[zg_2; \sigma_2] \rightarrow (Ag_2)[zg_2; \tau_2]$. Since $Ag_2 \cong M_2(\mathbb{F}_5)$ there is $u, v \in U(Ag_2)$ such that $\sigma_2(ag_2) = u^{-1}ag_2u$ and $\tau_2(ag_2) = v^{-1}ag_2v$, for all $a \in A$. Let $\chi_2(zg_2) = zg_2v^{-1}u$ and $\chi_2|_{Ag_2} = id_{Ag_2}$. Since $v^{-1}u$ is a unit in Ag_2 , then $g_1 + v^{-1}u$ is a unit in A and so $g_1 + v^{-1}u = k \cdot g$ for some $k \in \mathbb{F}_5 - \{0\}$ and $g \in S_3$. Hence $zg_2v^{-1}u = zg_2(g_1 + v^{-1}u) = (zg_2)kg$, i.e., χ_2 is weight-preserving. In order to see that χ_2 is a ring isomorphism we only have to check that $\chi_2(czg_2) = \chi_2(zg_2)\chi_2(\sigma_2(c))$ for all $c \in Ag_2$. But, $\chi_2(czg_2) = czg_2v^{-1}u = zg_2\tau_2(c)v^{-1}u = zg_2(v^{-1}cv)v^{-1}u = zg_2v^{-1}cu = zg_2v^{-1}u(u^{-1}cu) = \chi_2(zg_2)\chi_2(\sigma_2(c))$.

Now it is clear that the sum $\chi = \chi_1 + \chi_2 : A[z; \sigma] \rightarrow A[z; \tau]$ is a well-defined ring isometry.

When $\sigma(f_1) = f_1, \tau(f_1) = f_1$ we have

$$A[z; \sigma] = (A\varepsilon_1)[z\varepsilon_1; \sigma_1] \oplus (A\varepsilon_2)[z\varepsilon_2; \sigma_2] \oplus (Ag_2)[zg_2; \sigma_3]$$

and

$$A[z; \tau] = (A\varepsilon_1)[z\varepsilon_1; \tau_1] \oplus (A\varepsilon_2)[z\varepsilon_2; \tau_2] \oplus (Ag_2)[zg_2; \tau_3]$$

where σ_i and τ_i are the corresponding restriction automorphisms. However it is easy to see that $\sigma_i = id_{A\varepsilon_i}$, $\tau_i = id_{A\varepsilon_i}$ for $i = 1, 2$. Therefore, we simply take $\chi_i = id_{(A\varepsilon_1)[z\varepsilon_1]}$ for $i = 1, 2$. On the other hand, $Ag_2 \cong M_2(\mathbb{F}_5)$ so we can use the above idea to build an isometry $\chi_3 : (Ag_2)[zg_2; \sigma_3] \rightarrow (Ag_2)[zg_2; \tau_3]$. Then $\chi = \chi_1 + \chi_2 + \chi_3$ is the desired isometry.

4 Conclusions

All the minimal S_3 -convolutional codes over \mathbb{F}_5 have the parameters $(6, 1, t)$ or $(6, 2, 2t)$ (t an arbitrary positive integer). If we compare this with the parameters of minimal \mathbb{Z}_6 -convolutional codes (that is, σ -cyclic convolutional codes) we get the same result (see [5, Theorem 3.8]). Hence all minimal group convolutional codes of length 6 over the field of five elements have parameters $(6, 1, t)$ or $(6, 2, 2t)$. The positive integer t corresponds with the (constant) Forney indices of the code. Also note that general group codes are significantly more complicated than σ -cyclic convolutional ones. When $\sigma = id$, cyclic convolutional codes are always block codes, however, this is not the case for S_3 -convolutional codes. Finally, some free distances have been computed for these minimal S_3 -convolutional codes. The calculations show that MDS-convolutional codes (or optimal codes) appear frequently in this setting. It would be interesting to give some information on the free distance of group convolutional codes in terms of the algebraic structure of the groups.

Acknowledgements

This work have been supported by the grant BFM2002-02717 from DGES.

References

- [1] S. D. BERMAN. On the theory of group codes. *Kibernetika*, vol. 3, 31-39, (1967) .
- [2] C. W. CURTIS, I. REINER, Representation Theory of Finite Groups and Associative Algebras. *New York: Wile*, 1962.
- [3] I.D. DAMGARD, Concatenated group codes and their exponents. *IEEE Trans. Inform. Theory*, vol. IT-33, 849–854, (1987).

- [4] G. D. FORNEY JR. Convolutional codes I: Algebraic structure. *IEEE Trans. Inform. Theory*, vol. 16, 720-738, (1970).
- [5] H. GLUESING-LUERSSEN AND B. LANGFELD On the algebraic parameters of convolutional codes with cyclic structure. *J. Algebra Appl.* vol. 5, no. 1, 53-76, (2006).
- [6] H. GLUESING-LUERSSEN AND W. SCHMALE. On cyclic convolutional codes. *Acta Appl. Math.* vol. 82, no. 2, 183-237, (2004).
- [7] H. GLUESING-LUERSSEN AND W. SCHMALE. Distance bounds for convolutional codes and some optimal codes. *Preprint*, (2003).
- [8] N. JACOBSON. Structure of rings. *Amer. Math. Soc. Colloquium Publ.*, vol. 37, 1964.
- [9] A. V. JATEGAONKAR. Skew Polynomial Rings over Semisimple Rings. *J. Algebra*, vol. 19, 315-328, (1971).
- [10] P. LANDROCK AND I. B. DAMGARD Ideals and codes in group algebras. *Math. Inst. Univ. Aarhus*, preprint, vol. 12, (1987).
- [11] R. A. LIEBER AND K.-H. ZIMMERMANN. Combinatorial S_n -modules as codes. *J. Algeb. Combin.* vol. 4, 47-68, (1995).
- [12] F. J. MACWILLIAMS. Codes and ideals in groups algebras. *R.C. Bose and T. A. Dowling Eds., Combinatorial Mathematics and its Applications. Chapel Hill, NC: Univ. of North Carolina Press*, 317-329, (1969).
- [13] J. PERALTA AND B. TORRECILLAS, Graded codes. *Appl. Algebra Engrg. Comm. Comput.* vol. 13, no. 2, 107-120 (2002).
- [14] P. PIRET Structure and constructions of cyclic convolutional codes. *IEEE Trans. Inform. Theory*, vol. 22, 147-155, (1976).
- [15] P. PIRET Convolutional Codes; An Algebraic Approach. *MIT Press, Cambridge, MA*, 1988.
- [16] C. ROOS. On the structure of convolutional and cyclic convolutional codes. *IEEE Trans. Inform. Theory*, vol. 25, 676-683, (1979).

- [17] KARL-HEINZ ZIMMERMANN. On Generalization of Repeated-Root Cyclic Codes. *IEEE Trans. Inform. Theory*, vol. 42(2), 641-649, (1996).
- [18] KARL-HEINZ ZIMMERMANN. On the majority decodable distance of codes in filtrations of characteristic $p > 0$, *Arch. Math.*, vol. 59, 513–520, (1992).
- [19] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2004, (<http://www.gap-system.org>).