

SKEW CONSTACYCLIC CODES OVER FINITE CHAIN RINGS

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ABSTRACT. Skew polynomial rings over finite fields ([7] and [10]) and over Galois rings ([8]) have been used to study codes. In this paper, we extend this concept to finite chain rings. Properties of skew constacyclic codes generated by monic right divisors of $x^n - \lambda$, where λ is a unit element, are exhibited. When $\lambda^2 = 1$, the generators of Euclidean and Hermitian dual codes of such codes are determined together with necessary and sufficient conditions for them to be Euclidean and Hermitian self-dual. Of more interest are codes over the ring $\mathbb{F}_p^m + u\mathbb{F}_p^m$. The structure of all skew constacyclic codes is completely determined. This allows us to express generators of Euclidean and Hermitian dual codes of skew cyclic and skew negacyclic codes in terms of the generators of the original codes. An illustration of all skew cyclic codes of length 2 over $\mathbb{F}_3 + u\mathbb{F}_3$ and their Euclidean and Hermitian dual codes is also provided.

1. Introduction. In the early history of the art of Error Correcting Codes, codes were usually taken over finite fields. In the last two decades, interest has been shown in linear codes over rings. In an important work [17], Calderbank et al. showed that the Kerdock codes, the Preparata codes and Delsarte-Goethals codes can be obtained through the Gray images of linear codes over \mathbb{Z}_4 . In general, due to their rich algebraic structure, constacyclic codes have been studied over various finite chain rings (see, for example, [3], [6], [13], [14], [15], [16], [19], [20], [22] and [23]). In particular, successful applications of modular lattices using codes over a

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finite chain ring $\mathbb{F}_p + u\mathbb{F}_p$ [4] and constructions of good sequences from polynomial residue class rings [24] have motivated the study of constacyclic codes over a special family of finite chain rings of the form $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ (see, for example, [3], [6], [15], [16], [19] and [23]).

Classically, polynomial rings over finite fields or over finite rings and their ideals are key to determining the algebraic structures of constacyclic codes. In [7], skew (non-commutative) polynomial rings have been used to describe the structure of linear codes closed under a skew cyclic shift, namely, skew cyclic codes. Later on, in [10], more properties and good examples of such codes have been established. Recently, in [8], that approach has been extended to codes over Galois rings. Skew constacyclic codes have been studied for a particular case when codes are generated by monic right divisors of $x^n - \lambda$, where λ is a unit in the Galois ring fixed by a given automorphism.

Motivated by these works, we generalize the concept of skew constacyclic codes to over finite chain rings. We study the algebraic structure and properties of these codes and their Euclidean and Hermitian dual codes. For arbitrary finite chain rings, skew constacyclic codes with respect to a unit λ are studied in the case where their generator polynomials are right divisors of $x^n - \lambda$. Moreover, all skew constacyclic codes over a finite chain ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ are investigated.

This paper is organized as follows: Results concerning finite chain rings and skew polynomials over these are discussed in Section 2 along with some definitions and basic properties of skew constacyclic codes. In Section 3, the algebraic structure and some properties of skew constacyclic codes whose generator polynomials are monic right divisors of $x^n - \lambda$ are established. In many cases, the structures of their Euclidean and Hermitian dual codes are given. Necessary and sufficient conditions for them to be Euclidean and Hermitian self-dual are determined as well. A complete structural classification of skew constacyclic codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ comes in Section 4. Moreover, the structures of Euclidean and Hermitian dual codes of skew cyclic and skew-negacyclic codes are determined. Some illustration examples of skew cyclic codes are also provided.

2. Preliminaries. In this section, we recall and derive some useful results concerning finite chain rings and skew polynomials over such rings. The definition of a skew constacyclic code is introduced together with some basic properties.

2.1. Finite Chain Rings. A finite commutative ring with identity $1 \neq 0$ is called a *finite chain ring* if its ideals are linearly ordered by inclusion. It is known that every ideal of a finite chain ring is principal and its maximal ideal is unique (cf. [18]). Throughout, let \mathcal{R} denote a finite chain ring, γ a generator of its maximal ideal and \mathcal{K} the residue field $\mathcal{R}/\langle\gamma\rangle$. With these notations, the ideals of \mathcal{R} form the following chain

$$\mathcal{R} = \langle 1 \rangle \supseteq \langle \gamma \rangle \supseteq \langle \gamma^2 \rangle \supseteq \cdots \supseteq \langle \gamma^{e-1} \rangle \supseteq \langle \gamma^e \rangle = \langle 0 \rangle.$$

The integer e is called the *nilpotency index* of \mathcal{R} . If \mathcal{K} has q elements, then $|\mathcal{R}| = q^e$.

Typical examples of finite chain rings which are not fields are the integer residue ring \mathbb{Z}_{p^e} , the Galois ring $\text{GR}(p^e, m)$ and the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$, where p is a prime number and m, e are positive integers such that $e \geq 2$. Note that $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m} := \{\sum_{i=0}^{e-1} a_i u^i \mid a_i \in \mathbb{F}_{p^m}\}$ is a ring under the usual addition and multiplication of polynomials in indeterminate u together with the rule $u^e = 0$. This ring is isomorphic to $\mathbb{F}_{p^m}[u]/\langle u^e \rangle$, the only finite chain ring

of characteristic p , nilpotency index e , and residue field \mathbb{F}_{p^m} (cf. [12, Lemma 1]). The reader can find further details concerning finite chain rings in [5], [11], [12], [18] and [25].

In [1] and [2], the structure of the automorphism group $\text{Aut}(\mathcal{R})$ of every finite chain ring \mathcal{R} has been characterized. Many classes of finite chain rings have non-trivial automorphism groups. For examples, $\text{Aut}(\text{GR}(p^e, m))$ is non-trivial if and only if $m \geq 2$ (cf. [8] and [25]) and $\text{Aut}(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m})$ is non-trivial if and only if $m \geq 2$ or p is odd or $e \geq 3$ (cf. [1, Proposition 1]). Here, the automorphism group of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is given as a corollary of [1, Proposition 1], the complete characterization of the automorphism group of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$.

Corollary 2.1 ([1]). *For $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ and $\beta \in \mathbb{F}_{p^m}^*$, let*

$$\Theta_{\theta, \beta} : \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$$

be defined by

$$\Theta_{\theta, \beta}(a + bu) = \theta(a) + \beta\theta(b)u.$$

Then $\text{Aut}(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}) = \{\Theta_{\theta, \beta} \mid \theta \in \text{Aut}(\mathbb{F}_{p^m}) \text{ and } \beta \in \mathbb{F}_{p^m}^\}$.*

Note that $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and its automorphisms play an important role in later examples and in Section 4.

2.2. Skew Polynomial Rings over Finite Chain Rings. In [7], [8], [10] and [18], results concerning skew polynomial rings over finite fields and over Galois rings have been studied. Applying the ideas in these references, the following results over finite chain rings are given.

For a given automorphism Θ of \mathcal{R} , the set $\mathcal{R}[x; \Theta] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathcal{R} \text{ and } n \in \mathbb{N}_0\}$ of formal polynomials forms a ring under the usual addition of polynomials and where the multiplication is defined using the rule $xa = \Theta(a)x$. The multiplication is extended to all elements in $\mathcal{R}[x; \Theta]$ by associativity and distributivity. The ring $\mathcal{R}[x; \Theta]$ is called a *skew polynomial ring* over \mathcal{R} and an element in $\mathcal{R}[x; \Theta]$ is called a *skew polynomial*. It is easily seen that the ring $\mathcal{R}[x; \Theta]$ is non-commutative unless Θ is the identity automorphism on \mathcal{R} .

Based on the canonical reduction modulo $\gamma, \bar{\cdot} : \mathcal{R} \rightarrow \mathcal{K}$, the automorphism $\bar{\Theta}$ of \mathcal{K} is induced from Θ by

$$\bar{\Theta}(\bar{r}) = \overline{\Theta(r)} \text{ for all } r \in \mathcal{R}.$$

Then there is a natural ring epimorphism extension $\bar{\cdot} : \mathcal{R}[x; \Theta] \rightarrow \mathcal{K}[x; \bar{\Theta}]$ defined by

$$r_0 + r_1x + \cdots + r_nx^n \mapsto \bar{r}_0 + \bar{r}_1x + \cdots + \bar{r}_nx^n.$$

In other words, for each $f(x) \in \mathcal{R}[x; \Theta]$, $\overline{f(x)}$ denotes the componentwise reduction modulo γ of $f(x)$.

The ring $\mathcal{R}[x; \Theta]$ does not need to be a unique factorization ring. Moreover, the degrees of the irreducible factors are not unique up to permutation.

Example 2.1. Consider the automorphism $\Theta_{\text{id}, 2}$ of $\mathbb{F}_3 + u\mathbb{F}_3$, where $\Theta_{\text{id}, 2}(a + bu) = a + 2bu$. Here are two irreducible factorizations of $x^6 - 1$ in $(\mathbb{F}_3 + u\mathbb{F}_3)[x; \Theta_{\text{id}, 2}]$

$$\begin{aligned} x^6 - 1 &= (x + 1)^3(x + 2)^3 \\ &= (x^2 + ux + 2)^3. \end{aligned}$$

The skew polynomial ring $\mathcal{R}[x; \Theta]$ is neither left nor right Euclidean. However, left and right divisions can be defined for some suitable elements. Let $f(x) = a_0 + a_1x + \cdots + a_r x^r$ and $g(x) = b_0 + b_1x + \cdots + b_s x^s$, where b_s is a unit in \mathcal{R} . The *right division* of $f(x)$ by $g(x)$ is defined as follows:

If $r < s$, then $f(x) = 0g(x) + f(x)$. Suppose that $r \geq s$. First, note that the degree of

$$f(x) - a_r \Theta^{r-s} (b_s^{-1}) x^{r-s} g(x)$$

is less than the degree of $f(x)$. Then iterating the above procedure by subtracting further left multiples of $g(x)$ from the result until the degree is less than the degree of $g(x)$, we obtain skew polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x)g(x) + r(x) \text{ with } \deg(r(x)) < \deg(g(x)) \text{ or } r(x) = 0.$$

Note that $q(x)$ and $r(x)$ are unique and called the *right quotient* and *right remainder*, respectively. This algorithm is called the *Right Division Algorithm* in $\mathcal{R}[x; \Theta]$.

If $r(x) = 0$, we say that $g(x)$ is a *right divisor* of $f(x)$. In this case, denote by $\frac{f(x)}{g(x)}$ the right quotient $q(x)$ of $f(x)$ by $g(x)$. This implies

$$f(x) = \frac{f(x)}{g(x)} g(x). \quad (2.1)$$

Similarly, the *Left Division Algorithm* in $\mathcal{R}[x; \Theta]$ can be defined using the fact that the degree of

$$f(x) - g(x) \Theta^{-s} (a_r b_s^{-1}) x^{r-s}$$

is less than the degree of $f(x)$.

For a skew polynomial $f(x)$ in $\mathcal{R}[x; \Theta]$, let $\langle f(x) \rangle$ denote the left ideal of $\mathcal{R}[x; \Theta]$ generated by $f(x)$. Note that $\langle f(x) \rangle$ does not need to be two-sided. A sufficient condition for $\langle f(x) \rangle$ to be two-sided is given as follows:

Proposition 2.1. *If $f(x) = x^t g(x)$ such that $g(x)$ is central and $t \in \mathbb{N}_0$, then $\langle f(x) \rangle$ is a principal two-sided ideal in $\mathcal{R}[x; \Theta]$.*

Proof. Since $g(x)$ is central, for each skew polynomial $\sum_{i=0}^n a_i x^i$ in $\mathcal{R}[x; \Theta]$, we have $(\sum_{i=0}^n a_i x^i) (x^t g(x)) = x^t \sum_{i=0}^n \Theta^{-t} (a_i) x^i g(x) = (x^t g(x)) \sum_{i=0}^n \Theta^{-t} (a_i) x^i$. \square

Corollary 2.2. *If $f(x)$ is a monic central skew polynomial of degree n , then the skew polynomials of degree less than n are canonical representatives of the elements in $\mathcal{R}[x, \Theta] / \langle f(x) \rangle$.*

Proof. By Proposition 2.1, $\langle f(x) \rangle$ is two-sided and hence the quotient $\mathcal{R}[x, \Theta] / \langle f(x) \rangle$ is meaningful. The desired result follows from the Right Division Algorithm. \square

Proposition 2.2. *Let n be a positive integer and λ a unit in \mathcal{R} . Then the following statements are equivalent:*

- i) $x^n - \lambda$ is central in $\mathcal{R}[x, \Theta]$.
- ii) $\langle x^n - \lambda \rangle$ is two-sided.
- iii) n is a multiple of the order of Θ and λ is fixed by Θ .

Proof. i) \Rightarrow ii) follows directly from Proposition 2.1.

Next, we prove ii) \Rightarrow iii). Assume that $\langle x^n - \lambda \rangle$ is two-sided. Let $r \in \mathcal{R}$. Then $r x^n - r \lambda = r(x^n - \lambda) = (x^n - \lambda) s = \Theta^n(s) x^n - s \lambda$ for some $s \in \mathcal{R}$. Comparing the coefficients, we have $r \lambda = s \lambda$. As λ is a unit, it follows that $r = s$, and hence $r x^n - r \lambda = \Theta^n(r) x^n - r \lambda$. Thus, n is a multiple of the order of Θ . Next, we observe

that $x^{n+1} - \Theta(\lambda)x = x(x^n - \lambda) = (x^n - \lambda)(ax + b) = \Theta^n(a)x^{n+1} + \Theta^n(b)x^n - a\lambda x - b\lambda$, for some a and b in \mathcal{R} . Then $\Theta^n(a) = 1$ and $\Theta^n(b) = 0$. As Θ is an automorphism, we have $a = 1$ and $b = 0$, and hence $x^{n+1} - \Theta(\lambda)x = x^{n+1} - \lambda x$. Therefore, λ is fixed by Θ .

Finally, we prove $iii) \Rightarrow i)$. Assume that n is a multiple of the order of Θ and λ is fixed by Θ . Then $x(x^n - \lambda) = x^{n+1} - \Theta(\lambda)x = x^{n+1} - \lambda x = (x^n - \lambda)x$ and $(x^n - \lambda)t = \Theta^n(t)x^n - t\lambda = tx^n - t\lambda = t(x^n - \lambda)$, for all $t \in \mathcal{R}$. Consequently, $x^n - \lambda$ commutes with any skew polynomial in $\mathcal{R}[x; \Theta]$. \square

Proposition 2.3. *Let $h(x), g(x) \in \mathcal{R}[x; \Theta]$. If $h(x)g(x)$ is a monic central skew polynomial, then $h(x)g(x) = g(x)h(x)$. In particular, if $g(x)$ is a right divisor of a monic central skew polynomial $f(x)$, then $g(x)$ and the right quotient $\frac{f(x)}{g(x)}$ commute, i.e.*

$$g(x)\frac{f(x)}{g(x)} = f(x) = \frac{f(x)}{g(x)}g(x). \quad (2.2)$$

Proof. Assume that $h(x)g(x)$ is monic and central. Then the leading coefficient of $g(x)$ and $h(x)$ are units. Since $h(x)g(x)$ is central, we have

$$h(x)(h(x)g(x)) = (h(x)g(x))h(x) = h(x)(g(x)h(x)).$$

Thus $h(x)(h(x)g(x) - g(x)h(x)) = 0$. As the leading coefficient of $h(x)$ is a unit, $h(x)$ is not a zero divisor. Hence $h(x)g(x) = g(x)h(x)$ as desired. \square

The following discussion guarantees the existence of the right localization of $\mathcal{R}[x; \Theta]$ which plays a vital role in the study of dualities of codes. In the light of [21, Theorem 2], necessary and sufficient conditions for $\mathcal{R}[x; \Theta]$ to have the right localization are given as follows.

Theorem 2.1 ([21]). *Let $S = \{x^i \mid i \in \mathbb{N}\}$. Then $\mathcal{R}[x; \Theta]$ has the right localization at S if and only if both the following conditions hold:*

- i) For all $x^i \in S$ and $a(x) \in \mathcal{R}[x; \Theta]$, there exist $x^j \in S$ and $b(x) \in \mathcal{R}[x; \Theta]$ such that $a(x)x^i = x^j b(x)$.*
- ii) Given $a(x) \in \mathcal{R}[x; \Theta]$ and $x^i \in S$, if $x^i a(x) = 0$, then there exists $x^j \in S$ such that $a(x)x^j = 0$.*

Condition *i)* holds because the multiplication rule allows the shifting of powers of x from left to right by changing the coefficients. Note that, for each $x^i \in S$, it is never a left zero divisor. If $a(x) \in \mathcal{R}[x; \Theta]$ such that $x^i a(x) = 0$, then $a(x)$ must be zero and hence $a(x)x^j = 0$, for all $x^j \in S$. This obviously implies *ii)*. Then, by Theorem 2.1, the right localization $\mathcal{R}[x; \Theta]S^{-1}$ of $\mathcal{R}[x; \Theta]$ exists. We have $ax^{-1} = x^{-1}\Theta(a)$ where x^{-1} is the inverse of x in this right localization.

The following map is key to determining the structure of dual codes.

Proposition 2.4. *Let $\varphi : \mathcal{R}[x; \Theta] \rightarrow \mathcal{R}[x; \Theta]S^{-1}$ be defined by*

$$\varphi\left(\sum_{i=0}^t a_i x^i\right) = \sum_{i=0}^t x^{-i} a_i.$$

Then φ is a ring anti-monomorphism.

Proof. The proof is similar to a part of the argument used in the proof of [8, Theorem 4.4]. \square

2.3. Definitions and Basic Properties of Skew Constacyclic Codes over Finite Chain Rings. A *code of length n* over \mathcal{R} is a nonempty subset of \mathcal{R}^n . A code C is said to be *linear* if it is a submodule of the \mathcal{R} -module \mathcal{R}^n . In this paper, all codes are assumed to be linear unless otherwise stated.

Given an automorphism Θ of \mathcal{R} and a unit λ in \mathcal{R} , a code C is said to be *skew constacyclic*, or specifically, Θ - λ -*constacyclic* if C is closed under the Θ - λ -constacyclic shift $\rho_{\Theta, \lambda} : \mathcal{R}^n \rightarrow \mathcal{R}^n$ defined by

$$\rho_{\Theta, \lambda}((a_0, a_1, \dots, a_{n-1})) = (\Theta(\lambda a_{n-1}), \Theta(a_0), \dots, \Theta(a_{n-2})).$$

In particular, such codes are called *skew cyclic* and *skew negacyclic codes* when λ is 1 and -1 , respectively. When Θ is the identity automorphism, they become classical constacyclic, cyclic and negacyclic codes.

Analogous to the classical constacyclic codes, we characterize Θ - λ -constacyclic codes in terms of left ideals in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$. However, due to Proposition 2.2, $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$ is meaningful if only if $\langle x^n - \lambda \rangle$ is two-sided, or equivalently, n is a multiple of the order of Θ and λ is a unit fixed by Θ .

For this purpose, throughout, we restrict our study to the case where the length n of codes is a multiple of the order of Θ and λ is a unit in \mathcal{R}^Θ , where \mathcal{R}^Θ denotes the subring of \mathcal{R} fixed by Θ .

The *skew polynomial representation* of a code C is defined to be $\{c_0 + c_1x + \dots + c_{n-1}x^{n-1} \mid (c_0, c_1, \dots, c_{n-1}) \in C\}$. For convenience, it will be regarded as C itself. The next theorem is analogous to that for classical constacyclic codes. The proof is omitted.

Theorem 2.2. *A code C of length n over \mathcal{R} is Θ - λ -constacyclic if and only if the skew polynomial representation of C is a left ideal in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$.*

There are two inner products on \mathcal{R}^n that we are interested in. One is the *Euclidean inner product* defined by $\langle u, v \rangle = \sum_{i=0}^{n-1} u_i v_i$, for $u = (u_0, u_1, \dots, u_{n-1})$ and $v = (v_0, v_1, \dots, v_{n-1})$ in \mathcal{R}^n . When the order of Θ is 2, we can also consider the *Hermitian inner product* which is defined as $\langle u, v \rangle_H = \sum_{i=0}^{n-1} u_i \Theta(v_i)$. Vectors u and v are said to be *Euclidean orthogonal* (resp., *Hermitian orthogonal*) if $\langle u, v \rangle = 0$ (resp., $\langle u, v \rangle_H = 0$).

The *Euclidean dual code* of a code C of length n over \mathcal{R} is defined to be $C^\perp = \{v \in \mathcal{R}^n \mid \langle v, c \rangle = 0 \text{ for all } c \in C\}$. Similarly, the *Hermitian dual code* of C is defined as $C^{\perp_H} = \{v \in \mathcal{R}^n \mid \langle v, c \rangle_H = 0 \text{ for all } c \in C\}$. The code C is said to be *Euclidean self-dual* (resp., *Hermitian self-dual*) if $C = C^\perp$ (resp., $C = C^{\perp_H}$).

3. Skew Constacyclic Codes Generated by Monic Right Divisors of $x^n - \lambda$.

In this section, we focus on Θ - λ -constacyclic codes which are principal left ideals in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$ generated by monic right divisors of $x^n - \lambda$. We derived some useful tools and extend results on constacyclic codes over Galois rings [8, Sections 4-5, 7] to the case over an arbitrary finite chain ring \mathcal{R} . The main assumptions that λ is a unit in \mathcal{R}^Θ and the length n of codes is a multiple of the order of Θ are assumed.

3.1. Properties of Skew Constacyclic Codes Generated by Monic Right Divisors of $x^n - \lambda$.

Given a right divisor $g(x) = \sum_{i=0}^{n-k-1} g_i x^i + x^{n-k}$ of $x^n - \lambda$,

a generator matrix of the Θ - λ -constacyclic code C generated by $g(x)$ is given by

$$G = \begin{bmatrix} g_0 & \cdots & g_{n-k-1} & 1 & 0 & \cdots & 0 \\ 0 & \Theta(g_0) & \cdots & \Theta(g_{n-k-1}) & 1 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \Theta^2(g_{n-k-1}) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Theta^{k-1}(g_0) & \cdots & \Theta^{k-1}(g_{n-k-1}) & 1 \end{bmatrix}.$$

Then the rows of G are linearly independent, and hence the next proposition follows.

Proposition 3.1. *Let $g(x)$ be a right divisor of $x^n - \lambda$. Then the Θ - λ -constacyclic code C generated by $g(x)$ is a free \mathcal{R} -module with $|C| = |\mathcal{R}|^{n-\deg(g(x))}$.*

When Θ is the identity automorphism, a Θ - λ -constacyclic code becomes λ -constacyclic. However, the converse does not need to be true. Here, necessary and sufficient conditions for a Θ - λ -constacyclic code generated by a right divisor of $x^n - \lambda$ to be λ -constacyclic are given.

Proposition 3.2. *Let $g(x)$ be a monic right divisor of $x^n - \lambda$ in $\mathcal{R}[x; \Theta]$. The Θ - λ -constacyclic code generated by $g(x)$ is λ -constacyclic if and only if $g(x) \in \mathcal{R}^\Theta[x; \Theta]$.*

Proof. Suppose $g(x) = \sum_{i=0}^{n-k-1} g_i x^i + x^{n-k}$ and C is the Θ - λ -constacyclic code generated by $g(x)$.

Assume that C is λ -constacyclic. Then $xg(x), g(x)x \in C$. By the linearity of C , $xg(x) - g(x)x \in C$ and hence

$$(\Theta(g_0) - g_0)x + (\Theta(g_1) - g_1)x^2 + \cdots + (\Theta(g_{n-k-1}) - g_{n-k-1})x^{n-k} = p(x)g(x),$$

for some $p(x) \in \mathcal{R}[x; \Theta]$ such that $\deg(p(x)) < k$. Thus $\deg(p(x)g(x)) < n$ which implies that $p(x)$ is constant such that $p(x)g_0 = 0$. Since $g(x)$ is a right divisor of $x^n - \lambda$ and λ is a unit, g_0 is not a zero divisor. Thus $p(x)$ is zero and hence g_i is fixed by Θ for all i .

Conversely, assume that $g(x) \in \mathcal{R}^\Theta[x; \Theta]$. Then $g_i x = x g_i$ for all $i = 0, 1, \dots, n-k$. Thus $g(x)x = xg(x) \in C$, therefore, the result follows. \square

A parity-check matrix for C is determined in the next proposition.

Proposition 3.3. *Let C be the Θ - λ -constacyclic code generated by a monic right divisor $g(x)$ of $x^n - \lambda$ and $h(x) := \frac{x^n - \lambda}{g(x)}$. Then the following statements hold:*

- i) *For $c(x) \in \mathcal{R}[x; \Theta]$, $c(x) \in C$ if and only if $c(x)h(x) = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$.*
- ii) *If $h(x) = \sum_{i=0}^{k-1} h_i x^i + x^k$, then the following matrix*

$$H = \begin{bmatrix} 1 & \Theta(h_{k-1}) & \cdots & \Theta^k(h_0) & 0 & \cdots & 0 \\ 0 & 1 & \Theta^2(h_{k-1}) & \cdots & \Theta^{k+1}(h_0) & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \Theta^{n-k}(h_{k-1}) & \cdots & \Theta^{n-1}(h_0) \end{bmatrix}$$

is a parity-check matrix for C .

Proof. Since n is a multiple of the order of Θ and $\lambda \in \mathcal{R}^\Theta$, $x^n - \lambda$ is central and it follows from Proposition 2.3 that $x^n - \lambda = h(x)g(x) = g(x)h(x)$.

First, we prove *i*). Assume that $c(x) = p(x)g(x)$ for some $p(x)$ in $\mathcal{R}[x; \Theta]$. Then $c(x)h(x) = (p(x)g(x))h(x) = p(x)(x^n - \lambda) = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$.

Conversely, assume that $c(x)h(x) = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$. Then there exists $p(x) \in \mathcal{R}[x; \Theta]$ such that $c(x)h(x) = p(x)(x^n - \lambda) = p(x)g(x)h(x)$. As the leading coefficient of $h(x)$ is a unit, we then have $c(x) = p(x)g(x) \in C$.

To prove *ii*), let $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in C$ and let $[s_k \ s_{k+1} \ \cdots \ s_{n-1}] = [c_0 \ c_1 \ \cdots \ c_{n-1}]H^T$. Then, for $l \in \{k, k+1, \dots, n-1\}$,

$$s_l = c_{l-k} + \sum_{j=0}^{k-1} c_{l-j} \Theta^{l-j}(h_j)$$

which equals the coefficient of x^l in $c(x)h(x)$.

Since $c(x) \in C$, it follows from *i*) that $c(x)h(x) = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$. Then there exists $q(x) \in \mathcal{R}[x; \Theta]$ such that $q(x)(x^n - \lambda) = c(x)h(x)$ having degree less than $n+k$. Therefore, the coefficients of the monomials $x^k, x^{k+1}, \dots, x^{n-1}$ in this product must be zero, i.e., $[s_k \ s_{k+1} \ \cdots \ s_{n-1}]$ is the zero matrix.

Since the rank of H is $n-k$, the result follows. \square

3.2. Euclidean Dual Codes. We study Euclidean dual codes of Θ - λ -constacyclic codes over \mathcal{R} . Their characterization is given. When $\lambda^2 = 1$, a generator of the Euclidean dual code of a Θ - λ -constacyclic code is determined. Necessary and sufficient conditions for such a code to be Euclidean self-dual are given as well.

Lemma 3.1. *Let C be a code of length n over \mathcal{R} . Then C is Θ - λ -constacyclic if and only if C^\perp is Θ - λ^{-1} -constacyclic. In particular, if $\lambda^2 = 1$, then C is Θ - λ -constacyclic if and only if C^\perp is Θ - λ -constacyclic.*

Proof. Note that, for each unit α in \mathcal{R} , $\alpha \in \mathcal{R}^\Theta$ if and only if $\alpha^{-1} \in \mathcal{R}^\Theta$. Since $\lambda \in \mathcal{R}^\Theta$, so is λ^{-1} . Let $u = (u_0, u_1, \dots, u_{n-1}) \in C$ and $v = (v_0, v_1, \dots, v_{n-1}) \in C^\perp$. Since $(\Theta^{n-1}(\lambda u_1), \Theta^{n-1}(\lambda u_2), \dots, \Theta^{n-1}(\lambda u_{n-1}), \Theta^{n-1}(u_0)) = \rho_{\Theta, \lambda}^{n-1}(u) \in C$, we have

$$\begin{aligned} 0 &= \langle \rho_{\Theta, \lambda}^{n-1}(u), v \rangle \\ &= \langle (\Theta^{n-1}(\lambda u_1), \Theta^{n-1}(\lambda u_2), \dots, \Theta^{n-1}(\lambda u_{n-1}), \Theta^{n-1}(u_0)), (v_0, v_1, \dots, v_{n-1}) \rangle \\ &= \lambda \langle (\Theta^{n-1}(u_1), \Theta^{n-1}(u_2), \dots, \Theta^{n-1}(u_{n-1}), \Theta^{n-1}(\lambda^{-1}u_0)), (v_0, v_1, \dots, v_{n-1}) \rangle \\ &= \lambda (\Theta^{n-1}(\lambda^{-1}u_0)v_{n-1} + \sum_{i=1}^{n-1} \Theta^{n-1}(u_i)v_{i-1}). \end{aligned}$$

As n is a multiple of the order of Θ and λ^{-1} is fixed by Θ , it follows that

$$\begin{aligned} 0 &= \Theta(0) = \Theta(\lambda(\Theta^{n-1}(\lambda^{-1}u_0)v_{n-1} + \sum_{i=1}^{n-1} \Theta^{n-1}(u_i)v_{i-1})) \\ &= \lambda(u_0\Theta(\lambda^{-1}v_{n-1}) + \sum_{i=1}^{n-1} u_i\Theta(v_{i-1})) \\ &= \lambda \langle \rho_{\Theta, \lambda^{-1}}(v), u \rangle. \end{aligned}$$

Therefore, $\rho_{\Theta, \lambda^{-1}}(v) \in C^\perp$.

The converse follows from the fact that $(C^\perp)^\perp = C$.

In addition, assume that $\lambda^2 = 1$. Then $\lambda = \lambda^{-1}$ and hence the last statement follows immediately from the main result. \square

If $\lambda^2 = 1$, it follows from the previous lemma that the Euclidean dual C^\perp of a Θ - λ -constacyclic code C is again Θ - λ -constacyclic. In this case, a generator of C^\perp is given through the ring anti-monomorphism φ defined in Proposition 2.4, where $\varphi(\sum_{i=0}^t a_i x^i) = \sum_{i=0}^t x^{-i} a_i$. The next lemma is key to obtaining this result.

Lemma 3.2. *Assume that $\lambda^2 = 1$. Let $a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ and $b(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$ be in $\mathcal{R}[x; \Theta]$. Then the following statements are equivalent:*

- i) *The coefficient vector of $a(x)$ is Euclidean orthogonal to the coefficient vector of $x^i(x^{n-1}\varphi(b(x)))$ for all $i \in \{0, 1, \dots, n-1\}$.*
- ii) *$(a_0, a_1, \dots, a_{n-1})$ is Euclidean orthogonal to $(b_{n-1}, \Theta(b_{n-2}), \dots, \Theta^{n-1}(b_0))$ and all its Θ - λ -constacyclic shifts.*
- iii) *$a(x)b(x) = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$.*

Proof. i) if and only if ii) follows directly from the definition of φ . We prove ii) if and only if iii). Let $a(x)b(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in \mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$. Since $\lambda \in \mathcal{R}^\Theta$ such that $\lambda^2 = 1$ and n is a multiple of the order of Θ , it follows that, for each $k \in \{0, 1, \dots, n-1\}$,

$$\begin{aligned} c_k &= \sum_{\substack{i+j=k \\ 0 \leq i \leq n-1 \\ 0 \leq j \leq n-1}} a_i \Theta^i(b_j) + \sum_{\substack{i+j=k+n \\ 0 \leq i \leq n-1 \\ 0 \leq j \leq n-1}} \lambda a_i \Theta^i(b_j) \\ &= \lambda \left(\sum_{\substack{i+j=k \\ 0 \leq i \leq n-1 \\ 0 \leq j \leq n-1}} a_i \Theta^{k-j}(\lambda b_j) + \sum_{\substack{i+j=k+n \\ 0 \leq i \leq n-1 \\ 0 \leq j \leq n-1}} a_i \Theta^{n+k-j}(b_j) \right) \\ &= \lambda \langle (a_0, a_1, \dots, a_{n-1}), \\ &\quad (\lambda b_k, \Theta(\lambda b_{k-1}), \dots, \Theta^k(\lambda b_0), \Theta^{k+1}(b_{n-1}), \dots, \Theta^{n-1}(b_{k+1})) \rangle \\ &= \lambda \langle (a_0, a_1, \dots, a_{n-1}), (\Theta^{(n-k)+k}(\lambda b_k), \Theta^{(n-k+1)+k}(\lambda b_{k-1}), \dots, \\ &\quad \Theta^k(\lambda b_0), \Theta^{1+k}(b_{n-1}), \dots, \Theta^{(n-k-1)+k}(b_{k+1})) \rangle. \end{aligned}$$

Hence, $a(x)b(x) = 0$ if and only if $c_k = 0$ for all $k \in \{0, 1, \dots, n-1\}$, which is true if and only if $(a_0, a_1, \dots, a_{n-1})$ is Euclidean orthogonal to $(b_{n-1}, \Theta(b_{n-2}), \dots, \Theta^{n-1}(b_0))$ and all its Θ - λ -constacyclic shifts. \square

Theorem 3.3. *Assume that $\lambda^2 = 1$. Let $g(x)$ be a right divisor of $x^n - \lambda$ and $h(x) := \frac{x^n - \lambda}{g(x)}$. Let C be the Θ - λ -constacyclic code generated by $g(x)$. Then the following statements hold:*

- i) *The skew polynomial $x^{\deg(h(x))} \varphi(h(x))$ is a right divisor of $x^n - \lambda$.*
- ii) *The Euclidean dual C^\perp is a Θ - λ -constacyclic code generated by $x^{\deg(h(x))} \varphi(h(x))$.*

Proof. First, we prove *i*). Using the assumptions that n is a multiple of the order of Θ and $\lambda \in \mathcal{R}^\Theta$, we observe that

$$\begin{aligned}
\left(\varphi(g(x))(-\lambda)x^{n-\deg(h(x))}\right) \left(x^{\deg(h(x))}\varphi(h(x))\right) &= \varphi(g(x))(-\lambda)x^n\varphi(h(x)) \\
&= -\lambda x^n\varphi(g(x))\varphi(h(x)) \\
&= -\lambda x^n\varphi(h(x)g(x)), \\
&\text{(since } \varphi \text{ is a ring anti-monomorphism)} \\
&= -\lambda x^n\varphi(x^n - \lambda) \\
&= -\lambda x^n(x^{-n} - \lambda) \\
&= x^n - \lambda.
\end{aligned}$$

As $\varphi(g(x))(-\lambda)x^{n-\deg(h(x))}$ and $x^{\deg(h(x))}\varphi(h(x))$ belong to $\mathcal{R}[x; \Theta]$, $x^{\deg(h(x))}\varphi(h(x))$ is a right divisor of $x^n - \lambda$ in $\mathcal{R}[x; \Theta]$.

Next, we prove *ii*). Since $g(x)h(x) = x^n - \lambda = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$, by Lemma 3.2, $\langle x^{\deg(h(x))}\varphi(h(x)) \rangle \subseteq C^\perp$. As $x^{\deg(h(x))}\varphi(h(x))$ is a right divisor of $x^n - \lambda$, by Proposition 3.1, $|\langle x^{\deg(h(x))}\varphi(h(x)) \rangle| = |\mathcal{R}|^{n-\deg(h(x))} = |C^\perp|$. Therefore, $\langle x^{\deg(h(x))}\varphi(h(x)) \rangle = C^\perp$. \square

Necessary and sufficient conditions for a Θ - λ -constacyclic code to be Euclidean self-dual are given in the next theorem.

Theorem 3.4. *Assume that $\lambda^2 = 1$ and n is even, denoted by $n = 2k$. Let $g(x) = \sum_{i=0}^{k-1} g_i x^i + x^k$ be a right divisor of $x^n - \lambda$. Then the Θ - λ -constacyclic code generated by $g(x)$ is Euclidean self-dual if and only if*

$$\left(\sum_{i=0}^{k-1} g_i x^i + x^k\right)(\Theta^{-k}(g_0^{-1}) + \sum_{i=1}^{k-1} \Theta^{i-k}(g_0^{-1}g_{k-i})x^i + x^k) = x^n - \lambda. \quad (3.1)$$

Proof. Let C be the Θ - λ -constacyclic code generated by $g(x)$ and let $g^\perp(x)$ be the generator polynomial of the Euclidean dual code C^\perp . Denote by $h(x) := \sum_{i=0}^{k-1} h_i x^i + x^k$ the right quotient $\frac{x^n - \lambda}{g(x)}$. It follows from Theorem 3.3 that

$$g^\perp(x) = x^k\varphi(h(x)) = \Theta^k(h_0)x^k + \cdots + \Theta(h_{k-1})x + 1. \quad (3.2)$$

First, assume that C is Euclidean self-dual. It is easily seen that $g(x)$ is the unique monic generator of minimal degree in C . Then $g^\perp(x)$ is a scalar multiple of $g(x)$ of the form

$$g^\perp(x) = \Theta^k(h_0)g(x) = \Theta^k(h_0)\left(\sum_{i=0}^{k-1} g_i x^i + x^k\right). \quad (3.3)$$

Comparing the coefficients in (3.2) and (3.3), we obtain $\Theta^k(h_0)g_0 = 1$ and $\Theta^k(h_0)g_i = \Theta^i(h_{k-i})$, for all $i = 1, 2, \dots, k-1$. Consequently, $h_0 = \Theta^{-k}(g_0^{-1})$ and $h_i = \Theta^i(h_0)\Theta^{i-k}(g_{k-i}) = \Theta^{i-k}(g_0^{-1})\Theta^{i-k}(g_{k-i}) = \Theta^{i-k}(g_0^{-1}g_{k-i})$, for all $i = 1, 2, \dots, k-1$, and $h(x) = \Theta^{-k}(g_0^{-1}) + \sum_{i=1}^{k-1} \Theta^{i-k}(g_0^{-1}g_{k-i})x^i + x^k$. Therefore, (3.1) holds.

Conversely, assume that (3.1) holds. Then

$$h(x) = \Theta^{-k}(g_0^{-1}) + \sum_{i=1}^{k-1} \Theta^{i-k}(g_0^{-1}g_{k-i})x^i + x^k.$$

Hence, by Theorem 3.3,

$$g^\perp(x) = x^k \varphi(h(x)) = \sum_{i=1}^k (g_0^{-1} g_i) x^i + 1 = g_0^{-1} g(x).$$

This completes the proof. \square

Remark 3.1. From Theorem 3.4, we observe that if there is a Euclidean self-dual Θ - λ -constacyclic code, then $-\lambda = g_0 \Theta^{-k}(g_0^{-1}) = \Theta^k(g_0)g_0^{-1}$. Thus, if the order of Θ divides k and $\lambda \neq -1$, then there are no Euclidean self-dual Θ - λ -constacyclic codes of length $2k$. In particular, if Θ is the identity automorphism and $\lambda \neq -1$, then there are no Euclidean self-dual Θ - λ -constacyclic codes of any length.

3.3. Hermitian Dual Codes. Due to the constraint in the definition of the Hermitian inner product, the Hermitian dual codes of skew constacyclic codes are studied only when the order of Θ is 2. Using arguments similar to those in the previous proofs, the following results concerning the Hermitian duality are obtained.

Lemma 3.5. *Let C be a code of even length n over \mathcal{R} . Assume that the order of Θ is 2. Then C is Θ - λ -constacyclic if and only if C^{\perp_H} is Θ - λ^{-1} -constacyclic. In particular, if $\lambda^2 = 1$, then C is Θ - λ -constacyclic if and only if C^{\perp_H} is Θ - λ -constacyclic.*

When $\lambda^2 = 1$, a generator of the Hermitian dual code of a Θ - λ -constacyclic code is determined through the ring anti-monomorphism φ defined in Proposition 2.4 and a ring automorphism ϕ on $\mathcal{R}[x; \Theta]$ defined by

$$\phi\left(\sum_{i=0}^t a_i x^i\right) = \sum_{i=0}^t \Theta(a_i) x^i. \quad (3.4)$$

Lemma 3.6. *Assume that the order of Θ is 2 and $\lambda^2 = 1$. Let $a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ and $b(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$ be in $\mathcal{R}[x; \Theta]$. Then the following statements are equivalent:*

- i) *The coefficient vector of $a(x)$ is Hermitian orthogonal to the coefficient vector of $x^i \phi(x^{n-1} \varphi(b(x)))$ for all $i \in \{0, 1, \dots, n-1\}$.*
- ii) *$(a_0, a_1, \dots, a_{n-1})$ is Hermitian orthogonal to $(\Theta^{-1}(b_{n-1}), b_{n-2}, \dots, \Theta^{n-2}(b_0))$ and all its Θ - λ -constacyclic shifts.*
- iii) *$a(x)b(x) = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$.*

Theorem 3.7. *Assume that the order of Θ is 2 and $\lambda^2 = 1$. Let $g(x)$ be a right divisor of $x^n - \lambda$ and $h(x) := \frac{x^n - \lambda}{g(x)}$. Let C be the Θ - λ -constacyclic code generated by $g(x)$. Then the following statements hold:*

- i) *The skew polynomial $\phi(x^{\deg(h(x))} \varphi(h(x)))$ is a right divisor of $x^n - \lambda$.*
- ii) *The Hermitian dual C^{\perp_H} is a Θ - λ -constacyclic code generated by*

$$\phi(x^{\deg(h(x))} \varphi(h(x))).$$

Proof. From the proof of Theorem 3.3, we have

$$\varphi(g(x))(-\lambda x^{n-\deg(h)}) x^{\deg(h)} \varphi(h(x)) = x^n - \lambda.$$

Then $\phi(\varphi(g(x))(-\lambda x^{n-\deg(h)})) \phi(x^{\deg(h(x))} \varphi(h(x))) = \phi(x^n - \lambda) = x^n - \lambda$. Therefore, $\phi(x^{\deg(h(x))} \varphi(h(x)))$ is a right divisor of $x^n - \lambda$, which yields i).

Since $g(x)h(x) = x^n - \lambda = 0$ in $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$, by Lemma 3.6,

$$\langle \phi(x^{\deg(h(x))})\varphi(h(x)) \rangle \subseteq C^{\perp_H}.$$

Since $\phi(x^{\deg(h(x))})\varphi(h(x))$ is a right divisor of $x^n - \lambda$, by Proposition 3.1,

$$|\langle \phi(x^{\deg(h(x))})\varphi(h(x)) \rangle| = |\mathcal{R}|^{n-\deg(h(x))} = |C^{\perp_H}|.$$

Therefore, $\langle \phi(x^{\deg(h(x))})\varphi(h(x)) \rangle = C^{\perp_H}$. This proves *ii*. \square

Necessary and sufficient conditions for a Θ - λ -constacyclic code to be Hermitian self-dual are given. The proof follows as an application of the proof of Theorem 3.4.

Theorem 3.8. *Assume that the order of Θ is 2, $\lambda^2 = 1$ and n is even, denoted by $n = 2k$. Let $g(x) = \sum_{i=0}^{k-1} g_i x^i + x^k$ be a right divisor of $x^n - \lambda$. Then the Θ - λ -constacyclic code generated by $g(x)$ is Hermitian self-dual if and only if*

$$\left(\sum_{i=0}^{k-1} g_i x^i + x^k \right) (\Theta^{-k-1} g_0^{-1}) + \sum_{i=1}^{k-1} \Theta^{i-k-1} (g_0^{-1} g_{k-i}) x^i + x^k = x^n - \lambda.$$

Remark 3.2. Suppose there is a Hermitian self-dual Θ - λ -constacyclic code. Then, by Theorem 3.8, we have $-\lambda = g_0 \Theta^{-k-1} (g_0^{-1})$. Since λ is fixed by Θ , it follows that $\lambda = -\Theta^{k+1} (g_0) g_0^{-1}$. As the order of Θ is 2,

$$\lambda = \begin{cases} -1 & \text{if } k \text{ is odd,} \\ -\Theta(g_0)g_0^{-1} & \text{if } k \text{ is even.} \end{cases}$$

Therefore, if k is odd and $\lambda \neq -1$, then there are no Hermitian self-dual Θ - λ -constacyclic codes of length $2k$.

4. Skew Constacyclic Codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. The class of finite chain rings of the form $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ has widely been used as alphabet in certain constacyclic codes (see, for example, [3], [6], [15], [16], [19] and [23]). In this section, we characterize the structure of all Θ - λ -constacyclic codes over this ring under the conditions where λ is a unit in $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ fixed by a given automorphism Θ and the length n of codes is a multiple of the order of Θ . Moreover, the structures of Euclidean and Hermitian dual codes of skew cyclic and skew negacyclic codes over this ring are determined as well.

Recall that $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is a finite chain ring of nilpotency index 2 and characteristic p . Its only maximal ideal is $u\mathbb{F}_{p^m}$. The residue field \mathcal{K} of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ will be viewed as the subfield \mathbb{F}_{p^m} of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. Every automorphism of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is of the form $\Theta_{\theta, \beta}(a + bu) = \theta(a) + \beta\theta(b)u$, where $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ and $\beta \in \mathbb{F}_{p^m}^*$ (cf. Corollary 2.1 or [1, Proposition 1]). For simplicity, where no confusion arises, the subscripts θ and β will be dropped.

As the residue field \mathcal{K} of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is viewed as the subfield \mathbb{F}_{p^m} , the ring epimorphism $\overline{\cdot} : \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m}$ can be viewed as the reduction modulo u . For $f(x) \in (\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$, $\overline{f(x)}$ denotes the isomorphic image in $\mathbb{F}_{p^m}[x; \theta] \subsetneq (\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$ of the componentwise reduction modulo u of $f(x)$. Since every skew polynomial in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$ is viewed as $f_0(x) + uf_1(x)$, where $f_0(x), f_1(x) \in \mathbb{F}_{p^m}[x; \theta]$, we have $\overline{f_0(x) + uf_1(x)} = f_0(x) \in \mathbb{F}_{p^m}[x; \theta]$.

For $f(x)$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$, the multiplication rule allows the shifting of u and powers of x from the left to the right of $f(x)$ (and vice versa) by changing the coefficients of $f(x)$. Then, for $\Omega \in \{u, x^i \mid i \in \mathbb{N}\}$, it is meaningful to give the following notations:

- i) $\overleftarrow{f(x)}^\Omega$ denotes the skew polynomial such that $f(x)\Omega = \overleftarrow{\Omega f(x)}^\Omega$,
- ii) $\overrightarrow{f(x)}^\Omega$ denotes the skew polynomial such that $\Omega f(x) = \overrightarrow{f(x)}^\Omega \Omega$.

4.1. Classification of Skew Constacyclic Codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. In this subsection, the classification of Θ - λ -constacyclic codes is given in terms of generators of left ideals in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$. These generators are uniquely determined under some conditions. Their properties are also given.

Let C be a non-zero left ideal in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ and let A denote the set of all non-zero skew polynomials of minimal degree in C . Clearly, A is non-empty. We consider three cases: when there is a monic skew polynomial in A , when there are no monic skew polynomials in C , and when there are no monic skew polynomials in A but there is a monic skew polynomial in C .

Theorem 4.1. *Let C and A be as above. Then:*

- i) *If there exists a monic skew polynomial in A , then it is unique in A . In this case, $C = \langle g(x) \rangle$, where $g(x)$ is the unique such skew polynomial.*
- ii) *If there are no monic skew polynomials in C , then there exists a unique skew polynomial $g(x) = ug_1(x)$ in A with leading coefficient u . In this case, $C = \langle g(x) \rangle$.*
- iii) *If there are no monic skew polynomials in A but there exists a monic skew polynomial in C , then there exist a unique skew polynomial $g(x) = ug_1(x)$ in A with leading coefficient u and a unique monic skew polynomial $f(x) = f_0(x) + uf_1(x)$ of minimal degree in C such that $\deg(f_1(x)) < \deg(g_1(x))$. In this case, $C = \langle g(x), f(x) \rangle$.*

Proof. To prove i), assume that $g(x)$ and $g'(x)$ are monic skew polynomials in A . Then the degree of $g(x) - g'(x)$ is less than the degree of $g(x)$. By the minimality of $\deg(g(x))$, $g(x) - g'(x) = 0$. Hence, $g(x)$ is the unique monic skew polynomial in A .

Let $c(x) \in C$. Then by the Right Division Algorithm, there exist unique skew polynomials $q(x)$ and $r(x)$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$ such that

$$c(x) = q(x)g(x) + r(x),$$

and $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$. Then

$$r(x) = c(x) - q(x)g(x) \in C.$$

By the minimality of $\deg(g(x))$, $r(x) = 0$. Hence $c(x) = q(x)g(x)$, i.e., $C = \langle g(x) \rangle$.

To prove ii), assume there are no monic skew polynomials in C . Without loss of generality, let $g(x)$ be a skew polynomial in A with leading coefficient u . First, we show that $g(x)$ is a right multiple of u . Suppose that $g(x)$ has a unit coefficient a_i for some $i < \deg(g(x))$. Then $ug(x) \in C$ is a non-zero skew polynomial having degree less than $\deg(g(x))$, which contradicts the minimality of $\deg(g(x))$. Hence $g(x)$ is a right multiple of u , and we write $g(x) = ug_1(x)$, where $g_1(x)$ is a monic skew polynomial in $\mathbb{F}_{p^m}[x; \theta]$.

For the uniqueness, suppose that $g'(x)$ is a skew polynomial in A with leading coefficient u . Then the degree of $g(x) - g'(x)$ is less than the degree of $g(x)$. By the minimality of $\deg(g(x))$, $g(x) - g'(x) = 0$. Hence, $g(x) = ug_1(x)$ is the unique skew polynomial in A with leading coefficient u .

Now, we show that C is generated by $g(x) = ug_1(x)$. Suppose that there exists $h(x)$ in C of minimal degree ℓ which is not a left multiple of $g(x) = ug_1(x)$.

Moreover, $h(x)$ can be chosen to have leading coefficient u . Then

$$\begin{aligned} k(x) &:= h(x) - ux^{\ell - \deg(g(x))}g_1(x) \\ &= h(x) - \overrightarrow{x^{\ell - \deg(g(x))}}^u ug_1(x) \\ &= h(x) - \overrightarrow{x^{\ell - \deg(g(x))}}^u g(x) \in C. \end{aligned}$$

If $k(x) = 0$, then $h(x) = \overrightarrow{x^{\ell - \deg(g(x))}}^u g(x)$ which contradicts the assumption. Suppose $k(x) \neq 0$. Then the degree of $k(x)$ is less than ℓ and $k(x)$ is not a left multiple of $g(x)$, contradicting the choice of $h(x)$.

Finally, we prove *iii*). Assume there are no monic skew polynomials in A but there exists a monic skew polynomial in C . It can be shown as in *ii*) that there is a unique skew polynomial $g(x) = ug_1(x)$ in A with leading coefficient u .

Let $F(x)$ be a monic skew polynomial of minimal degree in C . We view $F(x) = F_0(x) + uF_1(x)$, where $F_0(x), F_1(x) \in \mathbb{F}_{p^m}[x; \theta]$. By the Right Division Algorithm, there exist unique skew polynomials $q(x)$ and $r(x)$ in $\mathbb{F}_{p^m}[x; \theta]$ such that

$$F_1(x) = q(x)g_1(x) + r(x),$$

and $r(x) = 0$ or $\deg(r(x)) < \deg(g_1(x))$. Thus

$$F(x) = F_0(x) + uF_1(x) = F_0(x) + uq(x)g_1(x) + ur(x).$$

We choose $f(x) = F(x) - uq(x)g_1(x)$, $f_0(x) = F_0(x)$ and $f_1(x) = r(x)$. Then $f(x) = f_0(x) + uf_1(x)$ is a monic skew polynomial of minimal degree in C such that $\deg(f_1(x)) < \deg(g_1(x))$.

The uniqueness of $ug_1(x)$ can be shown as in the proof of *ii*). Suppose $t_0(x) + ut_1(x)$ is a monic skew polynomial of minimal degree in C such that $\deg(t_1(x)) < \deg(g_1(x))$. Then $\langle uf_0(x) \rangle = uC = \langle ut_0(x) \rangle$. Hence, by the proof of *ii*), $f_0(x) = t_0(x)$. Note that $u(f_1(x) - t_1(x)) = (f_0(x) + uf_1(x)) - (t_0(x) + ut_1(x)) \in C$. Then $u(f_1(x) - t_1(x))$ is the zero or $\deg(f_1(x) - t_1(x)) \leq \max\{\deg(f_1(x)), \deg(t_1(x))\}$. If the later case occurs, then $\deg(f_1(x) - t_1(x)) < \deg(g_1(x))$, which contradicts the minimality of $\deg(g_1(x))$. Hence $f_1(x) - t_1(x) = 0$.

Let B be the set of all non-zero skew polynomials in C with degree less than $\deg(f(x))$. Then the leading coefficients of all skew polynomials in B are multiple of u . Since $ug_1 \in A$, we have $\deg(ug_1(x)) < \deg(f(x))$, and hence $ug_1(x) \in B$. Using arguments similar to the third paragraph in the proof of *ii*), B is contained in the left ideal generated by $ug_1(x)$.

To show that C is generated by $\{g(x) = ug_1(x), f(x) = g_0(x) + ug_1(x)\}$, let $c(x) \in C$. Then there exist unique skew polynomials $q'(x)$ and $r'(x)$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$ such that

$$c(x) = q'(x)f(x) + r'(x),$$

and $r'(x) = 0$ or $\deg(r'(x)) < \deg(f(x))$. If $r'(x) = 0$, we are done. Assume that $\deg(r'(x)) < \deg(f(x))$. Then $r'(x) \in B$ and hence $r'(x) = m(x)g(x)$ for some $m(x) \in (\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$. Hence

$$c(x) = q'(x)f(x) + r'(x) = q'(x)f(x) + m(x)g(x).$$

Therefore, C is generated by $\{g(x) = ug_1(x), f(x) = f_0(x) + uf_1(x)\}$. \square

For convenience, we split the left ideals of $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ into three types: Type LI-1 refers to the zero ideal or a left ideal satisfying Theorem 4.1 i),

type LI-2 refers to a left ideal satisfying Theorem 4.1 ii), and type LI-3 refers to a left ideal satisfying Theorem 4.1 iii).

More properties of left ideals of each type are given in the following propositions.

Proposition 4.1. *A left ideal of type LI-1 is principal and generated by a monic right divisor $g(x)$ of $x^n - \lambda$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$. Moreover, if we view $g(x) = g_0(x) + ug_1(x)$, where $g_0(x), g_1(x) \in \mathbb{F}_{p^m}[x; \theta]$, then $\deg(g_1(x)) < \deg(g_0(x))$ and $g_0(x)$ is a monic right divisor of $x^n - \bar{\lambda}$ in $\mathbb{F}_{p^m}[x; \theta]$.*

Proof. Let C be a left ideal of type LI-1. If $C = \{0\}$, then $C = \langle 0 \rangle = \langle x^n - \lambda \rangle$ has the desired properties.

Suppose C is non-zero. We prove that the generator polynomial $g(x)$ in Theorem 4.1 i) satisfies these properties. Recall that $g(x)$ is the unique monic skew polynomial in A , the set of all non-zero skew polynomials of minimal degree in C .

First, we show that $g(x)$ is a right divisor of $x^n - \lambda$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$. By the Right Division Algorithm, there exist unique skew polynomials $q(x)$ and $r(x)$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$ such that

$$x^n - \lambda = q(x)g(x) + r(x),$$

and $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$. Then

$$r(x) = -q(x)g(x) + (x^n - \lambda) \in C.$$

By the minimality of $\deg(g(x))$, $r(x) = 0$. Hence $g(x)$ is a right divisor of $x^n - \lambda$.

Finally, we write $g(x) = g_0(x) + ug_1(x)$, where $g_0(x), g_1(x) \in \mathbb{F}_{p^m}[x; \theta]$. Since $g(x)$ is monic, it is clear that $g_0(x)$ is monic and $\deg(g_1(x)) < \deg(g(x)) = \deg(g_0(x))$. As $g(x)$ is a right divisor of $x^n - \lambda$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$, there exists $p(x)$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$ such that

$$x^n - \lambda = p(x)(g_0(x) + ug_1(x)).$$

Computing modulo u , we have $x^n - \bar{\lambda} = \overline{p(x)}g_0(x)$ in $\mathbb{F}_{p^m}[x; \theta]$. This means $g_0(x)$ is a monic right divisor of $x^n - \bar{\lambda}$ in $\mathbb{F}_{p^m}[x; \theta]$. \square

Proposition 4.2. *A left ideal of type LI-2 is principal and generated by $g(x) = ug_1(x)$, where $g_1(x)$ is a monic right divisor of $x^n - \bar{\lambda}$ in $\mathbb{F}_{p^m}[x; \theta]$ such that $\deg(g_1(x)) < n$.*

Proof. Let C be a left ideal of type LI-2. We prove that the generator polynomial $g(x) = ug_1(x)$ in Theorem 4.1 ii) satisfies the desired properties. Recall that $g(x) = ug_1(x)$ is the unique skew polynomial with leading coefficient u in A , the set of all non-zero skew polynomials of minimal degree in C . Clearly, $\deg(g_1(x)) < n$. By the Right Division Algorithm, there exist unique skew polynomials $q(x)$ and $r(x)$ in $\mathbb{F}_{p^m}[x; \theta]$ such that

$$x^n - \bar{\lambda} = q(x)g_1(x) + r(x),$$

and $r(x) = 0$ or $\deg(r(x)) < \deg(g_1(x))$. Since $u(x^n - \bar{\lambda}) = u(x^n - \lambda)$, we have

$$\begin{aligned} ur(x) &= -uq(x)g_1(x) + u(x^n - \bar{\lambda}) \\ &= \overrightarrow{-q(x)}^u ug_1(x) + u(x^n - \lambda) \\ &= \overrightarrow{-q(x)}^u g(x) + u(x^n - \lambda) \in C. \end{aligned}$$

By the minimality of $\deg(g(x))$, $ur(x) = 0$. As $r(x) \in \mathbb{F}_{p^m}[x; \theta]$, $r(x) = 0$. Hence $g_1(x)$ is a right divisor of $x^n - \bar{\lambda}$ in $\mathbb{F}_{p^m}[x; \theta]$. \square

Proposition 4.3. *A left ideal of type LI-3 is generated by $\{g(x) = ug_1(x), f(x) = f_0(x) + uf_1(x)\}$, where $f_0(x), f_1(x), g_1(x) \in \mathbb{F}_{p^m}[x; \theta]$ satisfy the following properties:*

- i)* $g_1(x), f_0(x)$ are monic,
- ii)* $\deg(f_1(x)) < \deg(g_1(x)) < \deg(f_0(x)) < n$,
- iii)* $g_1(x)$ is a right divisor of $f_0(x)$ in $\mathbb{F}_{p^m}[x; \theta]$,
- iv)* $f_0(x)$ is a right divisor of $x^n - \bar{\lambda}$ in $\mathbb{F}_{p^m}[x; \theta]$.

Moreover, if $\lambda \in \mathbb{F}_{p^m}$, then $g_1(x)$ is a right divisor of $\left(\frac{x^n - \lambda}{f_0(x)}\right)^u f_1(x)$ in $\mathbb{F}_{p^m}[x; \theta]$.

Proof. Let C be a left ideal of type LI-3. We prove that the generator set $\{g(x) = ug_1(x), f(x) = f_0(x) + uf_1(x)\}$ in Theorem 4.1 *iii)* satisfies the desired properties. Recall that $g(x) = ug_1(x)$ is the unique skew polynomial with the leading coefficient u in A , the set of all non-zero skew polynomials of minimal degree in C and $f_0(x) + uf_1(x)$ is the unique monic skew polynomial of minimal degree in C such that $\deg(f_1(x)) < \deg(g_1(x))$.

Properties *i)* and *ii)* are clear. Property *iii)* can be proved by a similar argument in the case for Proposition 4.2 with $x^n - \bar{\lambda}$ replaced by $f_0(x)$.

Note that $uf_0(x)$ is a skew polynomial of minimal degree in $\langle uf_0(x) \rangle$. Using arguments similar to the proof of Proposition 4.2, $f_0(x)$ is a right divisor of $x^n - \bar{\lambda}$ in $\mathbb{F}_{p^m}[x; \theta]$. Hence, property *iv)* is proved.

Finally, it is straightforward to see that if $\lambda \in \mathbb{F}_{p^m}$, then $\bar{\lambda} = \lambda$. Thus

$$\begin{aligned} \frac{x^n - \lambda}{f_0(x)}(f_0(x) + uf_1(x)) &= \frac{x^n - \lambda}{f_0(x)}uf_1(x) \\ &= u \left(\frac{x^n - \lambda}{f_0(x)}\right)^u f_1(x) \\ &\in C \cap u((\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle). \end{aligned}$$

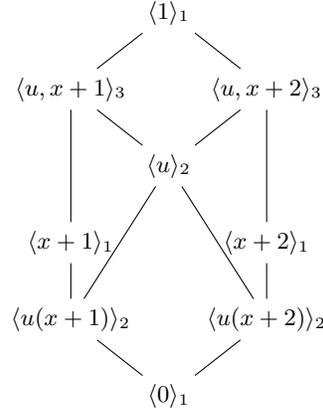
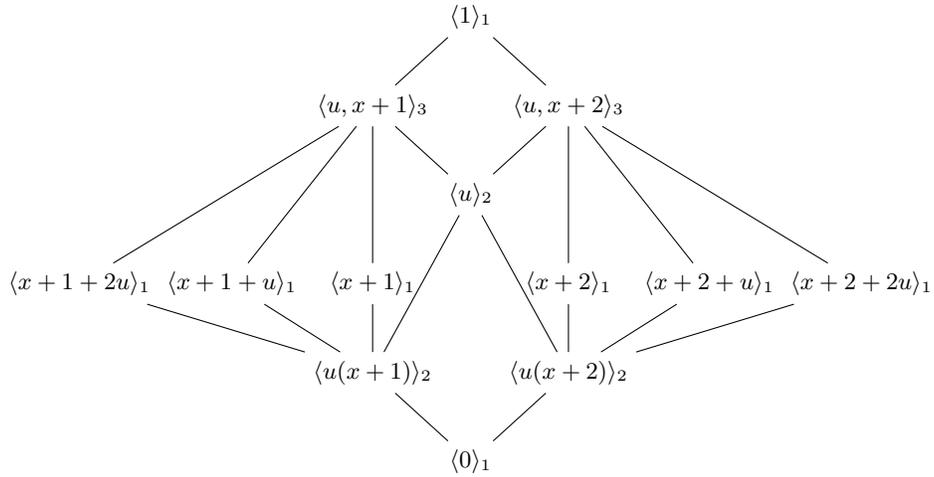
Note that $C \cap u((\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle)$ is a left ideal in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ containing $g(x) = ug_1(x)$ as a skew polynomial of minimal degree. Since $C \cap u((\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle)$ does not contain any monic element, by Proposition 4.2, it is generated by $g(x) = ug_1(x)$. Hence $g_1(x)$ is a right divisor of $\left(\frac{x^n - \lambda}{f_0(x)}\right)^u f_1(x)$. \square

Example 4.1. Figures 4.1 and 4.2 show the ideal lattices of $(\mathbb{F}_3 + u\mathbb{F}_3)[x]/\langle x^2 - 1 \rangle$ and $(\mathbb{F}_3 + u\mathbb{F}_3)[x; \Theta_{\text{id},2}]/\langle x^2 - 1 \rangle$, where $\Theta_{\text{id},2}(a + bu) = a + 2bu$ for all $a, b \in \mathbb{F}_3$. The subscripts 1, 2 and 3 indicate types LI-1, LI-2 and LI-3, respectively.

Note that Figure 4.1 is embedded in Figure 4.2.

4.2. Euclidean Dual Codes of Skew Cyclic and Skew Negacyclic Codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. We study the structures of the Euclidean dual codes of skew cyclic and skew negacyclic codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. For this purpose, we assume that $\lambda = \pm 1$. Since $\bar{\lambda} = \lambda \in \mathbb{F}_{p^m}$ is always fixed by any automorphism, Θ can be arbitrary. However, the length n of codes is assumed to be a multiple of the order of Θ .

As $\lambda^2 = 1$, by Lemma 3.1, the Euclidean dual codes of skew cyclic and skew negacyclic codes are again skew cyclic and skew negacyclic, respectively. Their generators are given through the unique representation of the original codes and


 FIGURE 4.1. The ideal lattice of $(\mathbb{F}_3 + u\mathbb{F}_3)[x]/\langle x^2 - 1 \rangle$

 FIGURE 4.2. The ideal lattice of $(\mathbb{F}_3 + u\mathbb{F}_3)[x; \Theta_{\text{id},2}]/\langle x^2 - 1 \rangle$

the ring anti-monomorphism φ defined in Proposition 2.4, where $\varphi(\sum_{i=0}^t a_i x^i) = \sum_{i=0}^t x^{-i} a_i$.

Theorem 4.2. *Let $\lambda \in \{-1, 1\}$. Then the Euclidean dual code of a left ideal in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ is also a left ideal in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ determined as follows:*

LI-1 $^\perp$. *If $C = \langle g_0(x) + ug_1(x) \rangle$, then $C^\perp = \langle x^{n-\deg(g_0(x))} \varphi \left(\frac{x^n - \lambda}{g_0(x) + ug_1(x)} \right) \rangle$.*

LI-2 $^\perp$. *If $C = \langle ug_1(x) \rangle$, then $C^\perp = \langle u, x^{n-\deg(g_1(x))} \varphi \left(\frac{x^n - \lambda}{g_1(x)} \right) \rangle$.*

LI-3 $^\perp$. If $C = \langle ug_1(x), f_0(x) + uf_1(x) \rangle$, then there exists $m(x) \in \mathbb{F}_{p^m}[x; \theta]$ such that

$$m(x)g_1(x) = \left(\frac{x^n - \lambda}{f_0(x)} \right) u f_1(x) \text{ and}$$

$$C^\perp = \langle x^{n-\deg(f_0(x))} \varphi \left(\frac{x^n - \lambda}{f_0(x)} u \right), x^{n-\deg(g_1(x))} \varphi \left(\frac{x^n - \lambda}{g_1(x)} - um(x) \right) \rangle.$$

For LI-1 $^\perp$, the Euclidean dual code of type LI-1 code is determined in Theorem 3.3 and it is shown to be type LI-1. Moreover, $(C^\perp)^\perp = C$ implies that C is type LI-1 if and only if C^\perp is type LI-1. However, this does not need to be true for types LI-2 and LI-3 (see Example 4.2).

In LI-2 $^\perp$ and LI-3 $^\perp$, $f_0(x)$, $g_1(x)$ are right divisors of $x^n - \lambda$ in $\mathbb{F}_{p^m}[x; \theta]$. Since $x^n - \lambda$ is central, it follows from (2.2) that

$$f_0(x) \frac{x^n - \lambda}{f_0(x)} = x^n - \lambda = \frac{x^n - \lambda}{f_0(x)} f_0(x), \quad (4.1)$$

$$g_1(x) \frac{x^n - \lambda}{g_1(x)} = x^n - \lambda = \frac{x^n - \lambda}{g_1(x)} g_1(x). \quad (4.2)$$

These two facts and the centrality of $x^n - \lambda$ will be frequently used in the following proofs.

Proof of LI-2 $^\perp$. Let $D := \langle u, x^{n-\deg(g_1(x))} \varphi \left(\frac{x^n - \lambda}{g_1(x)} \right) \rangle$. Clearly, $u \in C^\perp$. From

(4.2), it follows that $(ug_1(x)) \frac{x^n - \lambda}{g_1(x)} = u(x^n - \lambda) = 0$ in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \theta] / \langle x^n - \lambda \rangle$.

Hence $D \subseteq C^\perp$ is concluded via Lemma 3.2.

In the other direction, we note that C^\perp is of either type LI-2 or LI-3. If $C^\perp = \langle us_1(x) \rangle$ is of type LI-2, then $C^\perp \subseteq \langle u \rangle \subseteq D$. Suppose that $C^\perp := \langle us_1(x), t_0(x) + ut_1(x) \rangle$ is of type LI-3. Clearly, $us_1(x), ut_1(x) \in \langle u \rangle \subseteq D$.

Since $ug_1(x) \in C$ and $t_0(x) + ut_1(x) \in C^\perp$, it follows from Lemma 3.2

$$\begin{aligned} 0 &= (ug_1(x)) \varphi^{-1}(x^{-\deg(t_0(x))}(t_0(x) + ut_1(x))) \\ &= ug_1(x) \varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) \end{aligned}$$

in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \theta] / \langle x^n - \lambda \rangle$. Thus $g_1(x) \varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) = 0$. Hence, in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \theta]$,

$$g_1(x) \varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) = l_1(x)(x^n - \lambda) = (x^n - \lambda)l_1(x), \quad (4.3)$$

for some $l_1(x) \in \mathbb{F}_{p^m}[x; \theta]$. Note that

$$\deg(t_0(x)) = \deg(l_1(x)) + n - \deg(g_1(x)). \quad (4.4)$$

With the notation in (4.2), left cancellation of (4.3) by $g_1(x)$ gives

$$\frac{x^n - \lambda}{g_1(x)} l_1(x) = \varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)),$$

and hence, by (4.4),

$$\begin{aligned} t_0(x) &= x^{\deg(t_0(x))} \varphi \left(\frac{x^n - \lambda}{g_1(x)} l_1(x) \right) \\ &= x^{\deg(l_1(x)) + n - \deg(g_1(x))} \varphi(l_1(x)) \varphi \left(\frac{x^n - \lambda}{g_1(x)} \right) \\ &= x^{\deg(l_1(x))} \overrightarrow{\varphi(l_1(x))} x^{n - \deg(g_1(x))} \varphi \left(\frac{x^n - \lambda}{g_1(x)} \right) \in D. \end{aligned}$$

Consequently, $t_0(x) + ut_1(x) \in D$. As desired, $C^\perp \subseteq D$. \square

Proof of LI-3[⊥]. Since $\lambda \in \mathbb{F}_{p^m}$, it follows from Proposition 4.3 that $g_1(x)$ is a right divisor of $\overleftarrow{\left(\frac{x^n - \lambda}{f_0(x)} \right)^u} f_1(x)$. Then there exists $m(x) \in \mathbb{F}_{p^m}[x; \theta]$ such that

$$m(x)g_1(x) = \overleftarrow{\left(\frac{x^n - \lambda}{f_0(x)} \right)^u} f_1(x). \quad (4.5)$$

Let $D := \langle x^{n - \deg(f_0(x))} \varphi \left(\frac{x^n - \lambda}{f_0(x)} u \right), x^{n - \deg(g_1(x))} \varphi \left(\frac{x^n - \lambda}{g_1(x)} - um(x) \right) \rangle$. It follows from (4.5) that

$$um(x)g_1(x) = u \overleftarrow{\left(\frac{x^n - \lambda}{f_0(x)} \right)^u} f_1(x) = \frac{x^n - \lambda}{f_0(x)} u f_1(x). \quad (4.6)$$

Multiplying on the left of (4.6) by $f_0(x)$, we have

$$\begin{aligned} f_0(x)um(x)g_1(x) &= f_0(x) \frac{x^n - \lambda}{f_0(x)} u f_1(x) \\ &= (x^n - \lambda) u f_1(x) \quad (\text{using (4.1)}) \\ &= u f_1(x) (x^n - \lambda) \\ &= u f_1(x) \frac{x^n - \lambda}{g_1(x)} g_1(x) \quad (\text{using (4.2)}). \end{aligned}$$

Hence,

$$f_0(x)um(x) = u f_1(x) \frac{x^n - \lambda}{g_1(x)}, \quad (4.7)$$

and

$$\deg(m(x)) = n + \deg(f_1(x)) - \deg(f_0(x)) - \deg(g_1(x)). \quad (4.8)$$

Now, we observe the following:

a) Since $u^2 = 0$, we have

$$u g_1(x) \frac{x^n - \lambda}{f_0(x)} u = 0. \quad (4.9)$$

b) Using $u^2 = 0$ and (4.2), we conclude that

$$u g_1(x) \left(\frac{x^n - \lambda}{g_1(x)} - um(x) \right) = u g_1(x) \frac{x^n - \lambda}{g_1(x)} = u(x^n - \lambda). \quad (4.10)$$

c) It follows from $u^2 = 0$ and (4.1) that

$$(f_0(x) + uf_1(x))\left(\frac{x^n - \lambda}{f_0(x)}u\right) = f_0(x)\frac{x^n - \lambda}{f_0(x)}u = (x^n - \lambda)u = u(x^n - \lambda). \quad (4.11)$$

d) Since $g_1(x)$ is a right divisor of $f_0(x)$, by (2.1) and (4.2), we have

$$\begin{aligned} f_0(x)\frac{x^n - \lambda}{g_1(x)} &= \left(\frac{f_0(x)}{g_1(x)}g_1(x)\right)\frac{x^n - \lambda}{g_1(x)} = \frac{f_0(x)}{g_1(x)}\left(g_1(x)\frac{x^n - \lambda}{g_1(x)}\right) \\ &= \frac{f_0(x)}{g_1(x)}(x^n - \lambda). \end{aligned} \quad (4.12)$$

The next equation follows from $u^2 = 0$, (4.7) and (4.12)

$$\begin{aligned} (f_0(x) + uf_1(x))\left(\frac{x^n - \lambda}{g_1(x)} - um(x)\right) &= f_0(x)\frac{x^n - \lambda}{g_1(x)} + uf_1(x)\frac{x^n - \lambda}{g_1(x)} \\ &\quad - f_0(x)um(x) \\ &= \frac{f_0(x)}{g_1(x)}(x^n - \lambda). \end{aligned} \quad (4.13)$$

Equations (4.9)-(4.11) and (4.13) equal 0 in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$. Thus, by Lemma 3.2, $D \subseteq C^\perp$.

For the reverse inclusion, we note that C^\perp is of type LI-2 or LI-3. First, suppose that $C^\perp := \langle us_1(x) \rangle$ is of type LI-2. Since $f_0(x) + uf_1(x) \in C$ and $us_1(x) \in C^\perp$, the Euclidean orthogonality and Lemma 3.2 imply that

$$(f_0(x) + uf_1(x))\varphi^{-1}(x^{-\deg(s_1)}us_1(x)) = 0$$

in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$. Hence, in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$,

$$f_0(x)\varphi^{-1}(x^{-\deg(s_1)}us_1(x)) = ul(x)(x^n - \lambda) = (x^n - \lambda)ul(x), \quad (4.14)$$

for some $l(x) \in \mathbb{F}_{p^m}[x; \theta]$. Moreover, $\deg(s_1(x)) = n + \deg(l(x)) - \deg(f_0(x))$. It follows from (4.1) and (4.14) that

$$\varphi^{-1}(x^{-(n+\deg(l(x))-\deg(f_0(x)))}us_1(x)) = \varphi^{-1}(x^{-\deg(s_1)}us_1(x)) = \frac{x^n - \lambda}{f_0(x)}ul(x).$$

Since φ is a ring anti-monomorphism, we conclude that

$$x^{-(n+\deg(l(x))-\deg(f_0(x)))}us_1(x) = \varphi\left(\frac{x^n - \lambda}{f_0(x)}ul(x)\right) = \varphi(l(x))\varphi\left(\frac{x^n - \lambda}{f_0(x)}u\right).$$

Consequently,

$$\begin{aligned} us_1(x) &= x^{n+\deg(l(x))-\deg(f_0(x))}\varphi(l(x))\varphi\left(\frac{x^n - \lambda}{f_0(x)}u\right) \\ &= x^{\deg(l(x))}\overrightarrow{\varphi(l(x))}x^{n-\deg(f_0(x))}\varphi\left(\frac{x^n - \lambda}{f_0(x)}u\right) \in D. \end{aligned}$$

Next, suppose that $C^\perp := \langle us_1(x), t_0(x) + ut_1(x) \rangle$ is of type LI-3. Using arguments similar to those above, $f_0(x) + uf_1(x) \in C$ and $us_1(x) \in C^\perp$ imply $us_1(x) \in D$.

Since $ug_1(x) \in C$ and $t_0(x) + ut_1(x) \in C^\perp$, it follows from Lemma 3.2 that

$$0 = ug_1(x)\varphi^{-1}(x^{-\deg(t_0)}(t_0(x) + ut_1(x))) = ug_1(x)\varphi^{-1}(x^{-\deg(t_0)}t_0(x)),$$

in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$. Thus $g_1(x)\varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) = 0$, and hence, in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$,

$$g_1(x)\varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) = l_1(x)(x^n - \lambda) = (x^n - \lambda)l_1(x), \quad (4.15)$$

for some $l_1(x) \in \mathbb{F}_{p^m}[x; \theta]$. Note that

$$\deg(t_0(x)) = n + \deg(l_1(x)) - \deg(g_1(x)). \quad (4.16)$$

In the notation of (4.2), the left cancellation of (4.15) by $g_1(x)$ implies

$$\varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) = \frac{x^n - \lambda}{g_1(x)}l_1(x), \quad (4.17)$$

and hence

$$t_0(x) = x^{\deg(t_0(x))}\varphi\left(\frac{x^n - \lambda}{g_1(x)}l_1(x)\right) = x^{\deg(t_0(x))}\varphi(l_1(x))\varphi\left(\frac{x^n - \lambda}{g_1(x)}\right). \quad (4.18)$$

By Lemma 3.2, in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$,

$$\begin{aligned} 0 &= (f_0(x) + uf_1(x))\varphi^{-1}(x^{-\deg(t_0(x))}(t_0(x) + ut_1(x))) \\ &= f_0(x)\varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) + f_0(x)\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) \\ &\quad + uf_1(x)\varphi^{-1}(x^{-\deg(t_0(x))}t_0(x)) \\ &= f_0(x)\frac{x^n - \lambda}{g_1(x)}l_1(x) + f_0(x)\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) + uf_1(x)\frac{x^n - \lambda}{g_1(x)}l_1(x) \\ &\quad (\text{using (4.17)}) \\ &= \frac{f_0(x)}{g_1(x)}(x^n - \lambda)l_1(x) + f_0(x)\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) + f_0(x)um(x)l_1(x) \\ &\quad (\text{using (2.1), (4.2) and (4.7)}) \\ &= \frac{f_0(x)}{g_1(x)}l_1(x)(x^n - \lambda) + f_0(x)\left(\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) + um(x)l_1(x)\right) \\ &= f_0(x)(\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) + um(x)l_1(x)). \end{aligned}$$

Then there exists $l_2(x) \in \mathbb{F}_{p^m}[x; \theta]$ such that, in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]$,

$$\begin{aligned} f_0(x)(\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) + um(x)l_1(x)) &= ul_2(x)(x^n - \lambda) \\ &= (x^n - \lambda)ul_2(x). \end{aligned} \quad (4.19)$$

Using (4.8), (4.16) and the fact that $\deg(f_0(x)) > \deg(f_1(x))$, we conclude that

$$\deg(m(x)l_1(x)) \leq \deg(m(x)) + \deg(l_1(x)) < \deg(t_0(x)). \quad (4.20)$$

Hence, from (4.19) and (4.20),

$$\deg(t_0(x)) = n + \deg(l_2(x)) - \deg(f_0(x)). \quad (4.21)$$

The left cancellation of (4.19) by $f_0(x)$ implies

$$\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) + um(x)l_1(x) = \frac{x^n - \lambda}{f_0(x)}ul_2(x).$$

Hence $\varphi^{-1}(x^{-\deg(t_0(x))}ut_1(x)) = \frac{x^n - \lambda}{f_0(x)}ul_2(x) - um(x)l_1(x)$, i.e.,

$$ut_1(x) = x^{\deg(t_0(x))}\varphi\left(\frac{x^n - \lambda}{f_0(x)}ul_2(x) - um(x)l_1(x)\right). \quad (4.22)$$

Therefore,

$$\begin{aligned} t_0(x) + ut_1(x) &= x^{\deg(t_0(x))}\varphi(l_1(x))\varphi\left(\frac{x^n - \lambda}{g_1(x)}\right) \\ &\quad + x^{\deg(t_0(x))}\varphi\left(\frac{x^n - \lambda}{f_0(x)}ul_2(x) - um(x)l_1(x)\right) \quad (\text{using (4.18) and (4.22)}) \\ &= x^{\deg(t_0(x))}\varphi(l_1(x))\varphi\left(\frac{x^n - \lambda}{g_1(x)}\right) - x^{\deg(t_0(x))}\varphi(l_1(x))\varphi(um(x)) \\ &\quad + x^{\deg(t_0(x))}\varphi\left(\frac{x^n - \lambda}{f_0(x)}ul_2(x)\right) \\ &= x^{n+\deg(l_1(x))-\deg(g_1(x))}\varphi(l_1(x))\varphi\left(\frac{x^n - \lambda}{g_1(x)} - um(x)\right) \\ &\quad + x^{n+\deg(l_2(x))-\deg(f_0(x))}\varphi(l_2(x))\varphi\left(\frac{x^n - \lambda}{f_0(x)}u\right) \quad (\text{using (4.16) and (4.21)}) \\ &= x^{\deg(l_1(x))}\overbrace{\varphi(l_1(x))}^{x^{n-\deg(g_1(x))}}x^{n-\deg(g_1(x))}\varphi\left(\frac{x^n - \lambda}{g_1(x)} - um(x)\right) \\ &\quad + x^{\deg(l_2(x))}\overbrace{\varphi(l_2(x))}^{x^{n-\deg(f_0(x))}}x^{n-\deg(f_0(x))}\varphi\left(\frac{x^n - \lambda}{f_0(x)}u\right) \in D. \end{aligned}$$

As desired, $C^\perp \subseteq D$. \square

4.3. Hermitian Dual Codes of Skew Cyclic and Skew Negacyclic Codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. We assume that the order of Θ is 2 and determine the structure of the Hermitian dual codes of skew cyclic and skew negacyclic codes in terms of their unique representative generators, the ring anti-monomorphism φ defined in Proposition 2.4 and the ring automorphism ϕ defined in (3.4). Using Lemma 3.6 and arguments similar to those in the previous subsection, the next theorem follows.

Theorem 4.3. *Let $\lambda \in \{1, -1\}$ and let Θ be an automorphism of order 2. Then the Hermitian dual code of a left ideal in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ is again a left ideal in $(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x; \Theta]/\langle x^n - \lambda \rangle$ determined as follows:*

LI-1 $^{\perp H}$. If $C = \langle g_0(x) + ug_1(x) \rangle$, then $C^{\perp H} = \langle \phi(x^{n-\deg(g_0(x))})\varphi\left(\frac{x^n - \lambda}{g_0(x) + ug_1(x)}\right) \rangle$.

LI-2 $^{\perp H}$. If $C = \langle ug_1(x) \rangle$, then $C^{\perp H} = \langle u, \phi(x^{n-\deg(g_1(x))})\varphi\left(\frac{x^n - \lambda}{g_1(x)}\right) \rangle$.

LI-3 $^{\perp H}$. If $C = \langle ug_1(x), f_0(x) + uf_1(x) \rangle$, then there exists $m(x) \in \mathbb{F}_{p^m}[x; \theta]$ such

$$\text{that } m(x)g_1(x) = \left(\frac{x^n - \lambda}{f_0(x)}\right)_u f_1(x) \text{ and}$$

$$C^{\perp H} = \langle \phi(x^{n-\deg(f_0(x))})\varphi\left(\frac{x^n - \lambda}{f_0(x)}u\right), \phi(x^{n-\deg(g_1(x))})\varphi\left(\frac{x^n - \lambda}{g_1(x)} - um(x)\right) \rangle.$$

Example 4.2. Table 4.1 shows the Euclidean and Hermitian dual codes of the left ideals in $(\mathbb{F}_3 + u\mathbb{F}_3)[x; \Theta_{\text{id},2}]/\langle x^2 - 1 \rangle$ classified in Example 4.1. The dual codes are obtained via Theorems 4.2 and 4.3 and rewritten to satisfy the representation

in Proposition 4.1. The subscripts 1, 2 and 3 indicate types LI-1, LI-2 and LI-3, respectively.

TABLE 4.1. The left ideals in $(\mathbb{F}_3 + u\mathbb{F}_3)[x; \Theta_{\text{id},2}]/\langle x^2 - 1 \rangle$ and their Euclidean and Hermitian dual codes

C	C^\perp	$C^{\perp H}$
$\langle 0 \rangle_1$	$\langle 1 \rangle_1$	$\langle 1 \rangle_1$
$\langle u(x+1) \rangle_2$	$\langle u, x+2 \rangle_3$	$\langle u, x+2 \rangle_3$
$\langle u(x+2) \rangle_2$	$\langle u, x+1 \rangle_3$	$\langle u, x+1 \rangle_3$
$\langle u \rangle_2$	$\langle u \rangle_2$	$\langle u \rangle_2$
$\langle x+1+2u \rangle_1$	$\langle x+2+2u \rangle_1$	$\langle x+2+u \rangle_1$
$\langle x+1+u \rangle_1$	$\langle x+2+u \rangle_1$	$\langle x+2+2u \rangle_1$
$\langle x+1 \rangle_1$	$\langle x+2 \rangle_1$	$\langle x+2 \rangle_1$
$\langle x+2 \rangle_1$	$\langle x+1 \rangle_1$	$\langle x+1 \rangle_1$
$\langle x+2+u \rangle_1$	$\langle x+1+u \rangle_1$	$\langle x+1+2u \rangle_1$
$\langle x+2+2u \rangle_1$	$\langle x+1+2u \rangle_1$	$\langle x+1+u \rangle_1$
$\langle u, x+1 \rangle_3$	$\langle u(x+2) \rangle_2$	$\langle u(x+2) \rangle_2$
$\langle u, x+2 \rangle_3$	$\langle u(x+1) \rangle_2$	$\langle u(x+1) \rangle_2$
$\langle 1 \rangle_1$	$\langle 0 \rangle_1$	$\langle 0 \rangle_1$

5. Conclusion. The concept of coding with skew polynomial rings over finite fields [7] and [10] and over Galois rings [8] is extended to the case over finite chain rings. Given an automorphism Θ of a finite chain ring \mathcal{R} and a unit λ in \mathcal{R} , Θ - λ -constacyclic codes are introduced. Under the assumptions that λ is a unit fixed by Θ and the length n of codes is a multiple of the order of Θ , Θ - λ -constacyclic codes can be viewed as left ideals in the quotient ring $\mathcal{R}[x; \Theta]/\langle x^n - \lambda \rangle$. In particular, when the code is generated by a monic right divisor $g(x)$ of $x^n - \lambda$, its properties are exhibited. When $\lambda^2 = 1$, the generators of its Euclidean and Hermitian dual codes are given in terms of $h(x) := \frac{x^n - \lambda}{g(x)}$. Moreover, necessary and sufficient conditions for a Θ - λ -constacyclic code to be Euclidean and Hermitian self-dual are provided.

A typical example of a finite chain ring is $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$. In the case $e = 2$, a complete classification of Θ - λ -constacyclic codes over the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is given. For the special case when $\lambda = \pm 1$, the classification provides generators of the Euclidean and Hermitian dual codes of skew cyclic and skew negacyclic codes based on generators of the original codes. Moreover, an illustration of all skew cyclic codes of length 2 over $\mathbb{F}_3 + u\mathbb{F}_3$ and their Euclidean and Hermitian dual codes is also provided.

For further work, using the idea in [9], constructions and classification of skew constacyclic codes over finite chain rings could be considered as modules over the skew polynomial ring $\mathcal{R}[x; \Theta]$. This may lead to classification of codes of arbitrary lengths and constructions of more codes with good parameters.

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