

On the structure of non-full-rank perfect codes

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Abstract

The Krotov combining construction of perfect 1-error-correcting binary codes from 2000 and a theorem of Heden saying that every non-full-rank perfect 1-error-correcting binary code can be constructed by this combining construction is generalized to the q -ary case. Simply, every non-full-rank perfect code C is the union of a well-defined family of $\bar{\mu}$ -components $K_{\bar{\mu}}$, where $\bar{\mu}$ belongs to an “outer” perfect code C^* , and these components are at distance three from each other. Components from distinct codes can thus freely be combined to obtain new perfect codes. The Phelps general product construction of perfect binary code from 1984 is generalized to obtain $\bar{\mu}$ -components, and new lower bounds on the number of perfect 1-error-correcting q -ary codes are presented.

1. Introduction

Let F_q denote the finite field with q elements. A *perfect 1-error-correcting q -ary code of length n* , for short here a *perfect code*, is a subset C of the direct product F_q^n , of n copies of F_q , having the property that any element of F_q^n differs in at most one coordinate position from a unique element of C .

The family of all perfect codes is far from classified or enumerated. We will in this short note say something about the structure of these codes. We need the concept of rank.

We consider F_q^n as a vector space of dimension n over the finite field F_q . The *rank* of a q -ary code C , here denoted $\text{rank}(C)$, is the dimension of the linear span $\langle C \rangle$ of the elements of C . Trivial, and well known, counting arguments give that if there exists a perfect code in F_q^n then $n = (q^m - 1)/(q - 1)$, for some integer m , and $|C| = q^{n-m}$. So, for every perfect code C ,

$$n - m \leq \text{rank}(C) \leq n .$$

If $\text{rank}(C) = n$ we will say that C has *full rank*.

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We will show that every non-full-rank perfect code is a union of so called $\bar{\mu}$ -components $K_{\bar{\mu}}$, and that these components may be enumerated by some other perfect code C^* , i.e, $\bar{\mu} \in C^*$. Further, the distance between any two such components will be at least three. This implies that we will be completely free to combine $\bar{\mu}$ -components from different perfect codes of same length, to obtain other perfect codes. Generalizing a construction by Phelps of perfect 1-error correcting binary codes [8], we will obtain further $\bar{\mu}$ -components. As an application of our results we will be able to slightly improve the lower bound on the number of perfect codes given in [6].

Our results generalize corresponding results for the binary case. In [3] it was shown that a binary perfect code can be constructed as the union of different subcodes ($\bar{\mu}$ -components) satisfying some generalized parity-check property, each of them being constructed independently or taken from another perfect code. In [2] it was shown that every non-full-rank perfect binary code can be obtained by this combining construction.

2. Every non-full-rank perfect code is the union of $\bar{\mu}$ -components

We start with some notation. Assume we have positive integers n, t, n_1, \dots, n_t such that $n_1 + \dots + n_t \leq n$. Any q -ary word \bar{x} will be represented in the block form $\bar{x} = (\bar{x}_1 | \bar{x}_2 | \dots | \bar{x}_t | \bar{x}_0) = (\bar{x}_* | \bar{x}_0)$, where $\bar{x}_i = (x_{i1}, x_{i2}, \dots, x_{in_i})$, $i = 0, 1, \dots, t$, $n_0 = n - n_1 - \dots - n_t$, $\bar{x}_* = (\bar{x}_1 | \bar{x}_2 | \dots | \bar{x}_t)$. For every block \bar{x}_i , $i = 1, 2, \dots, t$, we define $\sigma_i(\bar{x}_i)$ by

$$\sigma_i(\bar{x}_i) = \sum_{j=1}^{n_i} x_{ij} ,$$

and, for \bar{x} ,

$$\bar{\sigma}(\bar{x}) = \bar{\sigma}(\bar{x}_*) = (\sigma_1(\bar{x}_1), \sigma_2(\bar{x}_2), \dots, \sigma_t(\bar{x}_t))$$

Recall that the Hamming *distance* $d(\bar{x}, \bar{y})$ between two words \bar{x}, \bar{y} of the same length means the number of positions in which they differ.

A *monomial transformation* is a map of the space F_q^n that can be composed by a permutation of the set of coordinate positions and the multiplication in each coordinate position with some non-zero element of the finite field F_q .

A q -ary code C is *linear* if C is a subspace of F_q^n . A linear perfect code is called a *Hamming code*.

Theorem 1. *Let C be any non-full-rank perfect code C of length $n = (q^m - 1)/(q - 1)$. To any integer $r < m$, satisfying*

$$1 \leq r \leq n - \text{rank}(C) ,$$

there is a q -ary Hamming code C^ of length $t = (q^r - 1)/(q - 1)$, such that for some monomial transformation ψ*

$$\psi(C) = \bigcup_{\bar{\mu} \in C^*} K_{\bar{\mu}} ,$$

where

$$K_{\bar{\mu}} = \{(\bar{x}_1 | \bar{x}_2 | \dots | \bar{x}_t | \bar{x}_0) : \bar{\sigma}(\bar{x}) = \bar{\mu}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_t \in F_q^{q^s}, \bar{x}_0 \in C_{\bar{\mu}}(\bar{x}_*)\} \quad (1)$$

for some family of perfect codes $C_{\bar{\mu}}(\bar{x})$, of length $1 + q + q^2 + \dots + q^{s-1}$, where $s = m - r$, and satisfying, for each $\bar{\mu} \in C^*$,

$$d(\bar{x}_*, \bar{x}'_*) \leq 2 \quad \implies \quad C_{\bar{\mu}}(\bar{x}_*) \cap C_{\bar{\mu}}(\bar{x}'_*) = \emptyset. \quad (2)$$

The code C^* will be called an *outer* code to $\psi(C)$. The subcodes $K_{\bar{\mu}}$ will be called $\bar{\mu}$ -*components* of $\psi(C)$. As the minimum distance of C is three, the distance between any two distinct $\bar{\mu}$ -components will be at least three.

Proof. Let D be any subspace of F_q^n containing $\langle C \rangle$, and of dimension $n - r$. By using a monomial transformation ψ of space we may achieve that the dual space of $\psi(D)$ is the nullspace of a $r \times n$ -matrix

$$H = \left[\begin{array}{c|ccc|c|ccc|c|ccc|c|ccc} \bar{\alpha}_{11} & \dots & \bar{\alpha}_{1n_1} & \bar{\alpha}_{21} & \dots & \bar{\alpha}_{2n_2} & \dots & \bar{\alpha}_{t1} & \dots & \bar{\alpha}_{tn_t} & \bar{0} & \dots & \bar{0} \\ \hline \end{array} \right]$$

where $\bar{\alpha}_{ij} = \bar{\alpha}_i$, for $i = 1, 2, \dots, t$, the first non-zero coordinate in each vector $\bar{\alpha}_i$ equals 1, $\bar{\alpha}_i \neq \bar{\alpha}_{i'}$, for $i \neq i'$, and where the columns of H are in lexicographic order, according to some given ordering of F_q .

To avoid too much notation we assume that C was such that $\psi = \text{id}$.

Let C^* be the null space of the matrix

$$H^* = \left[\begin{array}{c|c|c|c} \bar{\alpha}_1 & \bar{\alpha}_2 & \dots & \bar{\alpha}_t \\ \hline \end{array} \right]$$

Define, for $\bar{\mu} \in C^*$,

$$K_{\bar{\mu}} = \{(\bar{x}_1 | \bar{x}_2 | \dots | \bar{x}_t | \bar{x}_0) \in C : (\sigma_1(\bar{x}_1), \sigma_2(\bar{x}_2), \dots, \sigma_t(\bar{x}_t)) = \bar{\mu}\}.$$

Then,

$$C = \bigcup_{\bar{\mu} \in C^*} K_{\bar{\mu}}.$$

Further, since any two columns of H^* are linearly independent, for any two distinct words $\bar{\mu}$ and $\bar{\mu}'$ of C^*

$$d(K_{\bar{\mu}}, K_{\bar{\mu}'}) \geq 3. \quad (3)$$

We will show that $K_{\bar{\mu}}$ has the properties given in Equation (1).

Any word $\bar{x} = (\bar{x}_1 | \bar{x}_2 | \dots | \bar{x}_t | \bar{x}_0)$ must be at distance at most one from a word of C , and hence, the word $(\sigma_1(\bar{x}_1), \sigma_2(\bar{x}_2), \dots, \sigma_t(\bar{x}_t))$ is at distance at most one from some word of C^* . It follows that C^* is a perfect code, and as a consequence, as C^* is linear, it is a Hamming code with parity-check matrix H^* . As the number of rows of H^* is r , we then get that the number t of columns of H^* is equal to

$$t = \frac{q^r - 1}{q - 1} = 1 + q + q^2 + \dots + q^{r-1}.$$

For any word \bar{x}_* of $F_q^{n_1+n_2+\dots+n_t}$ with $\bar{\sigma}(\bar{x}_*) = \bar{\mu} \in C^*$, we now define the code $C_{\bar{\mu}}(\bar{x}_*)$ of length n_0 by

$$C_{\bar{\mu}}(\bar{x}_*) = \{ \bar{c} \in F_q^{n_0} : (\bar{x}_* \mid \bar{c}) \in C \} .$$

Again, using the fact that C is a perfect code, we may deduce that for any \bar{x}_* such that the set $C_{\bar{\mu}}(\bar{x}_*)$ is non empty, the set $C_{\bar{\mu}}(\bar{x}_*)$ must be a perfect code of length $n_0 = (q^s - 1)/(q - 1)$, for some integer s .

From the fact that the minimum distance of C equals three, we get the property in Equation (2).

Let \bar{e}_i denote a word of weight one with the entry 1 in the coordinate position i . It then follows that the two perfect codes $C_{\bar{\mu}}(\bar{x}_*)$ and $C_{\bar{\mu}}(\bar{x}_* + \bar{e}_1 - \bar{e}_i)$, for $i = 2, 3, \dots, n_1$, must be mutually disjoint. Hence, n_1 is at most equal to the number of perfect codes in a partition of $F_q^{n_0}$ into perfect codes, i.e.,

$$n_1 \leq (q - 1)n_0 + 1 = q^s .$$

Similarly, $n_i \leq q^s$, for $i = 2, 3, \dots, t$.

Reversing these arguments, using Equation (3) and the fact that C is a perfect code, we find that n_i , for each $i = 1, 2, \dots, t$, is at least equal to the number of words in an 1-ball of $F_q^{n_0}$.

We conclude that $n_i = q^s$, for $i = 1, 2, \dots, t$, and finally

$$n = q^s(1 + q + q^2 + \dots + q^{r-1}) + 1 + q + q^2 + \dots + q^{s-1} = 1 + q + q^2 + \dots + q^{r+s-1} .$$

Given r , we can then find s from the equality

$$n = 1 + q + q^2 + \dots + q^{m-1} .$$

△

3. Combining construction of perfect codes

In the previous section, it was shown that a perfect code, depending on its rank, can be divided onto small or large number of so-called $\bar{\mu}$ -components, which satisfy some equation with $\bar{\sigma}$. The construction described in the following theorem realizes the idea of combining independent $\bar{\mu}$ -components, differently constructed or taken from different perfect codes, in one perfect code.

A function $f : \Sigma^n \rightarrow \Sigma$, where Σ is some set, is called an n -ary (or *multary*) *quasigroup* of order $|\Sigma|$ if in the equality $z_0 = f(z_1, \dots, z_n)$ knowledge of any n elements of z_0, z_1, \dots, z_n uniquely specifies the remaining one.

Theorem 2. *Let m and r be integers, $m > r$, q be a prime power, $n = (q^m - 1)/(q - 1)$ and $t = (q^r - 1)/(q - 1)$. Assume that C^* is a perfect code in F_q^t and for every $\bar{\mu} \in C^*$ we have a distance-3 code $K_{\bar{\mu}} \subset F_q^n$ of cardinality $q^{n-m-(t-r)}$ that satisfies the following generalized parity-check law:*

$$\bar{\sigma}(\bar{x}) = (\sigma_1(x_1, \dots, x_t), \dots, \sigma_t(x_{t-l+1}, \dots, x_{lt})) = \bar{\mu}$$

for every $\bar{x} = (x_1, \dots, x_n) \in K_{\bar{\mu}}$, where $l = q^{m-r}$ and $\bar{\sigma} = (\sigma_1, \dots, \sigma_t)$ is a collections of l -ary quasigroups of order q . Then the union

$$C = \bigcup_{\bar{\mu} \in C^*} K_{\bar{\mu}}$$

is a perfect code in F_q^n .

Proof. It is easy to check that C has the cardinality of a perfect code. The distance at least 3 between different words \bar{x}, \bar{y} from C follows from the code distances of $K_{\bar{\mu}}$ (if \bar{x}, \bar{y} belong to the same $K_{\bar{\mu}}$) and C^* (if \bar{x}, \bar{y} belong to different $K_{\bar{\mu}'}, K_{\bar{\mu}''}, \bar{\mu}', \bar{\mu}'' \in C^*$). \triangle

The $\bar{\mu}$ -components $K_{\bar{\mu}}$ can be constructed independently or taken from different perfect codes. In the important case when all σ_i are linear quasigroups (e.g., $\sigma_i(y_1, \dots, y_l) = y_1 + \dots + y_l$) the components can be taken from any perfect code of rank at most $n - r$, as follows from the previous section (it should be noted that if $\bar{\sigma}$ is linear, then a $\bar{\mu}$ -component can be obtained from any $\bar{\mu}'$ -component by adding a vector \bar{z} such that $\bar{\sigma}(\bar{z}) = \bar{\mu} - \bar{\mu}'$).

In general, the existence of $\bar{\mu}$ -components that satisfy the generalized parity-check law for arbitrary $\bar{\sigma}$ is questionable. But for some class of $\bar{\sigma}$ such components exist, as we will see from the following two subsections.

Remark. It is worth mentioning that $\bar{\mu}$ -components can exist for arbitrary length t of $\bar{\mu}$ (for example, in the next two subsections there are no restrictions on t), if we do not require the possibility to combine them into a perfect code. This is especially important for the study of perfect codes of small ranks (close to the rank of a linear perfect code): once we realize that the code is the union of $\bar{\mu}$ -components of some special form, we may forget about the code length and consider $\bar{\mu}$ -components for arbitrary length of $\bar{\mu}$, which allows to use recursive approaches.

3.1. Mollard-Phelps construction

Here we describe the way to construct $\bar{\mu}$ -components derived from the product construction discovered independently in [7] and [9]. In terms of $\bar{\mu}$ -components, the construction in [9] is more general; it allows substitution of arbitrary multary quasigroups, and we will use this possibility in Section 4.

Lemma 1. Let $\bar{\mu} \in F_q^t$ and let $C^\#$ be a perfect code in F_q^k . Let v and h be $(q - 1)$ -ary quasigroups of order q such that the code $\{(\bar{y} \mid v(\bar{y}) \mid h(\bar{y})) : \bar{y} \in F_q^{q-1}\}$ is perfect. Let V_1, \dots, V_t and H_1, \dots, H_k be respectively $(k + 1)$ -ary and $(t + 1)$ -ary quasigroups of order q . Then the set

$$\begin{aligned} K_{\bar{\mu}} = & \left\{ (\underline{\bar{x}_{11}} \mid \dots \mid \underline{\bar{x}_{1k}} \mid \underline{y_1} \mid \underline{\bar{x}_{21}} \mid \dots \mid \underline{\bar{x}_{2k}} \mid \underline{y_2} \mid \dots \mid \underline{\bar{x}_{t1}} \mid \dots \mid \underline{\bar{x}_{tk}} \mid \underline{y_t} \mid \underline{z_1} \mid \underline{z_2} \mid \dots \mid \underline{z_k}) : \right. \\ & \bar{x}_{ij} \in F_q^{q-1}, \\ & (V_1(v(\bar{x}_{11}), \dots, v(\bar{x}_{1k}), y_1), \dots, V_t(v(\bar{x}_{t1}), \dots, v(\bar{x}_{tk}), y_t)) = \bar{\mu}, \\ & \left. (H_1(h(\bar{x}_{11}), \dots, h(\bar{x}_{t1}), z_1), \dots, H_k(h(\bar{x}_{1k}), \dots, h(\bar{x}_{tk}), z_k)) \in C^\# \right\} \end{aligned}$$

is a $\bar{\mu}$ -component that satisfies the generalized parity-check law with

$$\sigma_i(\cdot, \dots, \cdot, \cdot) = V_i(v(\cdot), \dots, v(\cdot), \cdot).$$

(The elements of $F_q^{(q-1)kt+k+t}$ in this construction may be thought of as three-dimensional arrays where the elements of \bar{x}_{ij} are z-lined, every underlined block is y-lined, and the tuple of blocks is x-lined. Naturally, the multary quasigroups V_i may be named “vertical” and H_i , “horizontal”.)

The proof of the code distance is similar to that in [9], and the other properties of a $\bar{\mu}$ -component are straightforward. The existence of admissible $(q-1)$ -ary quasigroups v and h is the only restriction on the q (this concerns the next subsection as well). If F_q is a finite field, there are linear examples: $v(y_1, \dots, y_{q-1}) = y_1 + \dots + y_{q-1}$, $v(y_1, \dots, y_{q-1}) = \alpha_1 y_1 + \dots + \alpha_{q-1} y_{q-1}$ where $\alpha_1, \dots, \alpha_{q-1}$ are all the non-zero elements of F_q . If q is not a prime power, the existence of a q -ary perfect code of length $q+1$ is an open problem (with the only exception $q=6$, when the nonexistence follows from the nonexistence of two orthogonal 6×6 Latin squares [1, Th. 6]).

3.2. Generalized Phelps construction

Here we describe another way to construct $\bar{\mu}$ -components, which generalizes the construction of binary perfect codes from [8].

Lemma 2. *Let $\bar{\mu} \in F_q^t$. Let for every i from 1 to $t+1$ the codes $C_{i,j}$, $j = 0, 1, \dots, qk-k$ form a partition of F_q^k into perfect codes and $\gamma_i : F_q^k \rightarrow \{0, 1, \dots, qk-k\}$ be the corresponding partition function:*

$$\gamma_i(\bar{y}) = j \iff \bar{y} \in C_{i,j}.$$

Let v and h be $(q-1)$ -ary quasigroups of order q such that the code $\{(\bar{y} \mid v(\bar{y}) \mid h(\bar{y})) : \bar{y} \in F_q^{q-1}\}$ is perfect. Let V_1, \dots, V_t be $(k+1)$ -ary quasigroups of order q and Q be a t -ary quasigroup of order $qk-k+1$.

$$K_{\bar{\mu}} = \left\{ (\bar{x}_{11} \mid \dots \mid \bar{x}_{1k} \mid \underline{y_1} \mid \bar{x}_{21} \mid \dots \mid \bar{x}_{2k} \mid \underline{y_2} \mid \dots \mid \bar{x}_{t1} \mid \dots \mid \bar{x}_{tk} \mid \underline{y_t} \mid \underline{z_1} \mid \underline{z_2} \mid \dots \mid \underline{z_k}) : \right. \\ \left. \begin{aligned} &\bar{x}_{ij} \in F_q^{q-1}, \\ &(V_1(v(\bar{x}_{11}), \dots, v(\bar{x}_{1k}), y_1), \dots, V_t(v(\bar{x}_{t1}), \dots, v(\bar{x}_{tk}), y_t)) = \bar{\mu}, \\ &Q(\gamma_1(h(\bar{x}_{11}), \dots, h(\bar{x}_{1k})), \dots, \gamma_t(h(\bar{x}_{t1}), \dots, h(\bar{x}_{tk}))) = \gamma_{t+1}(z_1, \dots, z_k) \end{aligned} \right\}$$

is a $\bar{\mu}$ -component that satisfies the generalized parity-check law with

$$\sigma_i(\cdot, \dots, \cdot, \cdot) = V_i(v(\cdot), \dots, v(\cdot), \cdot).$$

The proof consists of trivial verifications.

4. On the number of perfect codes

In this section we discuss some observations, which result in the best known lower bound on the number of q -ary perfect codes, $q \geq 3$. The basic facts are already contained in other known results: lower bounds on the number of multary quasigroups of order q , the

construction [9] of perfect codes from multary quasigroups of order q , and the possibility to choose the quasigroup independently for every vector of the outer code (this possibility was not explicitly mentioned in [9], but used in the previous paper [8]).

A general lower bound, in terms of the number of multary quasigroups, is given by Lemma 3. In combination with Lemma 4, it gives explicit numbers.

Lemma 3. *The number of q -ary perfect codes of length n is not less than*

$$Q\left(\frac{n-1}{q}, q\right)^{R_{\frac{n-1}{q}}}$$

where $Q(m, q)$ is the number of m -ary quasigroups of order q and where $R_{n'} = q^{n'}/(n'q - q + 1)$ is the cardinality of a perfect code of length n' .

Proof. Constructing a perfect code like in Theorem 2 with $t = \frac{n-1}{q}$, we combine $R_{\frac{n-1}{q}}$ different $\bar{\mu}$ -components.

Constructing every such a component as in Lemma 2, $k = 1$, $t = \frac{n-1}{q}$, we are free to choose the t -ary quasigroup Q of order q in $Q(t, q)$ ways. Clearly, different t -ary quasigroups give different components. (Equivalently, we can use Lemma 1 and choose the $(t+1)$ -ary quasigroup H_1 , but should note that the value of H_1 in the construction is always fixed when $k = 1$, because $C^\#$ consists of only one vertex; so we again have $Q(t, q)$ different choices, not $Q(t+1, q)$). \triangle

Lemma 4. *The number $Q(m, q)$ of m -ary quasigroups of order q satisfies:*

- (a) [5] $Q(m, 3) = 3 \cdot 2^m$;
- (b) [11] $Q(m, 4) = 3^{m+1} \cdot 2^{2^{m+1}}(1 + o(1))$;
- (c) [4] $Q(m, 5) \geq 2^{3^{n/3-0.072}}$;
- (d) [10] $Q(m, q) \geq 2^{((q^2-4q+3)/4)^{n/2}}$ for odd q (the previous bound [4] was $Q(m, q) \geq 2^{\lfloor q/3 \rfloor^n}$);
- (e) [4] $Q(m, q_1 q_2) \geq Q(m, q_1) \cdot Q(m, q_2)^{q_1^m}$.

For odd $q \geq 5$, the number of codes given by Lemmas 3 and 4(c,d) improves the constant c in the lower estimation of form $e^{c n^{1+o(1)}}$ for the number of perfect codes, in comparison with the last known lower bound [6]. Informally, this can be explained in the following way: the construction in [6] can be described in terms of mutually independent small modifications of the linear multary quasigroup of order q , while the lower bounds in Lemma 4(c,d) are based on a specially-constructed nonlinear multary quasigroup that allows a larger number of independent modifications. For $q = 3$ and $q = 2^s$, the number of codes given by Lemmas 3 and 4(a,b,e) also slightly improves the bound in [6], but do not affect on the constant c .

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