Conway's conjecture for monotone thrackles

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Abstract

A drawing of a graph in the plane is called a *thrackle* if every pair of edges meet precisely once, either at a common vertex or at a proper crossing. According to Conway's conjecture, every thrackle has at most as many edges as vertices. We prove this conjecture for *x-monotone* thrackles, that is, in the case when every edge meets every vertical line in at most one point.

1 Introduction

A *drawing* of a graph is a representation of the graph in the plane such that the vertices are represented by distinct points and the edges by (possibly crossing) simple continuous curves connecting the corresponding point pairs and not passing through any other point representing a vertex. If it leads to no confusion, we make no notational distinction between a drawing and the underlying abstract graph G. In the same vein, $V(G)$ and $E(G)$ will stand for the vertex set and edge set of G as well as for the sets of points and curves representing them.

A drawing of G is called a *thrackle* if every pair of edges meet precisely once, either at a common vertex or at a proper crossing. (A crossing p of two curves is *proper* if at p one curve passes from one side of the other curve to its other side. Two edges that share an endpoint cannot have any other point in common.)

More than *forty* years ago, Conway [1], [6], [13] conjectured that every thrackle has at most as many edges as vertices, and offered a bottle of beer for a solution. (The prize has since risen to a thousand dollars.) In spite of considerable efforts, Conway's thrackle conjecture is still open. If true, Conway's conjecture would be tight as any cycle of length at least *five* can be drawn as a thrackle, see [15]. Two thrackle drawings of C_5 and C_6 are shown in Figure 1. According to legend, Conway named these drawings after a peculiar term he heard from a fisherman during his holiday in Scotland: the man described his tangled fishing line as "thrackled."

The first linear upper bound for the number of edges of a thrackle of n vertices, $2n$, was established by Lovász, Pach, and Szegedy [10]. Cairns and Nikolayevsky improved this bound to $\frac{3}{2}n$ [2]. Presently, the best known bound, $\frac{167}{117}n < 1.428n$, is due to Fulek and Pach [7]. For related results, see [3], [4], [8], [11], [12].

The origins of the thrackle problem go back to the nineteen-thirties, when Hopf and Pannwitz [9] posed the following problem in the *Jahresbericht der Deutschen Mathematischen Vereinigung*. The *diameter* of a finite point set is the maximum distance between its elements. Let P be a set

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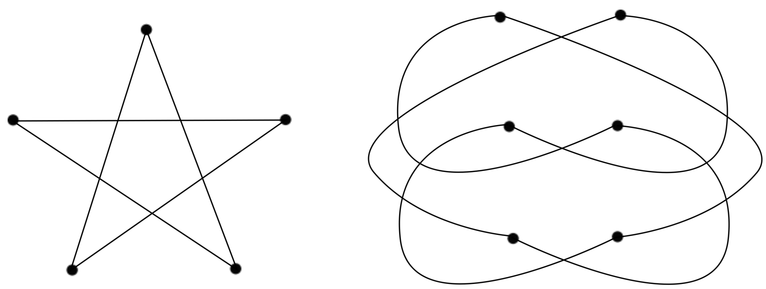

Figure 1: C_5 and C_6 drawn as thrackles

of n points in the plane with diameter *one*. Prove that the maximum number of point pairs in P, which determine distance *one* is n. Several solutions were submitted [14]. Much later, Erdős noticed that the statement follows from a more general result: Every graph drawn in the plane with straight-line edges so that each pair of edges either share an endpoint or properly cross has at most as many edges as vertices. Indeed, it is easy to verify that joining two points of a unit diameter set in the plane by a segment if they are at distance one, the resulting drawing satisfies the above condition. Using Conway's terminology, we obtain that Conway's conjecture is true for *straight-line thrackles*, that is, for thrackles drawn by straight-line edges on a set of vertices, no *three* of which are collinear. (We make the latter assumption to exclude that edges share a whole segment.)

Theorem 1. (Erdős) *Every straight-line thrackle has at most as many edges as vertices.*

The most elegant proof of Theorem 1 is due to Perles. Let G be a straight-line thrackle. We call a vertex $v \in V(G)$ *pointed* if all edges incident to v lie in a half-plane bounded by a line passing through v. (See Figure 2, part a.) For each pointed vertex v, delete from G the "leftmost" edge incident to v, that is, the first element in the clockwise order of edges around the half-plane at v. Notice that we deleted all edges, which proves the theorem. Indeed, if we are left with an edge $uv \in E(G)$, then it was not leftmost at u, nor at v. Thus, originally G contained two edges, uv' and $u'v$, with $\angle v'uv < \pi$ and $\angle u'vu < \pi$. These two edges must lie on opposite sides of the line uv. Hence, they are disjoint, contradicting the thrackle condition. See Figure 2, part b.

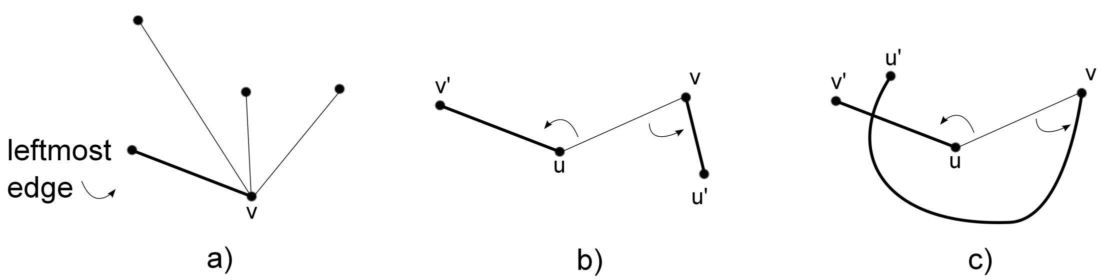

Figure 2: (a) a pointed vertex; (b) Perles' argument; and (3) why it breaks down for x-monotone thrackles.

The above proof applies *verbatim* to another special class of thrackles, for which it makes sense to speak of "leftmost" edges. A thrackle is called *outerplanar* if its vertices lie on a circle and its edges are represented by continuous curves running in the interior of this circle [5].

Theorem 2. *Every outerplanar thrackle has at most as many edges as vertices.*

Cairns and Nikolayevsky [5] have recently established the stronger statement that every outerplanar thrackle which has no vertex of degree at most *one*is an odd cycle. Woodall [15] characterized all thrackles, assuming that Conway's conjecture is true.

The aim of this note is to verify Conway's conjecture in another special case. We call a thrackle *x-monotone* if each curve representing an edge meets every vertical line in at most one point. In particular, every straight-line thrackle with no vertical edge is x-monotone.

Theorem 3. *Every x-monotone thrackle has at most as many edges as vertices.*

Theorem 3 is a generalization of Theorem 1. However, the argument of Perles fails in this case, because the edges uv' and $u'v$ may cross (see Figure 2, part c). Instead, we can explore the fact that there is a natural partial order on the edges.

2 Proof of Theorem 3

Let G be an x-monotone thrackle with n vertices and e edges. In the sequel, assume without loss of generality that G is in "general position," that is, no two vertices have the same x-coordinate and no three edges pass through the same point. The orthogonal projections of the edges to the x-axis are *closed* segments with the property that any *two* of them have at least one point in common. By Helly's theorem in dimension 1, we obtain that all of these segments share a point. Hence, there is a vertical line ℓ that meets all edges of G.

We distinguish two cases.

Case A: *The line ℓ does not pass through any vertex of G.*

In this case, G is a bipartite graph: all of its edges connect a point in the left half-plane bounded by ℓ to a point in the right half-plane. We show that the number of edges of G is strictly smaller than n. Indeed, otherwise G would contain a cycle of *even* length, contradicting the following lemma.

Lemma. *No even cycle can be drawn as an x-monotone thrackle.*

Proof. Suppose for a contradiction that there exists a drawing of an even cycle $C = v_0v_1 \dots v_{k-1}$ which can be drawn as a thrackle. (k is even and the indices are taken modulo k).

First of all, notice that we can assume without loss of generality that C meets the requirements of Case A: none of its vertices lies on ℓ. Indeed, if there existed such a vertex v_i, the two edges of C meeting at v_i would lie in the same half-plane bounded by ℓ, otherwise, by the evenness of k, one of them would be disjoint from another edge of C that lies entirely in the opposite open half-plane. If both $v_{i-1}v_i$ and v_iv_{i+1} lie in the same half-plane, then slightly translating ℓ we can bring it in a position where no vertex lies on it and ℓ (strictly) crosses every edge of C.

We say that the edge $v_{i-1}v_i$ lies *below* the edge v_iv_{i+1} if the intersection point of $v_{i-1}v_i$ with the line ℓ is below the intersection point of v_iv_{i+1} with ℓ. Notice that, by the definition of a thrackle, these two points cannot coincide, since the interiors of any two adjacent edges must be disjoint. If an edge lies below another edge adjacent to it, then we say that the latter edge lies *above* the first one.

Observe that if the edge $v_{i-1}v_i$ lies below v_iv_{i+1}, then the next edge along C, $v_{i+1}v_{i+2}$, must also lie below v_iv_{i+1}, otherwise $v_{i-1}v_i$ and $v_{i+1}v_{i+2}$ could not cross. In this case, we say that v_iv_{i+1}

is an *upper edge*. Otherwise, both $v_{i-1}v_i$ and $v_{i+1}v_{i+2}$ must lie above v_iv_{i+1}, and v_iv_{i+1} is called a *lower edge*. Obviously, the edges of C are alternately upper and lower edges. See Figure 3.

Consider now the cycle C as a closed self-intersecting curve embedded in the plane, which divides the plane into simply connected regions. Exactly one of these regions is unbounded. It is well known and easy to prove that one can color these regions with two colors, white and grey, say, such that no two regions that share a boundary arc are of the same color. (By our assumption of general position, each point at which C intersects itself belongs to the boundary of precisely *four* regions, two of which are white and the other two grey.)

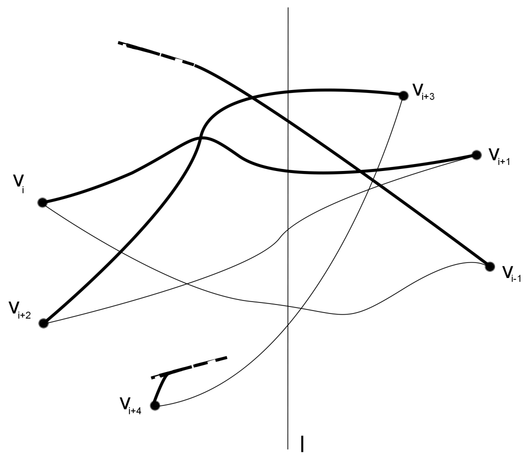

Figure 3: An even cycle. The upper edges are marked.

Let v_a and v_b be the leftmost and rightmost vertices of C, respectively. Assume, by symmetry, that $0 < a < b < k$. Since v_a and v_b are on different sides of ℓ and every edge of C crosses ℓ, we have that $b - a$ is *odd*. Assume, again by symmetry, that v_av_{a+1} is an upper edge. Since every upper edge is followed by a lower edge and *vice versa*, we obtain that $v_{b-1}v_b$ must also be an upper edge.

Let us trace now the curve $P = v_av_{a+1} \dots v_{b-1}v_b$ from left to right, and record the color changes along the right-hand side of P. Every time we arrive at a crossing where C intersects itself, the color changes. How many crossings are there along P? Each edge of C (including the edges of P) crosses $k - 3$ other edges, because each of them must cross every other edge except itself and the two adjacent edges. Since k is even, $k - 3$ is odd. We have seen that $b - a$ is odd, so that P consists of an odd number of edges, each of which is crossed by an odd number of other edges. Therefore, the total number of crossings along P must be *odd*. (Note that every crossing where P crosses itself is counted twice!) This implies that the color of the region lying on the right-hand side of the initial portion of P is different from the color on the right-hand side of its final portion. Since v_av_{a+1} and $v_{b-1}v_b$ are *upper* edges, this means that the regions directly *below* the initial and final portions of P are of *different* colors.

On the other hand, the points above v_a and the points above v_b belong to the (unique) unbounded region, and all of them are colored with the color of that region. Hence, the points above a small initial portion of P and the points above a small final portion of P are of the same color. This yields that the regions directly below the initial and final portions of P must be of the *same* color. This contradiction completes the proof of the lemma. $\square$

Case B: *The line ℓ passes through a vertex of $v \in V(G)$.*

In this case, replace v by two vertices, v' and v'', very close to each other and to the original position of v. Let v' lie in left half-plane, and let it be the new left endpoint of all edges that previously had v as their left endpoints. Analogously, let us redirect all edges that previously had v as their right endpoints to the point v'', which lies in the right half-plane. We can make sure that after this slight perturbation, every edge incident to v' will cross all edges incident to v'', and the resulting drawing G' remains an x-monotone thrackle with $n' = n + 1$ vertices and $e' = e$ edges.

Notice that all edges of G' meet ℓ and none of them lies on ℓ; that is, the conditions of Case A are satisfied. However, in Case A, we have shown that the number of edges is *strictly* smaller than the number of vertices. Thus, we have $e' < n'$, so that $e < n + 1$, and the proof of Theorem 3 is complete.

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References

[1] P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer-Verlag, New York, 2005.

[2] G. Cairns and Y. Nikolayevsky, Bounds for generalized thrackles, *Discrete Comput. Geom.* **23** (2000), 191–206.

[3] G. Cairns, M. McIntyre, and Y. Nikolayevsky, The thrackle conjecture for K_5 and $K_{3,3}$. In: *Towards a theory of Geometric Graphs, Contemp. Math.* **342**, Amer. Math. Soc., Providence, RI, 2004, 35–54.

[4] G. Cairns and Y. Nikolayevsky, Generalized thrackle drawings of non-bipartite graphs, *Discrete Comput. Geom.* **41** (2009), 119–134.

[5] G. Cairns and Y. Nikolayevsky, Outerplanar thrackles, manuscript, 2009.

[6] Unsolved problems. Chairman: P. Erdős, in: *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, Inst. Math. Appl., Southend-on-Sea, 1972, 351–363.

[7] R. Fulek and J. Pach, A computational approach to Conway's thrackle conjecture, submitted.

[8] J. E. Green and R. D. Ringeisen, Combinatorial drawings and thrackle surfaces. In: *Graph Theory, Combinatorics, and Algorithms, Vol. 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., Wiley, New York, 1995, 999–1009.

[9] H. Hopf and E. Pannwitz, Aufgabe Nr. 167, *Jahresbericht Deutsch. Math.-Verein.* **43** (1934), 114.

[10] L. Lovász, J. Pach, and M. Szegedy, On Conway's thrackle conjecture, *Discrete Comput. Geom.* **18** (1998), 369–376.

[11] A. Perlstein and R. Pinchasi, Generalized thrackles and geometric graphs in $\mathbb{R}^3$ with no pair of strongly avoiding edges, *Graphs Combin.* **24** (2008), 373–389.

[12] B. L. Piazza, R. D. Ringeisen, and S. K. Stueckle, Subthrackleable graphs and four cycles. In: *Graph theory and Applications (Hakone, 1990), Discrete Math.* **127** (1994), 265–276.

[13] R. D. Ringeisen, Two old extremal graph drawing conjectures: progress and perspectives, *Congressus Numerantium* **115** (1996), 91–103.

[14] J. W. Sutherland, Lösung der Aufgabe 167, *Jahresbericht Deutsch. Math.-Verein.* **45** (1935), 33–35.

[15] D. R. Woodall, Thrackles and deadlock, in: Combinatorial Mathematics and Its Applications (Welsh, D. J. A., ed.), Academic Press, 1969, 335-348.