# COSET INTERSECTION GRAPHS FOR GROUPS 

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Abstract. Let $H, K$ be subgroups of $G$. We investigate the intersection properties of left and right cosets of these subgroups.

If $H$ and $K$ are subgroups of $G$, then $G$ can be partitioned as the disjoint union of all left cosets of $H$, as well as the disjoint union of all right cosets of $K$. But how do these two partitions of $G$ intersect each other?

Definition 1. Let $G$ be a group, and $H$ a subgroup of $G$. A left transversal for $H$ in $G$ is a set $\left\{t_{\alpha}\right\}_{\alpha \in I} \subseteq G$ such that for each left coset $g H$ there is precisely one $\alpha \in I$ satisfying $t_{\alpha} H=g H$. A right transversal for $H$ in $G$ in defined in an analogous fashion. A left-right transversal for $H$ is a set $S$ which is simultaneously a left transversal, and a right transversal, for $H$ in $G$.

A useful tool for studying the way left and right cosets interact, and obtaining transversals, is the coset intersection graph which we introduce here.

Definition 2. Let $G$ be a group and $H, K$ subgroups of $G$. We define the coset intersection graph $\Gamma_{H, K}^{G}$ to be a graph with vertex set consisting of all left cosets of $H\left(\left\{l_{i} H\right\}_{i \in I}\right)$ together with all right cosets of $K\left(\left\{K r_{j}\right\}_{j \in J}\right)$, where $I, J$ are index sets. If a left coset of $H$ and right coset of $K$ correspond, they are still included twice. Edges (undirected) are included whenever any two of these cosets intersect, and the edge $a H-K b$ corresponds to the set $a H \cap K b$.

Observing that left (respectively, right) cosets do not intersect, we see that $\Gamma_{H, K}^{G}$ is a bipartite graph, split between $\left\{l_{i} H\right\}_{i \in I}$ and $\left\{K r_{j}\right\}_{j \in J}$.

For $H$ a finite index subgroup of $G$, the existence of a left-right transversal is well known, sometimes presented as an application of Hall's marriage theorem [3]. When $G$ is finite $H$ will have size $n$, so any set of $k$ left cosets of $H$ intersects at least $k$ right cosets of $H$ (or their union would have size $<k n$ ). Hence by Hall's theorem there is a matching on the bipartite graph $\Gamma_{H, H}^{G}$, and thus a leftright transversal (take one element from each edge in this matching). When $G$ is infinite the same argument applies to the finite quotient $G / \operatorname{core}(H)$ (The core of $H$, core $(H)$, is the intersection of all conjugates of $H$ in $G, \bigcap_{g \in G} g^{-1} H g$; it is always normal, and will be of finite index in $G$ whenever $H$ is).

The purpose of this paper is to show that in fact a much stronger result is true: we can completely describe the way that left and right cosets of $H$ intersect, without any need for Hall's theorem, but instead by studying and applying the properties of the coset intersection graph. We begin this now.
Theorem 3. $\Gamma_{H, K}^{G}$ is always a disjoint union of complete bipartite graphs.

Proof. We first show that for $a, b, c, d \in G$ if $a H-K b-c H-K d$ is a path in $\Gamma_{H, K}^{G}$ then there is an edge $a H-K d$. Note that there exist $h_{1}, h_{2}, h_{3} \in H$ and $k_{1}, k_{2}, k_{3} \in K$ such that $a h_{1}=k_{1} b, k_{2} b=c h_{2}, c h_{3}=k_{3} d$. Re-arranging gives $c=k_{3} d h_{3}^{-1}$, so $b=k_{2}^{-1} k_{3} d h_{3}^{-1} h_{2}$, so $a=k_{1} k_{2}^{-1} k_{3} d h_{3}^{-1} h_{2} h_{1}^{-1}$ and thus $a h_{1} h_{2}^{-1} h_{3}=k_{1} k_{2}^{-1} k_{3} d$. Hence $a H-K d$ as required.
Now take any $l_{i} H$, and some $K r_{j}$ in the connected component of $l_{i} H$ in $\Gamma_{H, K}^{G}$ (there is at least one such $K r_{j}$ ); we show $l_{i} H$ and $K r_{j}$ are connected by an edge. For if not, then there must be at least one finite path connecting them; take a minimal such path $\gamma$ from $l_{i} H$ to $K r_{j}$. Then $\gamma$ begins with $l_{i} H-K a-$ $b H-K c-\ldots$, where $K a \neq K r_{j}$. But by the previous remark, $l_{i} H$ and $K c$ must be joined by an edge, contradicting the minimality of $\gamma$. So $l_{i} H$ and $K r_{j}$ are joined by an edge, for every $K r_{j}$ in the connected component of $l_{i} H$.

Recall that $\mathbf{K}_{s, t}$ denotes the complete bipartite graph on ( $s, t$ ) vertices. By imposing finiteness conditions on subgroups (finite index, or finite size), the graph $\Gamma_{H, K}^{G}$ exhibits an even greater level of symmetry.

Theorem 4. Let $H, K<G$. Suppose that either $|H|=m,|K|=n$ (where both subgroups are finite), or $|G: H|=n,|G: K|=m$ (where both subgroups have finite index). Then the graph $\Gamma_{H, K}^{G}$ is a collection of disjoint, finite, complete bipartite graphs, where each component is of the form $\boldsymbol{K}_{s_{i}, t_{i}}$ with $s_{i} / t_{i}=n / m$.
Proof. Case 1: $|H|=m,|K|=n$. Take a connected component of $\Gamma_{H, K}^{G}$, which from theorem 3 must look like $\mathbf{K}_{s, t}$ (as $|H|,|K|$ are finite) with vertices given by $s$ left cosets of $H$ and $t$ right cosets of $K$. Thus, in $G$, the disjoint union of these $s$ left cosets must be set-wise equal to the disjoint union of these $t$ right cosets. So $s|H|=t|K|$, and hence $s / t=n / m$.
Case 2: $|G: H|=n,|G: K|=m$. Take core $(H \cap K)$, which must be finite index in $G$ (say $|G: \operatorname{core}(H \cap K)|=l$ ), as $H, K$ and hence $H \cap K$ are. Now form the quotient $G / \operatorname{core}(H \cap K)$. Set $H^{\prime}:=H / \operatorname{core}(H \cap K), K^{\prime}:=K / \operatorname{core}(H \cap K)$. Since $|G: \operatorname{core}(H \cap K)|=|G: H| \cdot|H: \operatorname{core}(H \cap K)|$, we have that $\left|H^{\prime}\right|=l / n$. Similarly, $\left|K^{\prime}\right|=l / m$. Now apply case 1 to $G / \operatorname{core}(H \cap K), H^{\prime}, K^{\prime}$.

Under the hypotheses of the above theorem, we see that sets of $s_{i}$ left cosets of $H$ completely intersect sets of $t_{i}$ right cosets of $K$, with $s_{i} / t_{i}$ constant over i. By drawing left cosets of $H$ as columns, and right cosets of $K$ as rows, we partition $G$ into irregular 'chessboards' (denoted $C_{i}$ ) each with edge ratio $n: m$. Each chessboard $C_{i}$ corresponds to the connected component $\mathbf{K}_{s_{i}, t_{i}}$ of $\Gamma_{H, K}^{G}$, and individual tiles in $C_{i}$ correspond to the nonempty intersection of a left coset of $H$ and a right coset of $K$ (i.e., edges in $\mathbf{K}_{s_{i}, t_{i}}$ ). By choosing one element from each tile on a leading diagonal of the $C_{i}$ 's (equivalently, one element from each edge in a maximum matching of the $\mathbf{K}_{s_{i}, t_{i}}$ 's), we deduce a stronger version of Hall's theorem for transversals:

Corollary 5. Let $H, K<G$ be of finite index, with $|G: H|=m$ and $|G: K|=$ $n$, where $m \leq n$. Then there exists a set $T \subseteq G$ which is a left transversal for $H$ in $G$, and which can be extended to a right transversal for $K$ in $G$. If $H=K$ in $G$, then $T$ becomes a left-right transversal for $H$.

We now compute the sizes of the complete bipartite components of $\Gamma_{H, K}^{G}$.
Proposition 6. Let $H, K<G$ and $g \in G$. Then the number of right cosets of $K$ intersecting $g H$ (call this $M_{g}$ ) satisfies:

1. $M_{g}=\frac{\left|G: g H g^{-1} \cap K\right|}{|G: H|}$ if $|G: H|,|G: K|$ are both finite.
2. $M_{g}=\frac{|H|}{\left|g H g^{-1} \cap K\right|}$ if $|H|,|K|$ are both finite.

A symmetric result applies for the number of left cosets of $H$ intersecting $K g$.
Proof. Let $N:=\operatorname{core}(H \cap K)$. We show that if $g H \cap K a \neq \emptyset$ for some $a \in G$, then the number of cosets of $N$ in $g H \cap K a$ is the same as the number in $g H g^{-1} \cap K$, independent of $a$ (in each of case 1 or 2 this number will be finite). So, as $g H \cap K a \neq \emptyset$, we must have $g h=k a$ for some $h \in H, k \in K$. As $N$ is normal, we have the the number of cosets of $N$ in $g H \cap K a$ is the same as the number in $g H a^{-1} \cap K=g H h^{-1} g^{-1} k \cap K=g H g^{-1} k \cap K$, which is the same as the number in $g \mathrm{Hg}^{-1} \cap K k^{-1}=g H g^{-1} \cap K$ (observe that this number will be $\left.\left|g \mathrm{Hg}^{-1} \cap K: N\right|\right)$. This immediately gives:
(number of cosets of $N$ in $g H$ ) $=\left(\right.$ number of cosets of $N$ in $\left.g H^{-1} \cap K\right) \cdot M_{g}$ Thus $M_{g}=\frac{|H: N|}{\left|g H g^{-1} \cap K: N\right|}$. Both cases of the proposition now follow.

All of our results can be derived from the work of Ore [5], who makes use of double cosets; partitions of $G$ into sets of the form KgH (where $H, K<G$ ). It follows that the complete bipartite components of $\Gamma_{H, K}^{G}$ from theorem 3 (the 'chessboards') correspond to the double cosets of $G$; a left coset $a H$ and a right coset $K b$ intersect if and only if they lie in the same double coset $K x H$. The symmetry exhibited by the coset intersection graph is not immediately obvious from Ore's use of terminology, and our exposition is more direct.

A Historical Remark (with contributions from Warren Dicks and Jack Schmidt). The results in this paper have a somewhat piecemeal historical origin. A weaker version of corollary 囵, that a subgroup of a finite group always has a left-right transversal, appeared in 1910 by Miller [4]. In 1913 Chapman [1] proved the same result; he then realised the existence of the proof by Miller and in 1914 issued a corrigendum [2]. In 1927 Scorza [6] proved corollary 5 for two separate subgroups $H, K$ but still taking $G$ to be finite (the first time such a proof used double cosets). By the time of Zassenhaus' text [8] in 1937, corollary 5 was known for finite index subgroups of infinite groups (the first time such a proof used Hall's theorem). In 1941 Shü [7] addressed this problem, in a way that leaves us somewhat confused. In 1958 Ore [5] expanded significantly on such ideas, and gives what is to-date the most complete treatment of these, as well as his own historical account.

## References

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