COSET INTERSECTION GRAPHS FOR GROUPS

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ABSTRACT. Let H, K be subgroups of G. We investigate the intersection properties of left and right cosets of these subgroups.

If H and K are subgroups of G, then G can be partitioned as the disjoint union of all left cosets of H, as well as the disjoint union of all right cosets of K. But how do these two partitions of G intersect each other?

Definition 1. Let G be a group, and H a subgroup of G. A left transversal for H in G is a set $\{t_{\alpha}\}_{\alpha \in I} \subseteq G$ such that for each left coset gH there is precisely one $\alpha \in I$ satisfying $t_{\alpha}H = gH$. A right transversal for H in G in defined in an analogous fashion. A left-right transversal for H is a set S which is simultaneously a left transversal, and a right transversal, for H in G.

A useful tool for studying the way left and right cosets interact, and obtaining transversals, is the coset intersection graph which we introduce here.

Definition 2. Let G be a group and H, K subgroups of G. We define the coset intersection graph $\Gamma_{H,K}^G$ to be a graph with vertex set consisting of all left cosets of H ($\{l_iH\}_{i\in I}$) together with all right cosets of K ($\{Kr_j\}_{j\in J}$), where I, J are index sets. If a left coset of H and right coset of K correspond, they are still included twice. Edges (undirected) are included whenever any two of these cosets intersect, and the edge aH - Kb corresponds to the set $aH \cap Kb$.

Observing that left (respectively, right) cosets do not intersect, we see that $\Gamma_{H,K}^G$ is a bipartite graph, split between $\{l_iH\}_{i\in I}$ and $\{Kr_j\}_{j\in J}$.

For H a finite index subgroup of G, the existence of a left-right transversal is well known, sometimes presented as an application of Hall's marriage theorem [3]. When G is finite H will have size n, so any set of k left cosets of H intersects at least k right cosets of H (or their union would have size < kn). Hence by Hall's theorem there is a matching on the bipartite graph $\Gamma_{H,H}^G$, and thus a leftright transversal (take one element from each edge in this matching). When G is infinite the same argument applies to the finite quotient $G/\operatorname{core}(H)$ (The core of H, $\operatorname{core}(H)$, is the intersection of all conjugates of H in G, $\bigcap_{g \in G} g^{-1}Hg$; it is always normal, and will be of finite index in G whenever H is).

The purpose of this paper is to show that in fact a much stronger result is true: we can completely describe the way that left and right cosets of Hintersect, without any need for Hall's theorem, but instead by studying and applying the properties of the coset intersection graph. We begin this now.

Theorem 3. $\Gamma_{H,K}^G$ is always a disjoint union of complete bipartite graphs.

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Proof. We first show that for $a, b, c, d \in G$ if aH - Kb - cH - Kd is a path in $\Gamma_{H,K}^G$ then there is an edge aH - Kd. Note that there exist $h_1, h_2, h_3 \in H$ and $k_1, k_2, k_3 \in K$ such that $ah_1 = k_1b, k_2b = ch_2, ch_3 = k_3d$. Re-arranging gives $c = k_3dh_3^{-1}$, so $b = k_2^{-1}k_3dh_3^{-1}h_2$, so $a = k_1k_2^{-1}k_3dh_3^{-1}h_2h_1^{-1}$ and thus $ah_1h_2^{-1}h_3 = k_1k_2^{-1}k_3d$. Hence aH - Kd as required.

Now take any l_iH , and some Kr_j in the connected component of l_iH in $\Gamma_{H,K}^G$ (there is at least one such Kr_j); we show l_iH and Kr_j are connected by an edge. For if not, then there must be at least one finite path connecting them; take a minimal such path γ from l_iH to Kr_j . Then γ begins with $l_iH - Ka - bH - Kc - \ldots$, where $Ka \neq Kr_j$. But by the previous remark, l_iH and Kcmust be joined by an edge, contradicting the minimality of γ . So l_iH and Kr_j are joined by an edge, for every Kr_j in the connected component of l_iH . \Box

Recall that $\mathbf{K}_{s,t}$ denotes the complete bipartite graph on (s,t) vertices. By imposing finiteness conditions on subgroups (finite index, or finite size), the graph $\Gamma_{H,K}^{G}$ exhibits an even greater level of symmetry.

Theorem 4. Let H, K < G. Suppose that either |H| = m, |K| = n (where both subgroups are finite), or |G:H| = n, |G:K| = m (where both subgroups have finite index). Then the graph $\Gamma_{H,K}^G$ is a collection of disjoint, finite, complete bipartite graphs, where each component is of the form \mathbf{K}_{s_i,t_i} with $s_i/t_i = n/m$.

Proof. Case 1: |H| = m, |K| = n. Take a connected component of $\Gamma_{H,K}^G$, which from theorem 3 must look like $\mathbf{K}_{s,t}$ (as |H|, |K| are finite) with vertices given by s left cosets of H and t right cosets of K. Thus, in G, the disjoint union of these s left cosets must be set-wise equal to the disjoint union of these t right cosets. So s|H| = t|K|, and hence s/t = n/m.

Case 2: |G:H| = n, |G:K| = m. Take $\operatorname{core}(H \cap K)$, which must be finite index in G (say $|G:\operatorname{core}(H \cap K)| = l$), as H, K and hence $H \cap K$ are. Now form the quotient $G/\operatorname{core}(H \cap K)$. Set $H' := H/\operatorname{core}(H \cap K)$, $K' := K/\operatorname{core}(H \cap K)$. Since $|G:\operatorname{core}(H \cap K)| = |G:H| \cdot |H:\operatorname{core}(H \cap K)|$, we have that |H'| = l/n. Similarly, |K'| = l/m. Now apply case 1 to $G/\operatorname{core}(H \cap K)$, H', K'. \Box

Under the hypotheses of the above theorem, we see that sets of s_i left cosets of H completely intersect sets of t_i right cosets of K, with s_i/t_i constant over i. By drawing left cosets of H as columns, and right cosets of K as rows, we partition G into irregular 'chessboards' (denoted C_i) each with edge ratio n : m. Each chessboard C_i corresponds to the connected component \mathbf{K}_{s_i,t_i} of $\Gamma_{H,K}^G$, and individual tiles in C_i correspond to the nonempty intersection of a left coset of H and a right coset of K (i.e., edges in \mathbf{K}_{s_i,t_i}). By choosing one element from each tile on a leading diagonal of the C_i 's (equivalently, one element from each edge in a maximum matching of the \mathbf{K}_{s_i,t_i} 's), we deduce a stronger version of Hall's theorem for transversals:

Corollary 5. Let H, K < G be of finite index, with |G : H| = m and |G : K| = n, where $m \le n$. Then there exists a set $T \subseteq G$ which is a left transversal for H in G, and which can be extended to a right transversal for K in G. If H = K in G, then T becomes a left-right transversal for H.

We now compute the sizes of the complete bipartite components of $\Gamma^G_{H,K}$.

Proposition 6. Let H, K < G and $g \in G$. Then the number of right cosets of K intersecting gH (call this M_a) satisfies:

- 1. $M_g = \frac{|G:gHg^{-1} \cap K|}{|G:H|}$ if |G:H|, |G:K| are both finite. 2. $M_g = \frac{|H|}{|gHg^{-1} \cap K|}$ if |H|, |K| are both finite.

A symmetric result applies for the number of left cosets of H intersecting Kg.

Proof. Let $N := \operatorname{core}(H \cap K)$. We show that if $qH \cap Ka \neq \emptyset$ for some $a \in G$, then the number of cosets of N in $gH \cap Ka$ is the same as the number in $qHq^{-1} \cap K$, independent of a (in each of case 1 or 2 this number will be finite). So, as $gH \cap Ka \neq \emptyset$, we must have gh = ka for some $h \in H$, $k \in K$. As N is normal, we have the number of cosets of N in $gH \cap Ka$ is the same as the number in $gHa^{-1} \cap K = gHh^{-1}g^{-1}k \cap K = gHg^{-1}k \cap K$, which is the same as the number in $gHg^{-1} \cap Kk^{-1} = gHg^{-1} \cap K$ (observe that this number will be $|gHg^{-1} \cap K:N|$). This immediately gives:

(number of cosets of N in gH) = (number of cosets of N in $gHg^{-1} \cap K) \cdot M_g$ Thus $M_g = \frac{|H:N|}{|gHg^{-1} \cap K:N|}$. Both cases of the proposition now follow.

All of our results can be derived from the work of Ore [5], who makes use of double cosets; partitions of G into sets of the form KgH (where H, K < G). It follows that the complete bipartite components of $\Gamma^G_{H,K}$ from theorem 3 (the 'chessboards') correspond to the double cosets of G; a left coset aH and a right coset Kb intersect if and only if they lie in the same double coset KxH. The symmetry exhibited by the coset intersection graph is not immediately obvious from Ore's use of terminology, and our exposition is more direct.

A Historical Remark (with contributions from Warren Dicks and Jack Schmidt). The results in this paper have a somewhat piecemeal historical origin. A weaker version of corollary 5, that a subgroup of a finite group always has a left-right transversal, appeared in 1910 by Miller [4]. In 1913 Chapman [1] proved the same result; he then realised the existence of the proof by Miller and in 1914 issued a corrigendum [2]. In 1927 Scorza [6] proved corollary 5 for two separate subgroups H, K but still taking G to be finite (the first time such a proof used double cosets). By the time of Zassenhaus' text [8] in 1937, corollary 5 was known for finite index subgroups of infinite groups (the first time such a proof used Hall's theorem). In 1941 Shü [7] addressed this problem, in a way that leaves us somewhat confused. In 1958 Ore [5] expanded significantly on such ideas, and gives what is to-date the most complete treatment of these, as well as his own historical account.

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