## EIGENVALUES OF REAL SYMMETRIC MATRICES

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ABSTRACT. We present a proof of the existence of real eigenvalues of real symmetric matrices which does not rely on any limit or compactness arguments, but only uses the notions of "sup", "inf".

Let  $M_n(\mathbb{R})$  denote the set of all real  $n \times n$ -matrices and  $\operatorname{Sym}_n(\mathbb{R})$  be the set of all  $A \in M_n(\mathbb{R})$ that are symmetric. A key ingredient of the "Spectral Theorem" is the existence of a real eigenvalue of a matrix  $A \in \operatorname{Sym}_n(\mathbb{R})$ . In some way, this uses limit or compactness arguments in  $\mathbb{R}^n$  (e.g., [2, Kap. 6, §2], [3]) or the fact that  $\mathbb{C}$  is algebraically closed (e.g., [1, §6.4]). Usually, none of these are available in a first course on linear algebra; in any case, it seems desirable to isolate the bare "analytic" prerequisites of this basic result about matrices. We present here a slight variation of the argument in [2], which refers at only one place to the completeness axiom for  $\mathbb{R}$ .

For  $v, w \in \mathbb{R}^n$  (column vectors) we let  $\langle v, w \rangle := {}^t v \cdot w$  denote the usual scalar product  $({}^t v$  is the transpose of v, that is, a row vector). The Euclidean norm of v is denoted by  $||v|| = \sqrt{\langle v, v \rangle}$ . We define the norm of a matrix  $A = (a_{ij}) \in M_n(\mathbb{R})$  by  $|A|_{\infty} = \max\{|a_{ij}|: 1 \leq i, j \leq n\}$ . All we need to know about these norms is the following inequality:

(†) 
$$||A \cdot v|| \le \sqrt{n^3} |A|_{\infty} ||v||$$
 for all  $v \in \mathbb{R}^n$ 

This easily follows from the inequalities  $|w|_{\infty} \leq ||w|| \leq \sqrt{n}|w|_{\infty}$  and  $|A \cdot w|_{\infty} \leq n|A|_{\infty}|w|_{\infty}$ ; we set  $|w|_{\infty} = \max\{|w_1|, \ldots, |w_n|\}$  for any  $w = {}^t(w_1, \ldots, w_n) \in \mathbb{R}^n$ .

**Remark 1.** Let  $A \in \text{Sym}_n(\mathbb{R})$ . By a finite sequence of row and column operations, A can be transformed by congruence into a diagonal matrix. That is, there is a nonsingular real matrix P and a real diagonal matrix D such that  $A = {}^t P \cdot D \cdot P$ ; e.g., [1, §6.7]. Since positive real numbers have square roots, we can further assume that all non-zero diagonal entries of D are  $\pm 1$ . Now assume that  $A \succeq 0$ , that is,  $\langle v, A \cdot v \rangle \geq 0$  for all  $v \in \mathbb{R}^n$ . (Such a matrix is called positive semidefinite.) Then all non-zero diagonal entries of D must be +1. Consequently, we have the implication:

$$A \succeq 0$$
 and  $\det(A) \neq 0 \Rightarrow A = {}^{t}P \cdot P$  with  $P \in M_{n}(\mathbb{R})$  invertible.

**Remark 2.** Let  $A = (a_{ij}) \in \text{Sym}_n(\mathbb{R})$ . If  $v \in \mathbb{R}^n$  is such that ||v|| = 1, then all components of v have absolute value  $\leq 1$  and so  $|\langle v, A \cdot v \rangle| \leq \sum_{i,j=1}^n |a_{ij}|$ . Hence, the set

$$S(A) := \{ \langle v, A \cdot v \rangle \colon v \in \mathbb{R}^n, \|v\| = 1 \} \subseteq \mathbb{R}$$

is bounded. In particular, this set has a greatest lower bound  $\mu(A) = \inf S(A)$ . We have

$$\langle v, A \cdot v \rangle \ge \mu(A) \|v\|^2$$
 for all  $v \in \mathbb{R}^n$ 

This inequality is clear if v = 0; if  $v \neq 0$ , then set w := v/||v|| and note that  $\langle w, A \cdot w \rangle \geq \mu(A)$ .

By a limit or compactness argument, one can deduce that there exists a vector  $v_0 \in \mathbb{R}^n$  such that  $||v_0|| = 1$  and  $\langle v_0, A \cdot v_0 \rangle = \mu(A)$ . It then follows easily that  $v_0$  is an eigenvector of A with eigenvalue  $\mu(A)$  (see [2, Kap. 6, §2, no. 4]). The proof below avoids this line of reasoning.

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**Theorem 3.** If  $A \in \text{Sym}_n(\mathbb{R})$ , then  $\mu(A)$  is an eigenvalue of A.

Proof. Let  $I_n$  be the identity matrix and set  $B := A - \mu(A)I_n \in \text{Sym}_n(\mathbb{R})$ . If  $\det(B) = 0$ , then  $\mu(A)$  is an eigenvalue of A. So let us now assume that  $\det(B) \neq 0$ . We have  $\langle v, B \cdot v \rangle = \langle v, A \cdot v \rangle - \mu(A) ||v||^2$  for all  $v \in \mathbb{R}^n$ . Remark 2 shows that  $B \succeq 0$  and  $\mu(B) = \inf S(B) = 0$ . Since also  $\det(B) \neq 0$ , we can write  $B = {}^t P \cdot P$ , where  $P \in M_n(\mathbb{R})$  is invertible (see Remark 1).

Now, for any  $v \in \mathbb{R}^n$ , we have  $\langle v, B \cdot v \rangle = {}^t v \cdot B \cdot v = \|P \cdot v\|^2$ . Furthermore, if  $\|v\| = 1$ , then  $1 = \|P^{-1} \cdot (P \cdot v)\| \le \sqrt{n^3} |P^{-1}|_{\infty} \|P \cdot v\|$ , using (†). Thus,  $\langle v, B \cdot v \rangle \ge 1/(n^3 |P^{-1}|_{\infty}^2) > 0$  for all  $v \in \mathbb{R}^n$  such that  $\|v\| = 1$ , contradicting  $\inf S(B) = 0$ .

**Remark 4.** The argument also works for Hermitian matrices  $A \in M_n(\mathbb{C})$ . One just has to use the Hermitian product  $\langle v, w \rangle = {}^t \overline{v} \cdot w$  for  $v, w \in \mathbb{C}^n$ , where the bar denotes complex conjugation. If A is Hermitian, then  $\langle v, A \cdot v \rangle \in \mathbb{R}$  for all  $v \in \mathbb{C}^n$ , so we can define  $\mu(A) = \inf S(A)$  as above.

## References

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