DILATED FLOOR FUNCTIONS THAT COMMUTE

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ABSTRACT. We determine all pairs of real numbers (α, β) such that the dilated floor functions $\lfloor \alpha x \rfloor$ and $\lfloor \beta x \rfloor$ commute under composition, i.e., such that $\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ holds for all real x.

1. INTRODUCTION

The *floor function* $\lfloor x \rfloor$ rounds a real number down to the nearest integer. The *ceiling function* $\lceil x \rceil$, which rounds up to the nearest integer, satisfies

$$\lceil x \rceil = -\lfloor -x \rfloor. \tag{1}$$

These two fundamental operations discretize (or quantize) real numbers in different ways. The names *floor function* and *ceiling function*, along with their notations, were coined in 1962 by Kenneth E. Iverson [5, p. 12], in connection with the programming language *APL*. Graham, Knuth, and Patashnik [4, Chap. 3] note this history and give many interesting properties of these functions.

We study the floor function applied to a linear function $\ell_\alpha(x) = \alpha x$, yielding the *dilated* floor function $f_\alpha(x) = \lfloor \alpha x \rfloor$, where α is a real number. Dilated floor functions arise in constructing digital straight lines, which are "lines" drawn on two-dimensional graphic displays using pixels, and are discussed further below. This note addresses the question: *When do two dilated floor functions commute under composition of functions?* Linear functions always commute under composition and satisfy the identities

$$\ell_\alpha \circ \ell_\beta(x) = \ell_\beta \circ \ell_\alpha(x) = \ell_{\alpha\beta}(x) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

However, discretization generally destroys such commutativity. We have the following.

Theorem 1. *The complete set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that*

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor$$

holds for all $x \in \mathbb{R}$ consists of:

(i) *three continuous families (α, α), $(\alpha, 0)$, $(0, \alpha)$ for all $\alpha \in \mathbb{R}$;*

(ii) *the infinite discrete family*

$$\left\{ (\alpha, \beta) = \left(\frac{1}{m}, \frac{1}{n} \right) : m, n \geq 1 \right\},$$

where m, n are positive integers. (The families overlap when $m = n$.)

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The interesting feature of this classification is the existence of the infinite discrete family (ii) of solutions where commutativity survives. The family (ii) fits together to form an infinite family of pairwise commuting functions $T_m(x) := f_{1/m}(x) = \lfloor \frac{1}{m}x \rfloor$ for integers $m \geq 1$. Moreover, these functions satisfy for all $m, n \geq 1$ the further relations

$$T_m \circ T_n(x) = T_n \circ T_m(x) = T_{mn}(x) \quad \text{for all } x \in \mathbb{R},$$

which are the same relations satisfied by composition of linear functions (2).

One can ask an analogous question for dilated ceiling functions: *When do two dilated ceiling functions commute?* The resulting classification turns out to be identical. To see this, set $g_\alpha(x) := \lceil \alpha x \rceil$. Using the identity (1), we deduce that for any α, β,

$$f_\alpha \circ f_\beta(x) = -g_\alpha \circ g_\beta(-x), \quad \text{for all } x \in \mathbb{R}.$$

Since $x \mapsto -x$ is a bijection of the domain $\mathbb{R}$ to itself, we see that g_α and g_β commute under composition if and only if f_α and f_β commute under composition.

The commuting family (ii) was noted by Cardinal [3, Lemma 6] in a number-theoretic context. He studied certain semigroups of integer matrices, constructed using the floor function, from which he constructed a family of symmetric integer matrices that he related to the Riemann hypothesis. Also from this number-theoretic perspective, symmetry properties of the solutions may be important. Both sets of solutions (i) and (ii) are invariant under exchange (α, β) to (β, α). However:

(1) The set of all continuous solution parameters (i) is invariant under the reflection symmetry taking (α, β) to $(-\alpha, -\beta)$, while the discrete solutions (ii) break this symmetry.
(2) If one restricts to strictly nonzero parameters, then the continuous solution parameters (i) are invariant under the symmetry taking (α, β) to $(\frac{1}{\alpha}, \frac{1}{\beta})$, while the discrete solutions (ii) break this symmetry.

In the next section we prove Theorem 1, and in the final section, we discuss the problem in the general context of digital straight lines.

2. Proof of Theorem 1

Two immediate cases where commutativity holds are $\alpha = 0$ or $\beta = 0$: In these cases, the functions f_α and f_β commute since their composition is the zero function. In what follows, we suppose that $\alpha\beta \neq 0$, and then we reparameterize the problem in terms of inverse parameters $(1/\alpha, 1/\beta)$, which will simplify the resulting formulas.

We prove Theorem 1 by a case analysis that depends on the signs of α and β. The proofs analyze the jump points in the graphs of $f_{1/\alpha} \circ f_{1/\beta}(x)$. We define for real y the *upper level set* at level y:

$$S_{1/\alpha, 1/\beta}(y) := \{x : f_{1/\alpha} \circ f_{1/\beta}(x) \geq y\} = (f_{1/\alpha} \circ f_{1/\beta})^{-1}[y, \infty).$$

The commutativity property asserts the equality $S_{1/\alpha,1/\beta}(n) = S_{1/\beta,1/\alpha}(n)$ of upper level sets for all $n \in \mathbb{Z}$, and the converse holds because the range of $f_{1/\alpha} \circ f_{1/\beta}$ is a subset of $\mathbb{Z}$. The key formulas are identities determining these upper level sets given in Lemmas 1 and 4, leading to formulas characterizing commutativity when $\alpha, \beta > 0$ and $\alpha, \beta < 0$ given in Lemmas 2 and 5, respectively.

Case 1. Both α and β are positive. We begin with a formula for the upper level sets at integer points.

Lemma 1. *For $\alpha, \beta > 0$ and each $n \in \mathbb{Z}$, the upper level set is*

$$S_{1/\alpha,1/\beta}(n) = [\beta\lceil n\alpha\rceil, \infty)\,.$$

Proof. We have the following implications:

$$\begin{aligned}
x \in S_{1/\alpha,1/\beta}(n) &\Leftrightarrow \left\lfloor \frac{1}{\alpha}\left\lfloor \frac{1}{\beta}x\right\rfloor\right\rfloor \geq n && \text{(by definition)}\\
&\Leftrightarrow \frac{1}{\alpha}\left\lfloor \frac{1}{\beta}x\right\rfloor \geq n && \text{(the right side is in } \mathbb{Z})\\
&\Leftrightarrow \left\lfloor \frac{1}{\beta}x\right\rfloor \geq n\alpha && \text{(since } \alpha > 0)\\
&\Leftrightarrow \left\lfloor \frac{1}{\beta}x\right\rfloor \geq \lceil n\alpha\rceil && \text{(the left side is in } \mathbb{Z})\\
&\Leftrightarrow \frac{1}{\beta}x \geq \lceil n\alpha\rceil && \text{(the right side is in } \mathbb{Z})\\
&\Leftrightarrow x \geq \beta\lceil n\alpha\rceil && \text{(since } \beta > 0).
\end{aligned}$$

□

Lemma 2. *For $\alpha, \beta > 0$, the function $f_{1/\alpha}$ commutes with $f_{1/\beta}$ if and only if the equality*

$$\beta\lceil n\alpha\rceil = \alpha\lceil n\beta\rceil \tag{3}$$

holds for all integers $n \in \mathbb{Z}$.

Proof. By Lemma 1, we have $x \in S_{1/\alpha,1/\beta}(n)$ if and only if $x \geq \beta\lceil n\alpha\rceil$. Similarly, $x \in S_{1/\beta,1/\alpha}(n)$ if and only if $x \geq \alpha\lceil n\beta\rceil$, so that commutativity of the functions is equivalent to the desired equality of ceiling functions. □

Lemma 3. *For $\alpha, \beta > 0$, the function $f_{1/\alpha}$ commutes with $f_{1/\beta}$ if and only if either $\alpha = \beta$ or if α and β are both positive integers.*

Proof. If $\alpha = \beta$ then commutativity clearly holds. If α, β are both (positive) integers, then the relation (3) holds for all $n \in \mathbb{Z}$ since the ceiling functions have no effect. Hence, commutativity holds.

The remaining case is that where at least one of α, β is not an integer; without loss of generality, assume α is not an integer. We write $\lceil\alpha\rceil = A \geq 1$, with $A > \alpha$, and $\lceil\beta\rceil = B \geq 1$. We show that commutativity occurs only if $\alpha = \beta$.

Starting from Lemma 2, the relation (3) can be rewritten

$$\frac{\alpha}{\beta} = \frac{\lceil n\alpha\rceil}{\lceil n\beta\rceil}, \tag{4}$$

whenever the term $\lceil n\beta\rceil$ is non-vanishing; here $\lceil n\beta\rceil \geq 1$ holds for $n \geq 1$.

Since $\alpha < A$, there exists a finite $n \geq 2$ such that $\lceil k\alpha\rceil = kA$ for $1 \leq k \leq n-1$, while $\lceil n\alpha\rceil = nA-1$. Now, (4) requires

$$\frac{\alpha}{\beta} = \frac{A}{B} = \frac{\lceil k\alpha\rceil}{\lceil k\beta\rceil} \quad \text{for all } k \geq 1.$$

By induction on $k \geq 1$, this relation implies $\lceil k\beta \rceil = kB$ for $1 \leq k \leq n-1$. It also implies that $\lceil n\beta \rceil = nB$ or $nB - 1$. The relation (4) for $k = n$ becomes

$$\frac{\alpha}{\beta} = \frac{A}{B} = \frac{\lceil n\alpha \rceil}{\lceil n\beta \rceil} = \frac{nA - 1}{\lceil n\beta \rceil},$$

which rules out $\lceil n\beta \rceil = nB$. Thus, $\lceil n\beta \rceil = nB - 1$, and we now have

$$\frac{A}{B} = \frac{nA - 1}{nB - 1}.$$

Clearing denominators yields $nAB - A = nAB - B$, whence $A = B$. Thus, we have $\frac{\alpha}{\beta} = \frac{A}{B} = 1$, so that $\alpha = \beta$ as asserted. □

Case 2. Both α and β are negative. We obtain a criterion which parallels Lemma 2 in the positive case.

Lemma 4. *For $\alpha, \beta < 0$ and each $n \in \mathbb{Z}$, the upper level set is*

$$S_{1/\alpha,1/\beta}(n) = (\beta\lfloor n\alpha \rfloor + \beta, \infty).$$

Proof. We have the following implications:

$$\begin{aligned}
x \in S_{1/\alpha,1/\beta}(n) &\Leftrightarrow \left\lfloor \frac{1}{\alpha} \left\lfloor \frac{1}{\beta} x \right\rfloor \right\rfloor \geq n && \text{(by definition)} \\
&\Leftrightarrow \frac{1}{\alpha} \left\lfloor \frac{1}{\beta} x \right\rfloor \geq n && \text{(the right side is in } \mathbb{Z}) \\
&\Leftrightarrow \left\lfloor \frac{1}{\beta} x \right\rfloor \leq n\alpha && \text{(since } \alpha < 0) \\
&\Leftrightarrow \left\lfloor \frac{1}{\beta} x \right\rfloor \leq \lfloor n\alpha \rfloor && \text{(the left side is in } \mathbb{Z}) \\
&\Leftrightarrow \frac{1}{\beta} x < \lfloor n\alpha \rfloor + 1 && \text{(the right side is in } \mathbb{Z}) \\
&\Leftrightarrow x > \beta\lfloor n\alpha \rfloor + \beta && \text{(since } \beta < 0).
\end{aligned}$$

□

Lemma 5. *For $\alpha, \beta < 0$, the function $f_{1/\alpha}(x)$ commutes with $f_{1/\beta}(x)$ if and only if the equality*

$$\beta\lfloor n\alpha \rfloor + \beta = \alpha\lfloor n\beta \rfloor + \alpha$$

holds for all integers $n \in \mathbb{Z}$.

Proof. By Lemma 4, we have $x \in S_{1/\alpha,1/\beta}(n)$ if and only if $x > \beta\lfloor n\alpha \rfloor + \beta$. Similarly, we have $x \in S_{1/\beta,1/\alpha}(n)$ if and only if $x > \alpha\lfloor n\beta \rfloor + \alpha$, so that commutativity of the functions is equivalent to the desired equality. □

Lemma 6. *For $\alpha, \beta < 0$, the function $f_{1/\alpha}$ commutes with $f_{1/\beta}$ if and only if $\alpha = \beta$.*

Proof. Choose $n = 0$ in Lemma 5. We obtain that $\alpha = \beta$ is a necessary condition for commutativity. But this condition is obviously sufficient. □

Case 3. α **and** β **are of opposite signs.**

Lemma 7. *For* (α,β) *with* $\alpha\beta < 0$*, the function* $f_{1/\alpha}(x)$ *never commutes with* $f_{1/\beta}(x)$*.*

Proof. Without loss of generality, we may consider $\alpha > 0$ and $\beta < 0$. It suffices to show $S_{1/\alpha,1/\beta}(n) \neq S_{1/\beta,1/\alpha}(n)$. We will see that both of these upper level sets start at $-\infty$ and have a finite right endpoint.

We first compute $S_{1/\alpha,1/\beta}(n)$. We can follow the same steps as in Lemma 1, except in the last step where we have instead that $x \in S_{1/\alpha,1/\beta}(n)$ if and only if $x \leq \beta\lceil n\alpha\rceil$ since $\beta < 0$. We obtain for $\alpha > 0$ and $\beta < 0$ that

$$S_{1/\alpha,1/\beta}(n) = (-\infty, \beta\lceil n\alpha\rceil]$$

is a closed interval.

Next, we compute $S_{1/\beta,1/\alpha}(n)$. We can follow the same steps as in Lemma 4, except in the last step where we have instead that $x \in S_{1/\beta,1/\alpha}(n)$ if and only if $x < \alpha\lfloor n\beta\rfloor + \alpha$ since $\alpha > 0$. We find in this case that

$$S_{1/\alpha,1/\beta}(n) = (-\infty, \alpha\lfloor n\beta\rfloor + \alpha)$$

is an open interval. It follows that the two functions cannot commute. ◻

The case analysis is complete, and Theorem 1 follows.

3. Digital Straight Lines

The mathematical study of digital straight lines, which are "lines" drawn on two-dimensional graphic displays represented by pixels, was initiated by A. Rosenfeld [9] in 1974. For more recent work, see Klette and Rosenfeld [7] and Kiselman [6]. In drawing a digital image of the line $\ell_{\alpha,\gamma}(x) := \alpha x + \gamma$, a simple recipe is to associate to the abscissa $n = \lfloor x\rfloor$ the pixel $(\lfloor x\rfloor, \lfloor \alpha\lfloor x\rfloor + \gamma\rfloor) \in \mathbb{Z}^2$ (more complicated recipes are used in practice). Bruckstein [2] noted self-similar features of digital straight lines, relating them to the continued fraction expansion of their slopes; see also McIlroy [8]. In contrast, our proof of Theorem 1 does not require continued fractions.

From the digital straight line viewpoint, one can view $\lfloor\alpha\lfloor\beta x\rfloor\rfloor$ as a step function approximation to the straight line $\ell_{\alpha\beta}(x) := \alpha\beta x$ in the sense that the difference function

$$h_{\alpha,\beta}(x) := \lfloor\alpha\lfloor\beta x\rfloor\rfloor - \alpha\beta x$$

is a bounded function. This difference function is explicitly given by a combination of iterated fractional part functions $h_{\alpha,\beta}(x) = -\alpha\{\beta x\} - \{\alpha(\beta x - \{\beta x\})\}$ so is a bounded *generalized polynomial* in the sense of Bergelson and Leibman [1]. The commutativity problem studied here is that of determining when the generalized polynomial $h_{\alpha,\beta}(x) - h_{\beta,\alpha}(x)$ is identically zero.

Commutativity questions under composition can be considered for general digital straight lines such as $f_{\alpha,\gamma}(x) := \lfloor\alpha x + \gamma\rfloor$. However, general linear functions $\ell_{\alpha,\gamma}(x) = \alpha x + \gamma$ with distinct nonzero γ do not commute under composition. We do not know whether any interesting new commuting pairs occur in this more general context.

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