

THE FUNDAMENTAL THEOREM OF ALGEBRA: A MOST ELEMENTARY PROOF

Oswaldo Rio Branco de Oliveira

Abstract. This paper shows an elementary and direct proof of the Fundamental Theorem of Algebra, via Bolzano-Weierstrass Theorem on Minima and the Binomial Formula, that avoids: any root extraction other than the one used to define the modulus function over \mathbb{C} , trigonometry, differentiation, integration, series, arguments by induction and ϵ – δ type arguments.

Mathematics Subject Classification: 12D05, 30A10

Key words and phrases: Fundamental Theorem of Algebra, Inequalities in \mathbb{C} .

The aim of this article is, by combining an inequality proved in [9] and a lemma by Estermann [4], to show a very elementary proof of the FTA that requires no other n th root than the square root implicit in the modulus function. Following a suggestion given by Littlewood [8], see also Remmert [11], the proof requires a minimum amount of “limit processes lying outside algebra proper”. Hence, the proof avoids differentiation, integration, series, angle and the transcendental functions (i.e., non-algebraic functions) $\cos \theta$, $\sin \theta$ and $e^{i\theta}$, $\theta \in \mathbb{R}$. Another reason to avoid these functions is justified by the fact that the theory of transcendental functions is more profound than that of the FTA (a polynomial result), see Burckel [3]. Also avoided are arguments by induction and ϵ – δ type arguments.

Many elementary proofs of the FTA, implicitly assuming the modulus function $|z| = \sqrt{z\bar{z}}$, where $z \in \mathbb{C}$, assume either the Bolzano-Weierstrass Theorem on Minima or the Intermediate Value Theorem, plus polynomial continuity. Then, along the proof it is used further root extraction in \mathbb{R} or in \mathbb{C} (see Argand [1] and [2], Estermann [4], Fefferman [5], Kochol [7], Littlewood [8], Oliveira [9], Redheffer [10], Remmert [11], Searcoid [12], Vaggione [13]). Beginning with Littlewood [8], some of these proofs include a proof by induction of the existence of every n th root, $n \in \mathbb{N}$, of every complex number (see [7], [11] and [12]).

Besides the modulus function (derived from the inner product $z\bar{w}$, with $z, w \in \mathbb{C}$) and the Binomial Formula $(z + w)^n = \sum_{j=0}^n \binom{n}{j} z^j w^{n-j}$, $z \in \mathbb{C}$, $w \in \mathbb{C}$, $n \in \mathbb{N}$, $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ and $0! = 1$, it is assumed, without proof, only:

- Polynomial continuity.
- Bolzano-Weierstrass Theorem: *Any continuous function $f : D \rightarrow \mathbb{R}$, D a bounded and closed disc, has a minimum on D .*

Right below we show, for the case k even, $k \geq 2$, a pair of inequalities that Estermann [4] proved for every $k \in \mathbb{N} \setminus \{0\}$. The proof, via binomial formula, is a simplification of the one made by induction and given by Estermann. The case k odd can be proved similarly, if one wishes.

Lemma (Estermann). For $\zeta = \left(1 + \frac{i}{k}\right)^2$ and k even, $k \geq 2$, we have

$$\operatorname{Re}[\zeta^k] < 0 < \operatorname{Im}[\zeta^k] .$$

Proof. Since $k = 2m$ and $2k = 4m$, for some $m \in \mathbb{N}$, applying the formulas

$$\begin{aligned} \operatorname{Re} \left[\left(1 + \frac{i}{k}\right)^{2k} \right] &= 1 - \binom{2k}{2} \frac{1}{k^2} + \binom{2k}{4} \frac{1}{k^4} + \sum_{\text{odd } j, j=3}^{k-1} \left[-\binom{2k}{2j} \frac{1}{k^{2j}} + \binom{2k}{2j+2} \frac{1}{k^{2j+2}} \right] \text{ and ,} \\ \operatorname{Im} \left[\left(1 + \frac{i}{k}\right)^{2k} \right] &= \sum_{\text{odd } j, j=1}^{k-1} \left[\binom{2k}{2j-1} \frac{1}{k^{2j-1}} - \binom{2k}{2j+1} \frac{1}{k^{2j+1}} \right] , \end{aligned}$$

we end the proof by noticing that for every $j \in \mathbb{N}$, $1 \leq j \leq k-1$, we have

$$\begin{aligned} 1 - \binom{2k}{2} \frac{1}{k^2} + \binom{2k}{4} \frac{1}{k^4} &= 1 - \left(2 - \frac{1}{k}\right) \left(\frac{2}{3} + \frac{5}{6k} - \frac{1}{2k^2}\right) \leq \\ &\leq 1 - \frac{3}{2} \left(\frac{2}{3} + \frac{5k-3}{6k^2}\right) = -\frac{3}{2} \cdot \frac{5k-3}{6k^2} < 0 , \end{aligned}$$

$$-\binom{2k}{2j} \frac{1}{k^{2j}} + \binom{2k}{2j+2} \frac{1}{k^{2j+2}} = -\frac{(2k)!}{(2j)! k^{2j} (2k-2j-2)!} \left[\frac{1}{(2k-2j)(2k-2j-1)} - \frac{1}{(2kj+2k)(2kj+k)} \right] < 0 ,$$

$$\binom{2k}{2j-1} \frac{1}{k^{2j-1}} - \binom{2k}{2j+1} \frac{1}{k^{2j+1}} = \frac{(2k)!}{(2j-1)! (2k-2j-1)! k^{2j-1}} \left[\frac{1}{(2k-2j+1)(2k-2j)} - \frac{1}{(2kj+k)(2kj)} \right] > 0 .$$

Theorem. Let P be a complex polynomial, with $\text{degree}(P) = n \geq 1$. Then, P has a zero.

Proof. Putting $P(z) = a_0 + a_1z + \dots + a_nz^n$, where $a_j \in \mathbb{C}$, $0 \leq j \leq n$, $a_n \neq 0$, we have $|P(z)| \geq |a_n||z|^n - |a_{n-1}||z|^{n-1} - \dots - |a_0||z|^n$. Hence, $|P(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$ and, by continuity, $|P|$ has a global minimum at some $z_0 \in \mathbb{C}$. We can suppose without loss of generality $z_0 = 0$. Hence,

$$(1) \quad |P(z)|^2 - |P(0)|^2 \geq 0, \quad \forall z \in \mathbb{C},$$

and $P(z) = P(0) + z^k Q(z)$, for some $k \in \{1, \dots, n\}$, where Q is a polynomial and $Q(0) \neq 0$. Substituting this equation, at $z = r\zeta$, where $r \geq 0$ and ζ is arbitrary in \mathbb{C} , in inequality (1), we arrive at

$$2r^k \text{Re}[\overline{P(0)}\zeta^k Q(r\zeta)] + r^{2k} |\zeta^k Q(r\zeta)|^2 \geq 0, \quad \forall r \geq 0, \quad \forall \zeta \in \mathbb{C},$$

and, dividing the above inequality by $r^k > 0$, we find the inequality

$$2\text{Re}[\overline{P(0)}\zeta^k Q(r\zeta)] + r^k |\zeta^k Q(r\zeta)|^2 \geq 0, \quad \forall r > 0, \quad \forall \zeta \in \mathbb{C},$$

whose left side is a continuous function of r , $r \in [0, +\infty)$.

Thus, taking the limit as $r \rightarrow 0$ we find,

$$(2) \quad 2\text{Re}[\overline{P(0)}Q(0)\zeta^k] \geq 0, \quad \forall \zeta \in \mathbb{C}.$$

Let $\alpha = \overline{P(0)}Q(0) = a + ib$, where $a, b \in \mathbb{R}$. If k is odd then, substituting $\zeta = \pm 1$ and $\zeta = \pm i$ in (2), we conclude that $a = 0$ and $b = 0$. Hence $\alpha = 0$ and then, $P(0) = 0$. Thus, the case k odd is proved. Next, let us suppose k even. Taking $\zeta = 1$ in (2), we conclude that $a \geq 0$. Picking ζ as in the lemma, let us write $\zeta^k = x + iy$, with $x < 0$ and $y > 0$. Substituting ζ^k and $\overline{\zeta^k} = \overline{\zeta}^k$ in (2) it follows that $\text{Re}[\alpha(x \pm iy)] = ax \mp by \geq 0$. Hence $ax \geq 0$ and (since $x < 0$) we conclude that $a \leq 0$. So, $a = 0$. Therefore, we get that $\mp by \geq 0$. Hence, since $y \neq 0$, we conclude that $b = 0$. Hence $\alpha = 0$ and then, $P(0) = 0$. Thus, the case k even is proved. The theorem is proved.

Remarks

- (1) The almost algebraic ‘‘Gauss’ Second Proof’’ (see [6]) of the FTA uses only that ‘‘every real polynomial of odd degree has a real zero’’ and the existence of a positive square root of every positive real number. Nevertheless, this proof by Gauss is not elementary.

- (2) It is possible to rewrite a small part of the given proof of the FTA so that the polynomial continuity is used only to guarantee the existence of z_0 , a point of global minimum of $|P|$. In fact, to avoid extra use of polynomial continuity, let us keep the notation of the proof and indicate $Q(z) = Q(0) + zR(z)$, with R a polynomial. Then, substituting this expression for $Q(z)$ only in the first parcel in the left side of the inequality $2r^k \operatorname{Re}[\overline{P(0)}\zeta^k Q(r\zeta)] + r^{2k} |\zeta^k Q(r\zeta)|^2 \geq 0$, $\forall r \geq 0$, $\forall \zeta \in \mathbb{C}$, that appeared just above inequality (2), we get the inequality

$$2\operatorname{Re}[\overline{P(0)}\zeta^k Q(0)] + 2r\operatorname{Re}[\overline{P(0)}\zeta^{k+1} R(r\zeta)] + r^k |\zeta^k Q(r\zeta)|^2 \geq 0,$$

$\forall r > 0$, $\forall \zeta \in \mathbb{C}$. Fixing ζ arbitrary in \mathbb{C} , it is clear that there exists $M = M(\zeta) > 0$ such that $\max(|P(0)\zeta^{k+1} R(r\zeta)|, |\zeta^k Q(r\zeta)|^2) \leq M$, $\forall r \in (0, 1)$. Hence,

$$-2\operatorname{Re}[\overline{P(0)}\zeta^k Q(0)] \leq 2rM + r^k M \leq 3rM, \quad \forall r \in (0, 1).$$

So, we conclude that $-2\operatorname{Re}[\overline{P(0)}\zeta^k Q(0)] \leq 0$, with ζ arbitrary in \mathbb{C} . Now, obviously, the proof continues as in the proof of the theorem.

- (3) It is worth to point out that this proof of the FTA easily implies an independent proof of the existence of a unique positive n th root, $n \geq 3$, of any number $a \geq 0$. In fact, given $z \in \mathbb{C}$ such that $z^n = a$, we have that $|z|^n = a$, with $|z| \geq 0$. The uniqueness of such n th root is very trivial.

Acknowledgments

I thank Professors J. V. Ralston and Paulo A. Martin for very valuable comments and suggestions, J. Aragona for reference [5] and R. B. Burckel for references [7] and [13].

References

- [1] Argand, J. R., *Imaginary Quantities: Their Geometrical Interpretation*, University Michigan Library, 2009.
- [2] Argand, J. R., *Essay Sur Une Manière de Représenter Les Quantités Imaginaires Dans Les Contructions Géométriques*, Nabu Press, 2010.

- [3] Burckel, R. B., “ A Classical Proof of The Fundamental Theorem of Algebra Dissected”, *Mathematical Newsletter of the Ramanujan Mathematical Society* 7, no. 2 (2007), 37-39.
- [4] Estermann, T., “On The Fundamental Theorem of Algebra”, *J. London Mathematical Society* 31 (1956), 238-240.
- [5] Fefferman, C., “An Easy Proof of the Fundamental Theorem of Algebra”, *American Mathematical Monthly* 74 (1967), 854-855.
- [6] Gauss, C. F., *Werke*, Volume 3, 33-56 (in latin; English translation available at <http://www.cs.man.ac.uk/~pt/misc/gauss-web.html>).
- [7] Kochol, M., “An Elementary Proof of The Fundamental Theorem of Algebra”, *International Journal of Mathematical Education in Science and Technology*, 30 (1999), 614-615.
- [8] Littlewood, J. E., “Mathematical notes (14): Every Polynomial has a Root”, *J. London Mathematical Society* 16 (1941), 95-98.
- [9] Oliveira, O. R. B., “The Fundamental Theorem of Algebra: An Elementary and Direct Proof”, *The Mathematical Intelligencer* 33, No. 2, (2011), 1-2.
- [10] Redheffer, R. M., “What! Another Note Just on the Fundamental Theorem of Algebra?”, *American Mathematical Monthly* 71 (1964), 180-185.
- [11] Remmert, R., “The Fundamental Theorem of Algebra”. In H.-D. Ebbinghaus, et al., *Numbers*, Graduate Texts in Mathematics, no. 123, Springer-Verlag, New York, 1991. Chapters 3 and 4.
- [12] Searcoid, M. O., *Elements of Abstract Analysis*, Springer-Verlag, London, 2003.
- [13] Vaggione, D., “On The Fundamental Theorem of Algebra”. *Colloquium Mathematicum* 73 No. 2 (1997), 193-194.

Departamento de Matemática
 Universidade de São Paulo
 Rua do Matão 1010, CEP 05508-090
 São Paulo, SP
 Brasil
 e-mail: oliveira@ime.usp.br