- from L. Carroll 'Alice's Adventures in Wonderland'


# A FEW RIDDLES BEHIND ROLLE'S THEOREM 

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## Getting started

First year undergraduates usually learn about the classical Rolle's theorem saying that between two consecutive zeros of a smooth function $f$ one can always find at least one zero of its derivative $f^{\prime}$. In this paper we study a generalization of Rolle's theorem dealing with the zeros of higher derivatives for a class of smooth functions which we call $n$-nice. For such functions we obtain some mysteriously looking additional inequalities governing the mutual arrangement of the zeros of different derivatives. The considered topic is so classical that we find it impossible to be sure that our results are new. However, we were unable to trace anything similar in the literature although such facts might have been known to at least Chevalier Augustin Cauchy if not to Sir Isaac Newton himself.

Consider a smooth function $f$ that on a certain interval has $n$ distinct real zeros which we denote by $x_{1}^{(0)}<x_{2}^{(0)}<\ldots<x_{n}^{(0)}$. Then, by the classical Rolle's theorem, $f^{\prime}$ has at least $(n-1)$ zeros, $f^{\prime \prime}$ has at least $(n-2)$ zeros, $\ldots, f^{(n-1)}$ has at least one zero on $\left(x_{1}^{(0)}, x_{n}^{(0)}\right)$.

We call a smooth function $f$ having $n$ simple real zeros on some interval an $n$-nice function if for all $i=0, \ldots, n$ the $i$-th derivative $f^{(i)}$ has on the same interval exactly $(n-i)$ zeros denoted by $x_{1}^{(i)}<x_{2}^{(i)}<\ldots<x_{n-i}^{(i)}$. Note that we require, in particular, that $f^{(n)}$ is nonvanishing! Observe also that if $f$ is $n$-nice on $I$ then for all $i<n$ its derivative $f^{(i)}$ is $(n-i)$-nice on the same interval. As a natural example of an $n$-nice function on the whole $\mathbb{R}$ one can take any polynomial of degree $n$ with only real and distinct zeros. In the above notation the following system of inequalities holds

$$
\begin{equation*}
x_{l}^{(i)}<x_{l}^{(j)}<x_{l+j-i}^{(i)} ; \quad i<j \leq n-l . \tag{1}
\end{equation*}
$$

We call this system the standard Rolle's restrictions.
With any $n$-nice function $f$ we can associate the arrangement $\mathcal{A}_{f}$ of all $\binom{n+1}{2}$ zeros $\left\{x_{l}^{(i)}\right\}$ of $f^{(i)}, i=0, \ldots, n-1 ; 1 \leq l \leq n-i$, say, taking first all $x_{l}^{(0)}$ then all $x_{l}^{(1)}$ etc.

The main problem we address in this note is as follows.
Question. What additional restrictions besides (1) exist on the arrangements $\mathcal{A}_{f}=\left\{x_{l}^{(i)}\right\}$ for $n$-nice functions? Or even more ambitiously, given an arrangement of $\binom{n+1}{2}$ real numbers $\mathcal{A}=\left\{x_{l}^{(i)} \mid i=0, \ldots, n-1 ; l=1, \ldots n-i\right\}$ satisfying the standard Rolle's restrictions is it possible to say if there exist an $n$-nice function $f$ such that $\mathcal{A}_{f}=\mathcal{A}$ ?

Our choice of the class of $n$-nice functions is motivated by the (easy to formalize) idea that as soon as one allows several real zeros of $f^{\prime}$ in between two consecutive zeros of $f$ then no interesting additional restrictions are possible. Also this class seems to be the natural generalization of the well-studied class of real polynomials with all real zeros.

We will soon discover that the fact that a smooth function is $n$-nice implies additional inequalities on the components of $\mathcal{A}_{f}$. Notice that the set of all $n$-nice functions on a given interval forms the subset of a larger class of functions which in several aspects behave similarly to real polynomials of degree $n$. Let us now define this smooth analog of polynomials. We postpone the discussion of the usual polynomials until the next section.

Main Definition. A smooth real-valued function $f$ defined on some interval $I$ is called a pseudopolynomial of degree $n$ on $I$ if $f^{(n)}$ never vanishes.

The usual Rolle's theorem immediately implies that any pseudopolynomial $f$ of degree $n$ has at most $n$ real zeros (counted with multiplicitites). The set of all $n$-nice functions on a given interval coincides with the set of all pseudopolynomials of degree $n$ with exactly $n$ real and distinct zeros.

Let $\mathcal{N}_{n}(I)$ denote the set of all $n$-nice functions on an interval $I$. (Since the particular choice of $I$ is unimportant in the formulation below we will often use $\mathcal{N}_{n}$ instead of $\mathcal{N}_{n}(I)$.) Our result below answers the posed question for $n=3$, i.e., the case of 3 -nice functions. (Obviously, there are no additional restrictions for $n=2$, i.e. on the arrangements of the two zeros of a smooth function $f$ and one zero of $f^{\prime}$.) In order to make our notation more readable denote the three zeros of a 3 -nice function $f$ by $x_{1}<x_{2}<x_{3}$, the two zeros of $f^{\prime}$ by $y_{1}<y_{2}$ and the only zero of $f^{\prime \prime}$ by $z_{1}$.

Inequality Theorem. For any 3-nice function $f$ its arrangement $\mathcal{A}_{f}=\left(x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, z_{1}\right)$ satisfies the following inequalities:

$$
\left\{\begin{array}{l}
x_{1}<y_{1}<x_{2}<y_{2}<x_{3}  \tag{2}\\
y_{1}<z_{1}<y_{2} \\
y_{1}-x_{1}<\min \left(x_{2}-y_{1}, \sqrt{\left(z_{1}-y_{1}\right)^{2}+2\left(z_{1}-y_{1}\right)\left|z_{1}-x_{2}\right|}\right) \\
x_{3}-y_{2}<\min \left(y_{2}-x_{2}, \sqrt{\left(y_{2}-z_{1}\right)^{2}+2\left(y_{2}-z_{1}\right)\left|z_{1}-x_{2}\right|}\right)
\end{array}\right.
$$

And, conversely, for any 6-tuple satisfying the above inequalities there exists a 3 -nice function $f$ with that arrangement $\mathcal{A}_{f}$ of the zeros of $f, f^{\prime}, f^{\prime \prime}$ respectively. (The geometrical meaning of the additional inequalities will be quite clear from the proof of the theorem below.)

The inequalities on the first two lines of (2) are the standard Rolle's restrictions. Notice that two remaining new inequalities interchange places under the substitution $x \mapsto-x$.

Proof. Take some $f \in \mathcal{N}_{3}$. Without loss of generality we can assume $f^{\prime \prime \prime}>0$ implying that $f^{\prime}$ is convex (otherwise multiply $f$ by -1 ). As above, the three zeros of $f$ are denoted by $x_{1}<x_{2}<x_{3}$, the two zeros of $f^{\prime}$ by $y_{1}<y_{2}$ and the only zero of $f^{\prime \prime}$ by $z_{1}$. Let us first consider the case $x_{2}<z_{1}$; see Fig.1.


Fig. 1. Derivative of $f$ and its accompanying elementary con-
figuration.

We will immediately derive the additional inequalities:

$$
\begin{equation*}
y_{1}-x_{1}<x_{2}-y_{1} \quad \text { and } \quad x_{3}-y_{2}<\sqrt{\left(y_{2}-z_{1}\right)^{2}+2\left(y_{2}-z_{1}\right)\left(z_{1}-x_{2}\right)} \tag{3}
\end{equation*}
$$

The case $z_{1}<x_{2}$ is completely analogous and leads to:

$$
\begin{equation*}
x_{3}-y_{2}<y_{2}-x_{2} \quad \text { and } \quad y_{1}-x_{1}<\sqrt{\left(z_{1}-y_{1}\right)^{2}+2\left(z_{1}-y_{1}\right)\left(x_{2}-z_{1}\right)} \tag{4}
\end{equation*}
$$

The union of (3) and (4) gives exactly the required new inequalities in (2).
Indeed, the convexity of $f^{\prime}$ implies that
i) on $\left(x_{1}, y_{1}\right)$ the graph of $f^{\prime}$ lies above the line segment $\left\{a, y_{1}\right\}$ which is tangent to it at $y_{1}$; ii) on $\left(y_{1}, x_{2}\right)$ the graph of $f^{\prime}$ lies between the $x$-axis and the line segment $\left\{y_{1}, b\right\}$ which is tangent to it at $y_{1}$;
iii) on $\left(x_{2}, y_{2}\right)$ the graph of $f^{\prime}$ lies between the $x$-axis and the broken line segment $\left\{c, d, y_{2}\right\}$ where $(c, d)$ is horizontal;
iv) on ( $y_{2}, x_{3}$ ) the graph of $f^{\prime}$ lies above the line segment $\left\{y_{2}, e\right\}$ which is tangent to it at $y_{2}$.

Note that since $x_{1}, x_{2}, x_{3}$ are consecutive real zeros of $f$ then

$$
\int_{x_{1}}^{x_{2}} f^{\prime} d x=\int_{x_{2}}^{x_{3}} f^{\prime} d x=0
$$

implying

$$
\int_{x_{1}}^{y_{1}} f^{\prime} d x=-\int_{y_{1}}^{x_{2}} f^{\prime} d x \quad \text { and } \quad \int_{x_{2}}^{y_{2}} f^{\prime} d x=-\int_{y_{2}}^{x_{3}} f^{\prime} d x
$$

Therefore,

$$
\operatorname{Ar}\left(\triangle_{a x_{1} y_{1}}\right)<\operatorname{Ar}\left(\triangle_{y_{1} b x_{2}}\right) \quad \text { and } \quad \operatorname{Ar}\left(\triangle_{e y_{2} x_{3}}\right)<\operatorname{Ar}\left(\square_{x_{2} c d y_{2}}\right),
$$

where $A r$ stands for the area of the corresponding figures. (The figure $\square_{x_{2} c d y_{2}}$ is a trapezoid.) Using our knowledge of high-school mathematics we get

$$
\operatorname{Ar}\left(\triangle_{a x_{1} y_{1}}\right)=\frac{1}{2}\left(y_{1}-x_{1}\right)^{2} \tan \alpha, \operatorname{Ar}\left(\triangle_{e y_{2} x_{3}}\right)=\frac{1}{2}\left(x_{3}-y_{2}\right)^{2} \tan \beta, \operatorname{Ar}\left(\triangle_{y_{1} b x_{2}}\right)=\frac{1}{2}\left(x_{2}-y_{2}\right)^{2} \tan \alpha
$$ and,

$$
\begin{gathered}
\operatorname{Ar}\left(\square_{x_{2} c d y_{2}}\right)=\operatorname{Ar}\left(\triangle_{z_{1} d y_{2}}\right)+\operatorname{Ar}\left(\square_{x_{2} c d z_{1}}\right)=\frac{1}{2}\left(y_{2}-z_{1}\right)^{2} \tan \beta+\left(y_{2}-z_{1}\right)\left(z_{1}-x_{2}\right) \tan \beta= \\
=\frac{1}{2}\left(\left(y_{2}-z_{1}\right)^{2}+2\left(y_{2}-z_{1}\right)\left(z_{1}-x_{2}\right)\right) \tan \beta .
\end{gathered}
$$

These relations immediately imply the required inequalities.
To finish the proof, pick any 6 -tuple of real numbers $x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, z_{1}$ satisfying (2). We again assume that $x_{2}<z_{1}$. (The case $z_{1} \leq x_{2}$ is completely analogous.) Draw a piecewise linear function $g$ as shown on Fig. 2.


Fig. 2. Constructing an appropriate function.
(The only difference with Fig. 1 is that we force $c=b$.) The inequalities (2) imply that $\operatorname{Ar}\left(\triangle_{a x_{1} y_{1}}\right)<\operatorname{Ar}\left(\triangle_{y_{1} b x_{2}}\right)$ and $\operatorname{Ar}\left(\triangle_{e y_{2} x_{3}}\right)<\operatorname{Ar}\left(\square_{x_{2} b d y_{2}}\right)$. It is easy to approximate $g$ by a convex function $h$ so that
i) $y_{1}$ and $y_{2}$ ) are the zeros of $h$ and $z_{1}$ is the zero of $h^{\prime}$;
ii) $\int_{x_{1}}^{y_{1}} h d x<-\int_{y_{1}}^{x_{2}} h d x$ and $\int_{x_{2}}^{y_{2}} h d x<-\int_{y_{2}}^{x_{3}} h d x$.

Keeping the function $h$ as it is on the interval $\left(y_{1}, y_{2}\right)$ we can increase it on the intervals $\left(x_{1}, y_{1}\right)$ and $\left(y_{2}, x_{3}\right)$ and construct a new convex function $\tilde{h}$ with the properties
i) $y_{1}$ and $y_{2}$ are the zeros of $\tilde{h}$ and $z_{1}$ is the zero of $\tilde{h}^{\prime}$;
ii) $\int_{x_{1}}^{y_{1}} \tilde{h} d x=-\int_{y_{1}}^{x_{2}} \tilde{h} d x$ and $\int_{x_{2}}^{y_{2}} \tilde{h} d x=-\int_{y_{2}}^{x_{3}} \tilde{h} d x$.

The function $f \in \mathcal{N}_{3}$ we were looking for is now obtained as $f=-\int_{y_{1}}^{x} \tilde{h}(t) d t$.
As an illustration of the Inequality Theorem consider the polynomial

$$
p(x)=x(x-1)(x-4) .
$$

The polynomial $p(x)$ and its first two derivatives are shown on Fig. 3 below. As any polynomial of degree 3 with real and distinct zeros $p(x)$ is 3 -nice. Elementary calculation gives that in the notation of the theorem we have

$$
x_{1}=0, x_{2}=1, x_{3}=4, y_{1}=\frac{5-\sqrt{13}}{3}, y_{2}=\frac{5+\sqrt{13}}{3}, z_{1}=\frac{5}{3} .
$$

Therefore, $x_{2}<z_{1}$ and we are interested in checking the validity of (3). Indeed, $y_{1}-x_{1}=$ $\frac{5-\sqrt{13}}{3} \simeq 0.464816$ which is smaller than $x_{2}-y_{1}=\frac{\sqrt{13}-2}{3} \simeq 0.535184$. Next, $x_{3}-y_{2}=$ $\frac{7-\sqrt{13}}{3} \simeq 1.13148$ which is smaller than $\sqrt{\left(z_{1}-y_{1}\right)^{2}+2\left(z_{1}-y_{1}\right)\left(x_{2}-z_{1}\right)}=\frac{\sqrt{13+4 \sqrt{13}}}{3} \simeq$ 1.74554 .


Fig. 3. Illustration of the theorem.

## UsUal Polynomials

Let us now try to find meaningful restrictions on arrangements $\mathcal{A}_{p}$ for the usual polynomials with all real and distinct zeros. (Note that inequalities (2) remain valid but they clearly do not give all the restrictions since for a usual polynomial its zeros define all the zeros of all its higher derivatives in a unique way.) Let us assume that all $n$ zeros $x_{1}^{(0)}<x_{2}^{(0)}<\ldots<x_{n}^{(0)}$ of a polynomial $p$ of degree $n$ are real and distinct. Assume additionally that all $x_{l}^{(i)}$ are pairwise different. (One can see that this extra condition holds for almost all polynomials with real zeros.) We call such polynomials strictly $n$-nice. For a strictly $n$-nice polynomial $p$ its whole arrangement $\mathcal{A}_{p}$ is naturally ordered on the real line. Substituting each zero of $p$ by the symbol 0 , each zero of $p^{\prime}$ by $1, \ldots$, each zero of $p^{(n-1)}$ by $(n-1)$ respectively we get a symbolic sequence of $p$ of length $\binom{n+1}{2}$ with $n$ occurrences of $0,(n-1)$ occurrences of $1, \ldots$, one occurence of $(n-1)$ and satisfying the condition that between any two consecutive occurrences of the symbol $i$ it has exactly one occurrence of the symbol $i+1$.

For example, there are only two possible symbolic sequences for $n=3$, namely, 012010 and 010210 . For $n=4$ there are 12 such sequences 0123012010 , 0120312010, 0120132010, 0102312010, 0102132010, 0123010210, 0120310210, 0120130210, 0120103210, 0123010210, 0102130210, 0102103210. A patient reader will find the for $n=5$ there are 286 such sequences.

If we denote by $b_{n}$ the number of all possible symbolic sequences of length $n$ then actually, this number is possible to calculate. It turns out to be equal to

$$
b_{n}=\binom{n+1}{2}!\frac{1!2!\ldots(n-1)!}{1!3!\ldots(2 n-1)!}
$$

But since this calculation is a content of a different story we refer the interested reader to [6].
We can now formulate a natural discrete analog of the main problem from the previous section which makes sense for the usual polynomials.

Question. What symbolic sequences can occur for strictly $n$-nice polynomials of degree $n$ ? (We will call such sequences realizable.)

As we will see shortly already the first nontrivial case of $n=4$ shows that the number $\sharp_{n}$ of all realizable symbolic sequences is strictly smaller than the corresponding $b_{n}$, namely, $10=\sharp_{4} \neq b_{4}=12$.

The fact that $\sharp n \neq b_{n}$ was apparently observed by a number of authors but the only relevant reference we found is [1] published in 1993. An explanation of this phenomenon for $n=4$ is as follows. (See further generalizations in [4].)

Theorem, see [1]. A polynomial $p$ of degree 4 with real zeros $x_{1}<x_{2}<x_{3}<x_{4}$ satisfying the inequalities $x_{2}<z_{1}$ and $x_{3}<z_{2}$, satisfies additionally the inequality $y_{2}<t_{1}$. Here $y_{1}<y_{2}<y_{3}$ are the zeros of $f^{\prime}$; the zeros of $f^{\prime \prime}$ are $z_{1}<z_{2}$ and $t_{1}$ is the zero of $f^{\prime \prime \prime}$. In other words, the symbolic sequences 0102310210 and 0120132010 are non-realizable.

This is easy to check once you know what to prove! Indeed, any monic polynomial of degree 4 with all real zeros can be put in the form $x^{4}-x^{2}+u x+v$ by a linear change of $x$ and scaling. Namely, by shifting $x \mapsto x+\alpha$ we can always get rid of the $x^{3}$-term. Since the second derivative of the obtained polynomial has two real zeros, the coefficient at $x^{2}$ should be negative. Appropriate scaling now puts $p$ in the above form. Note that $p^{\prime \prime}=12 x^{2}-2$, its zeros being $\pm \sqrt{\frac{1}{6}}$. The assumptions $x_{2}<z_{1}, x_{3}<z_{2}$ together with $p$ having real zeros imply $p\left(-\sqrt{\frac{1}{6}}\right)>0, p\left(\sqrt{\frac{1}{6}}\right)<0$ (draw the graph of $p$ ). Noting that $p^{\prime \prime \prime}=24 x$ what we need to prove is that $p^{\prime}(0)<0$ (draw the graph of $p^{\prime}$ ). The last inequality is equivalent to $u<0$. Expanding $p\left(-\sqrt{\frac{1}{6}}\right)>0$ and $p\left(\sqrt{\frac{1}{6}}\right)<0$ we get $\frac{1}{36}-\frac{1}{6}-\frac{u}{\sqrt{6}}+v>0$ and $\frac{1}{36}-\frac{1}{6}+\frac{u}{\sqrt{6}}+v<0$. Subtracting the former from the later implies $\frac{2 u}{\sqrt{6}}<0$.

The next case $n=5$ was considered in [2]. V. Kostov was able to show that among 286 possible symbolic sequences only 116 are realizable by strictly $n$-nice polynomials. Very recently the same author considered the similar question which symbolic sequences are realizable for the case of $n$-nice functions; see [3]. It turned out that for $n=4$ all 12 symbolic sequences are realizable but already for $n=5$ there are non-realizable sequences. The situation does not seem to change much if we extend the class of polynomials but $n$-nice functions.

To finish the section let us present a tempting problem posed by the famous mathematician Vladimir Arnold after the talk given by V. Kostov on his seminar.

Problem. Is it true that $\lim _{n \rightarrow \infty} \frac{\sharp_{n}}{b_{n}}=0$ ? If yes, how fast does the quotient $\frac{\sharp_{n}}{b_{n}}$ decrease?

## Periodic functions

At the end let us briefly discuss what happens with periodic functions, i.e., functions defined on a circle. In the previous sections we defined the class of $n$-nice functions - a generalization of polynomial of degree $n$ with $n$ distinct real roots - and found some inequalities involving the roots of higher derivatives valid for any 3-nice smooth function. It seems quite natural to try to develop a similar concept for periodic functions. The periodic analog of polynomials of degree $n$ are trigonometric polynomials of degree $n$, i.e., expressions of the form $a_{0}+\sum_{k=1}^{n} a_{k} \cos k x+b_{k} \sin k x$. Any trigonometric polynomial of degree $n$ has at most $2 n$ real zeros on a period.

Observe that if we take such a trigonometric polynomial with exactly $2 n$ real and distinct zeros then its derivative of any order will also be a trigonometric polynomial of the same degree $n$. Moreover, by the usual Rolle's theorem it will also have exactly $2 n$ real and distinct zeros on a period. So it seems tempting to define a periodic analog of an $n$-nice function as a periodic function such that it and its derivatives of any order have exactly $2 n$ real zeros. But (for not completely clear reasons) the situation with periodic functions turns out to be much more rigid than with the functions on an interval.

To explain the situation we have to invoke the following famous classical result of G. Polya and N. Wiener; see [5].

Theorem. Any periodic function $f$ such that the number of real zeros of the $i$-th derivative $f^{(i)}$ on a period remains bounded as $i \rightarrow \infty$ is a trigonometric polynomial.

In particular, any (conjectural) $n$-nice periodic function must necessarily be an actual trigonometric polynomial. On the other hand, one can define a periodic analog of symbolic sequences from the previous section and ask which of those are realizable by trigonometric polynomials with all real zeros. Namely, for a positive integer $k$ consider a sequence of integers of length $n k$ written on a circle and containing $n$ zeros, $n$ ones, $n$ twos, $\ldots, n$ copies of ( $k-1$ ). We call such a sequence possible periodic if for any $i=0,1, \ldots, k-2$ in between any two consecutive (on the circle) copies of $i$ the sequence contains exactly one copy of $i+1$. Now we can ask:

Question. What possible periodic sequences can occur as the sequences of zeros of $f$ and its higher derivatives of order up to $k-1$ where $f$ is a trigonometric polynomial of degree $n$ with all real and distinct zeros? (Here as before the integer $i$ substitutes a real root of the $i$-th derivative of $f$.)

Unfortunately, at the moment there is no nontrivial information available about the latter problem.
Acknowledments. The authors are sincerely grateful to the anonymous referees whose important suggestions allowed us to substantially improve the quality of exposition.

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