# A Class of Continued Radicals 

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#### Abstract

We compute the limits of a class of continued radicals extending the results of a previous note in which only periodic radicals of the class were considered.


## 1 Introduction.

In [1] the author discussed the values for a class of periodic continued radicals of the form

$$
\begin{equation*}
a_{0} \sqrt{2+a_{1} \sqrt{2+a_{2} \sqrt{2+a_{3} \sqrt{2+\cdots}}}}, \tag{1}
\end{equation*}
$$

where for some positive integer $n$,

$$
a_{n+k}=a_{k}, \quad k=0,1,2, \ldots,
$$

and

$$
a_{k} \in\{-1,+1\}, \quad k=0,1, \ldots, n-1
$$

It was also shown that the radicals given by equation (1) have limits two times the fixed points of the Chebycheff polynomials $T_{2^{n}}(x)$, thus unveiling an interesting relation between these topics.

In [3], the authors defined the set $S_{2}$ of all continued radicals of the form (1) (with $a_{0}=1$ ) and they investigated some of their properties by assuming that the limit of the radicals exists. In particular, they showed that all elements of $S_{2}$ lie between 0 and 2 , any two radicals cannot be equal to each other, and $S_{2}$ is uncountable.

My previous note hence partially bridged this gap but left unanswered the question 'what are the limits if the radicals are not periodic?' I answer the question in this note. The result is easy to establish, but I realized it only as I was reading the proof of my previous note. Such is the working of the mind!

## 2 The Limits.

Towards the desired result, I present the following lemma from [2], also used in the periodic case, which is an extension of the well known trigonometric formulas of the angles $\pi / 2^{n}$.

Lemma 1. For $a_{i} \in\{-1,1\}$, with $i=0,1, \ldots, n-1$, we have that

$$
2 \sin \left[\left(a_{0}+\frac{a_{0} a_{1}}{2}+\cdots+\frac{a_{0} a_{1} \cdots a_{n-1}}{2^{n-1}}\right) \frac{\pi}{4}\right]=a_{0} \sqrt{2+a_{1} \sqrt{2+a_{2} \sqrt{2+\cdots+a_{n-1} \sqrt{2}}}} .
$$

The lemma is proved in [2] using induction.
According to this lemma, the partial sums of the continued radical (1) are given by

$$
x_{n}=2 \sin \left[\left(a_{0}+\frac{a_{0} a_{1}}{2}+\cdots+\frac{a_{0} a_{1} \cdots a_{n-1}}{2^{n-1}}\right) \frac{\pi}{4}\right]
$$

The series

$$
a_{0}+\frac{a_{0} a_{1}}{2}+\cdots+\frac{a_{0} a_{1} \cdots a_{n-1}}{2^{n-1}}+\cdots
$$

is absolutely convergent and thus it converges to some number $a$. Therefore, the original continued radical converges to the real number

$$
x=2 \sin \frac{a \pi}{4}
$$

We can find a concise formula for $x$. For this calculation it is more useful to use the products

$$
P_{m}=\prod_{k=0}^{m} a_{k}, \quad \text { for } m=0,1,2, \ldots
$$

which take the values $\pm 1$. We will refer to these as partial parities. (When the pattern is periodic of period $n$ only the first $n$ parities $P_{0}, P_{1}, \ldots, P_{n-1}$ are independent.) Using the notation with the partial parities, set

$$
a=P_{0}+\frac{P_{1}}{2}+\frac{P_{2}}{2^{2}}+\cdots+\frac{P_{n-1}}{2^{n-1}}+\frac{P_{n}}{2^{n}}+\cdots
$$

We now define

$$
Q_{m}=\frac{1+P_{m}}{2}
$$

Since $P_{m} \in\{-1,1\}$, it follows that $Q_{m} \in\{0,1\}$. Inversely, $P_{m}=2 Q_{m}-1$. Thus

$$
a=\sum_{m=0}^{\infty} \frac{P_{m}}{2^{m}}=\sum_{m=0}^{\infty} \frac{Q_{m}}{2^{m-1}}-\sum_{m=0}^{\infty} \frac{1}{2^{m}}=4 \sum_{m=0}^{\infty} \frac{Q_{m}}{2^{m+1}}-2
$$

Notice that the sum

$$
Q=\sum_{m=0}^{\infty} \frac{Q_{m}}{2^{m+1}}
$$

in the previous equation is the number $Q$ whose binary expression is $0 . Q_{0} Q_{1} \cdots Q_{n-3} Q_{n-2} \cdots$. Therefore $a=4 Q-2$. In [3], the authors noticed that all continued radicals of the form (1) (with $a_{0}=1$ ) are in one-to-one correspondence with the set of decimals between 0 and 1 as written in binary notation (and that's how they determined that the set $S_{2}$ is uncountable). But, with the above calculation, this correspondence is made deeper. It gives the limit of the radical (1) as follows

$$
x=-2 \cos (Q \pi)
$$

For example, if $a_{k}=1$ for all $k$, then also $Q_{k}=1$ for all $k$ and the number $Q=$ $0.111111111 \cdots$ written in the binary system is the number $Q=1$ in the decimal system; hence $x=2$. We thus recover the well known result

$$
2=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} .
$$

## 3 Conclusion.

Having found the limit of (1), the next obvious question is to determine the limit of the radical

$$
a_{0} \sqrt{y+a_{1} \sqrt{y+a_{2} \sqrt{y+a_{3} \sqrt{y+\cdots}}}},
$$

for values of the variable $y$ that make the radical (and the limit) well defined. However, a direct application of the above method fails and so far a convenient variation has been elusive. Therefore, the limit of the last radical in the general case remains an open problem although it is known in at least two cases [3].

## References

[1] C. J. Efthimiou, A class of periodic continued radicals, Amer. Math. Monthly 118 (2012) 52.
[2] D. O. Shklarsky, N. N. Chentzov, and I. M. Yaglom, The USSR Problem Book: Selected Problems and Theorems of Elementary Mathematics, Dover, New York, 1993.
[3] S. Zimmerman and C. W. Ho, On infinitely nested radicals, Math. Mag. 81 (2008) 3.

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