## We compute the limits of a class of continued radicals extending the results of a previous note in which only periodic radicals of the class were considered. Introduction. 1

In [1] the author discussed the values for a class of periodic continued radicals of the form

A Class of Continued Radicals

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Abstract

$$a_0\sqrt{2+a_1\sqrt{2+a_2\sqrt{2+a_3\sqrt{2+\cdots}}}},$$
 (1)

where for some positive integer n,

$$a_{n+k} = a_k$$
,  $k = 0, 1, 2, \dots$ 

and

$$a_k \in \{-1, +1\}, \quad k = 0, 1, \dots, n-1.$$

It was also shown that the radicals given by equation (1) have limits two times the fixed points of the Chebycheff polynomials  $T_{2^n}(x)$ , thus unveiling an interesting relation between these topics.

In [3], the authors defined the set  $S_2$  of all continued radicals of the form (1) (with  $a_0 = 1$ ) and they investigated some of their properties by assuming that the limit of the radicals exists. In particular, they showed that all elements of  $S_2$  lie between 0 and 2, any two radicals cannot be equal to each other, and  $S_2$  is uncountable.

My previous note hence partially bridged this gap but left unanswered the question 'what are the limits if the radicals are not periodic?' I answer the question in this note. The result is easy to establish, but I realized it only as I was reading the proof of my previous note. Such is the working of the mind!

## The Limits. $\mathbf{2}$

Towards the desired result, I present the following lemma from [2], also used in the periodic case, which is an extension of the well known trigonometric formulas of the angles  $\pi/2^n$ .

**Lemma 1.** For  $a_i \in \{-1, 1\}$ , with i = 0, 1, ..., n - 1, we have that

$$2\sin\left[\left(a_0 + \frac{a_0a_1}{2} + \dots + \frac{a_0a_1\cdots a_{n-1}}{2^{n-1}}\right)\frac{\pi}{4}\right] = a_0\sqrt{2 + a_1\sqrt{2 + a_2\sqrt{2 + \dots + a_{n-1}\sqrt{2}}}}$$

The lemma is proved in [2] using induction.

According to this lemma, the partial sums of the continued radical (1) are given by

$$x_n = 2\sin\left[\left(a_0 + \frac{a_0a_1}{2} + \dots + \frac{a_0a_1 \cdots a_{n-1}}{2^{n-1}}\right)\frac{\pi}{4}\right] .$$

The series

$$a_0 + \frac{a_0 a_1}{2} + \dots + \frac{a_0 a_1 \cdots a_{n-1}}{2^{n-1}} + \dots$$

is absolutely convergent and thus it converges to some number a. Therefore, the original continued radical converges to the real number

$$x = 2\sin\frac{a\pi}{4} \ .$$

We can find a concise formula for x. For this calculation it is more useful to use the products

$$P_m = \prod_{k=0}^m a_k$$
, for  $m = 0, 1, 2, \dots$ ,

which take the values  $\pm 1$ . We will refer to these as partial parities. (When the pattern is periodic of period *n* only the first *n* parities  $P_0, P_1, \ldots, P_{n-1}$  are independent.) Using the notation with the partial parities, set

$$a = P_0 + \frac{P_1}{2} + \frac{P_2}{2^2} + \dots + \frac{P_{n-1}}{2^{n-1}} + \frac{P_n}{2^n} + \dots$$

We now define

$$Q_m = \frac{1+P_m}{2}$$

Since  $P_m \in \{-1, 1\}$ , it follows that  $Q_m \in \{0, 1\}$ . Inversely,  $P_m = 2Q_m - 1$ . Thus

$$a = \sum_{m=0}^{\infty} \frac{P_m}{2^m} = \sum_{m=0}^{\infty} \frac{Q_m}{2^{m-1}} - \sum_{m=0}^{\infty} \frac{1}{2^m} = 4 \sum_{m=0}^{\infty} \frac{Q_m}{2^{m+1}} - 2.$$

Notice that the sum

$$Q = \sum_{m=0}^{\infty} \frac{Q_m}{2^{m+1}}$$

in the previous equation is the number Q whose binary expression is  $0.Q_0Q_1 \cdots Q_{n-3}Q_{n-2} \cdots$ . Therefore a = 4Q - 2. In [3], the authors noticed that all continued radicals of the form (1) (with  $a_0 = 1$ ) are in one-to-one correspondence with the set of decimals between 0 and 1 as written in binary notation (and that's how they determined that the set  $S_2$  is uncountable). But, with the above calculation, this correspondence is made deeper. It gives the limit of the radical (1) as follows

$$x = -2\cos\left(Q\pi\right)$$

For example, if  $a_k = 1$  for all k, then also  $Q_k = 1$  for all k and the number  $Q = 0.111111111\cdots$  written in the binary system is the number Q = 1 in the decimal system; hence x = 2. We thus recover the well known result

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}.$$

## 3 Conclusion.

Having found the limit of (1), the next obvious question is to determine the limit of the radical

$$a_0\sqrt{y+a_1\sqrt{y+a_2\sqrt{y+a_3\sqrt{y+\cdots}}}}$$

for values of the variable y that make the radical (and the limit) well defined. However, a direct application of the above method fails and so far a convenient variation has been elusive. Therefore, the limit of the last radical in the general case remains an open problem although it is known in at least two cases [3].

## References

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- [2] D. O. Shklarsky, N. N. Chentzov, and I. M. Yaglom, The USSR Problem Book: Selected Problems and Theorems of Elementary Mathematics, Dover, New York, 1993.
- [3] S. Zimmerman and C. W. Ho, On infinitely nested radicals, Math. Mag. 81 (2008) 3.

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