ADDITIVE SYSTEMS AND A THEOREM OF DE BRUIJN

MELVYN B. NATHANSON

In memoriam Nicolaas Govert de Bruijn

ABSTRACT. This paper proves a theorem of de Bruijn that classifies additive systems for the nonnegative integers, that is, families $\mathcal{A} = (A_i)_{i \in I}$ of sets of nonnegative integers, each set containing 0, such that every nonnegative integer can be written uniquely in the form $\sum_{i \in I} a_i$ with $a_i \in A_i$ for all i and $a_i \neq 0$ for only finitely many i.

1. Additive systems

Let **N**, **N**₀, and **Z** denote the sets of positive integers, nonnegative integers, and all integers, respectively. For integers a and b with a < b, we define the *intervals of integers* $[a, b] = \{n \in \mathbf{Z} : a \le n \le b\}$ and $[a, b) = \{n \in \mathbf{Z} : a \le n < b\}$. For $A \subseteq \mathbf{Z}$ and $g \in \mathbf{Z}$, the *dilation* of the set A by g is the set $g * A = \{ga : a \in A\}$.

Let I be a nonempty finite or infinite set, and let $\mathcal{A} = (A_i)_{i \in I}$ be a family of sets of integers with $0 \in A_i$ and $|A_i| \geq 2$ for all $i \in I$. We may also call \mathcal{A} a sequence if $I = \mathbf{N}$ or if I is an interval of integers. Each set A_i can be finite or infinite. We say that a set X belongs to \mathcal{A} if $X = A_i$ for some $i \in I$. The sumset $S = \sum_{i \in I} A_i$ is the set of all integers n that can be represented in the form $n = \sum_{i \in I} a_i$, where $a_i \in A_i$ for all $i \in I$ and $a_i \neq 0$ for only finitely many $i \in I$. If every element of S has a unique representation in the form $n = \sum_{i \in I} a_i$, then we call \mathcal{A} a unique representation system for S, and we write $S = \bigoplus_{i \in I} A_i$. If \mathcal{A} is a unique representation system for S, then $A_i \cap A_j = \{0\}$ for all $i \neq j$. The condition $|A_i| \geq 2$ for all $i \in I$ implies that $A_i = S$ for some $i \in I$ if and only if |I| = 1. Moreover, if $I^{\flat} \subseteq I$ and $S = \sum_{i \in I^{\flat}} A_i$, then $S = \bigoplus_{i \in I^{\flat}} A_i$ and $I = I^{\flat}$.

The family $\mathcal{A} = (A_i)_{i \in I}$ is an *additive system* if \mathcal{A} is a unique representation system for the set of nonnegative integers. Equivalently, \mathcal{A} is an additive system if $\mathbf{N}_0 = \bigoplus_{i \in I} A_i$.

The object of this paper is to prove a beautiful theorem of deBruijn in additive number theory that completely classifies additive systems. The only number theory used in the proof is the *division algorithm*: For every positive integer g and for every integer n there exist unique integers x and r with $r \in [0, g)$ such that n = gx + r.

Example 1: For $g \ge 2$, let

$$A_1 = \{0, 1, 2, \dots, g - 1\} = [0, g]$$

and

$$A_2 = \{0, g, 2g, 3g, 4g, \ldots\} = g * \mathbf{N}_0.$$

²⁰¹⁰ Mathematics Subject Classification. Primary 11A05, 11B75.

Supported in part by a grant from the PSC-CUNY Research Award Program.

The division algorithm implies that $\mathcal{A} = (A_i)_{i \in [1,2]}$ is an additive system. More generally, let $\mathcal{A} = (A_i)_{i \in I}$ be an additive system. Let $I_1 = I \cup \{i_1\}$, where $i_1 \notin I$, and define the sets $A'_{i_1} = [0,g)$ and $A'_i = g * A_i$ for all $i \in I$. Again, the division algorithm implies that $\mathcal{A}' = (A'_i)_{i \in I_1}$ is an additive system. We call \mathcal{A}' the *dilation* of the additive system \mathcal{A} by the integer g, and we write $\mathcal{A}' = g * \mathcal{A}$.

Example 2: For i = 1, 2, 3, ..., let

$$B_i = \{0, 2^{i-1}\} = 2^{i-1} * [0, 2].$$

Because every nonnegative integer can be written uniquely as a finite sum of pairwise distinct powers of 2, the family $\mathcal{B} = (B_i)_{i \in \mathbb{N}}$ is an additive system, called the *binary number system*. More generally, for any integer $g \geq 2$, let

$$C_i = g^{i-1} * [0,g)$$

for i = 1, 2, 3, ... The additive system $C = \{C_i\}_{i \in \mathbb{N}}$ is the *g*-adic number system. Example 3: Let

$$M_1 = \{0, 1, 2, 3, \dots, 11\} = [0, 12)$$

$$M_2 = \{0, 12, 24, 36, \dots, 228\} = 12 * [0, 20)$$

$$M_3 = \{0, 240, 480, 720, 960, \dots\} = 240 * \mathbf{N}_0$$

Applying the division algorithm with r = 2, $g_1 = 12$ and $g_2 = 20$, we see that $\mathcal{M} = (M_i)_{i \in [1,3]}$ is an additive system. For example,

$$835 = 7 + 108 + 720 = 1 \cdot 7 + 12 \cdot 9 + 240 \cdot 3 \in \sum_{i \in [1,3]} M_i.$$

In pre-1971 British currency, there were 20 shillings in a pound and 12 pence (or pennies) in a shilling, hence 240 pence in a pound. Thus, 835 pence were equal to 3 pounds, 9 shillings, and 7 pence. The additive system \mathcal{M} is the old *British monetary system*.

The following result generalizes Example 3.

Theorem 1. Let $r \in \mathbf{N}$ and let $(g_i)_{i \in [1,r]}$ be a finite sequence of not necessarily distinct integers such that $g_i \geq 2$ for all $i \in [1,r]$. Let $G_0 = 1$ and $G_i = \prod_{j=1}^i g_j$ for $i \in [1,r]$. Then

(1)
$$[0,G_r) = \bigoplus_{i \in [1,r]} G_{i-1} * [0,g_i)$$

and

(2)
$$\mathbf{N}_0 = \bigoplus_{i \in [1,r]} G_{i-1} * [0,g_i) \oplus G_r * \mathbf{N}_0.$$

Thus, the family $(G_{i-1} * [0, g_i))_{i \in [1,r]}$ is a unique representation system for the interval $[0, G_r)$, and this family together with the set $G_r * \mathbf{N}_0$ is an additive system.

Proof. The proof is by induction on r. The case r = 1 is Example 1.

Let $r \ge 2$ and assume the Theorem holds for r-1. For $n \in \mathbf{N}_0$ there are unique integers $x_1, \ldots, x_{r-1}, x'_r$ with $x_i \in [0, g_i)$ for $i \in [1, r-1)$ and $x'_r \in \mathbf{N}_0$ such that

$$n = \sum_{i=1}^{r-1} G_{i-1} x_i + G_{r-1} x'_r.$$

Applying the division algorithm to x'_r , we obtain unique integers $x_r \in [0, g_r)$ and $x_{r+1} \in \mathbf{N}_0$ such that $x'_r = x_r + g_r x_{r+1}$, and so

$$n = \sum_{i=1}^{r-1} G_{i-1}x_i + G_{r-1}(x_r + g_r x_{r+1}) = \sum_{i=1}^r G_{i-1}x_i + G_r x_{r+1}.$$

The inequality

$$0 \le \sum_{i=1}^{r} G_{i-1} x_i \le \sum_{i=1}^{r} G_{i-1} (g_i - 1) = \sum_{i=1}^{r} G_i - \sum_{i=1}^{r} G_{i-1} = G_r - 1$$

implies that $n \in [0, G_r)$ if and only if $x_{r+1} = 0$. This completes the proof.

2. DILATION AND CONTRACTION

In this section we describe two operations on additive systems that produce new additive systems. Let $\mathcal{A} = (A_i)_{i \in I}$ be an additive system, Without loss of generality, and for simplicity of notation, we shall assume that $I \cap \mathbf{N} = \emptyset$.

In Example 1 we described the dilation of the additive system by an integer $g \ge 2$. We define *dilation by a finite family* $(g_i)_{i \in [1,r]}$ of integers $g_i \ge 2$ by iterated dilation by integers:

$$(g_i)_{i \in [1,r]} * \mathcal{A} = g_1 * (g_2 * (\dots * (g_{r-1} * (g_r * \mathcal{A}))) \dots)) = (A'_i)_{i \in [1,r] \cup I}$$

where

$$A'_{i} = \begin{cases} G_{i-1} * [0, g_{i}) & \text{ if } i \in [1, r] \\ G_{r} * A_{i} & \text{ if } i \in I. \end{cases}$$

and $G_0 = 1$ and $G_i = \prod_{j \in [1,i]} g_j$ for $i \in [1,r]$.

Note that dilation of additive systems by finite families of integers is not commutative. For example, if $g_1 \neq g_2$, then $g_1 * (g_2 * \mathcal{A})$ consists of $(g_1g_2 * A_i)_{i \in I}$ and the sets $[0, g_1)$ and $g_1 * [0, g_2)$, while $g_2 * (g_1 * \mathcal{A})$ consists of the sets $(g_1g_2 * A_i)_{i \in I}$ and the sets $[0, g_2)$ and $g_2 * [0, g_1)$. Because $[0, g_1) \neq [0, g_2)$, it follows that $(g_i)_{i \in [1,2]} * \mathcal{A} \neq (g_{3-i})_{i \in [1,2]} * \mathcal{A}$.

The following two lemmas follow immediately from the definition of dilation and the definition of additive system, respectively. The first lemma shows that the dilation of a dilation is a dilation, or, equivalently, that dilation is associative. The second shows that partitioning an index set produces a new additive system.

Lemma 1. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be additive systems. If the additive system \mathcal{A} is a dilation of the additive system \mathcal{B} by the finite sequence $(g_i)_{i \in [1,r]}$, and if \mathcal{B} is a dilation of the additive system \mathcal{C} by the finite sequence $(g'_j)_{j \in [1,s]}$, then \mathcal{A} is a dilation of the additive system \mathcal{C} dilated by $(g_i)_{i \in [1,r+s]}$, where $g_{r+j} = g'_j$ for $j \in [1,s]$.

Lemma 2. Let $\mathcal{B} = (B_j)_{j \in J}$ be an additive system. If $\{J_i\}_{i \in I}$ is a partition of J into pairwise disjoint nonempty sets, and if

$$A_i = \sum_{j \in J_i} B_j$$

then $\mathcal{A} = (A_i)_{i \in I}$ is an additive system.

An additive system \mathcal{A} obtained from an additive system \mathcal{B} by the partition procedure described in Lemma 2 is called a *contraction* of \mathcal{B} . (In [4], de Bruijn called \mathcal{A} a *degeneration* of \mathcal{B} .) The set I in Lemma 2 can be finite or infinite. If I = J and σ is a permutation of J such that $J_i = \{\sigma(i)\}$ for all $i \in J$, then \mathcal{A} and \mathcal{B} contain exactly the same sets. Thus, every additive system is a contraction of itself. An additive system \mathcal{A} is a *proper contraction* of \mathcal{B} if at least one set $A_i \in \mathcal{A}$ is the sum of at least two sets in \mathcal{B} .

If $I = \{1\}$ and $J_1 = J$, then $A_1 = \mathbf{N}_0$. Thus, the additive system (\mathbf{N}_0) is a contraction of every additive system.

The following Lemma shows that the contraction of a contraction is a contraction:

Lemma 3. If $\mathcal{A} = (A_i)_{i \in I}$, $\mathcal{B} = (B_j)_{j \in J}$, and $\mathcal{C} = (C_k)_{k \in K}$ are additive systems such that \mathcal{A} is a contraction of \mathcal{B} and \mathcal{B} is a contraction of \mathcal{C} , then \mathcal{A} is a contraction of \mathcal{C} .

Proof. Because \mathcal{A} is a contraction of \mathcal{B} , there exists a partition $\{J_i : i \in I\}$ of J such that $A_i = \sum_{j \in J_i} B_j$ for all $i \in I$. Because \mathcal{B} is a contraction of \mathcal{C} , there exists a partition $\{K_j : j \in J\}$ of K such that $B_j = \sum_{k \in K_j} C_k$ for all $j \in J$. Then

$$A_i = \sum_{j \in J_i} B_j = \sum_{j \in J_i} \sum_{k \in K_j} C_k = \sum_{k \in L_i} C_k$$

where

$$L_i = \bigcup_{j \in J_i} K_j \subseteq K$$

and

$$\bigcup_{i \in I} L_i = \bigcup_{i \in I} \bigcup_{j \in J_i} K_j = \bigcup_{j \in J} K_j = K.$$

We shall show the sets in $\{L_i : i \in I\}$ are pairwise disjoint.

Let $k \in K$. If $i_1, i_2 \in I$ and $k \in L_{i_1} \cap L_{i_2}$, then $k \in K_{j_1}$ for some $j_1 \in J_{i_1}$ and $k \in K_{j_2}$ for some $j_2 \in J_{i_2}$. Because $\{K_j : j \in J\}$ is a set of pairwise disjoint sets and $K_{j_1} \cap K_{j_2} \neq \emptyset$, it follows that $j_1 = j_2$ and so $J_{i_1} \cap J_{i_2} \neq \emptyset$. Because $\{J_i : i \in I\}$ is a set of pairwise disjoint sets, it follows that $i_1 = i_2$, and so the sets in $\{L_i : i \in I\}$ are pairwise disjoint. Thus, $\{L_i : i \in I\}$ is a partition of K, and the additive system \mathcal{A} is a contraction of \mathcal{C} . This completes the proof. \Box

Let \mathcal{A} and \mathcal{B} be additive systems, let $r \in \mathbf{N}$, and let $(g_i)_{i \in [1,r]}$ be a finite sequence of integers $g_i \geq 2$. The expression " \mathcal{A} is a contraction of \mathcal{B} dilated by $(g_i)_{i \in [1,r]}$ " means that \mathcal{A} is the additive system obtained by first dilating \mathcal{B} by $(g_i)_{i \in [1,r]}$ and then contracting the dilated system. It was not hard to prove that a "dilation of a dilation" is a dilation (Lemma 1) or that a "contraction of a contraction is a contraction" (Lemma 3). It is more challenging to prove that a "contraction of a dilation.

Lemma 4. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be additive systems. If the additive system \mathcal{A} is a contraction of the additive system \mathcal{B} dilated by the finite sequence $(g_i)_{i \in [1,r]}$, and if \mathcal{B} is a contraction of the additive system \mathcal{C} dilated by the finite sequence $(g'_j)_{j \in [1,s]}$, then \mathcal{A} is a contraction of the additive system \mathcal{C} dilated by $(g_i)_{i \in [1,r+s]}$, where $g_{r+j} = g'_j$ for $j \in [1, s]$.

Proof. See Appendix A.

Lemma 5. Let $(\mathcal{A}_i)_{i \in [0,n]}$ be a sequence of additive systems and let $(g_i)_{i \in [1,n]}$ be a finite sequence of integers $g_i \geq 2$ such that \mathcal{A}_{i-1} is a contraction of \mathcal{A}_i dilated by g_i for all $i \in [1, n]$. Then \mathcal{A}_0 is a contraction of \mathcal{A}_n dilated by the sequence $(g_i)_{i \in [1,n]}$.

Proof. This follows from Lemma 4 by induction on n.

3. British number systems

In this section we describe certain additive systems that de Bruijn called *British* number systems. A British number system is an additive system constructed from an infinite sequence of integers according to the algorithm in the following theorem.

Theorem 2. Let $(g_i)_{i \in \mathbb{N}}$ be an infinite sequence of integers such that $g_i \geq 2$ for all $i \geq 1$. Let $G_0 = 1$ and, for $i \in \mathbb{N}$, let $G_i = \prod_{i=1}^{i} g_i$ and

$$A_i = \{0, G_{i-1}, 2G_{i-1}, \dots, (g_i - 1)G_{i-1}\} = G_{i-1} * [0, g_i].$$

Then $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ is an additive system.

Proof. If $n \in \mathbf{N}_0$, then $n \in [0, G_r)$ for some sufficiently large integer r. By Theorem 1, there exist unique integers $a_i \in A_i$ for $i = 1, \ldots, r$ such that $n = \sum_{i=1}^r a_i \in \bigoplus_{i=1}^r A_i$. Because $a \ge G_r$ for all $a \in \left(\bigcup_{i \in \mathbf{N} \setminus [1,r]} A_i\right) \setminus \{0\}$, it follows that n has a unique representation in the form $\sum_{i \in \mathbf{N}} a_i$ with $a_i \in A_i$ for all $i \in \mathbf{N}$, and so $\mathcal{A} = (A_i)_{i \in \mathbf{N}}$ is an additive system.

We write that the sequence $(g_i)_{i \in \mathbb{N}}$ generates the British number system \mathcal{A} if \mathcal{A} is constructed from $(g_i)_{i \in \mathbb{N}}$ according to the algorithm in Theorem 2.

Lemma 6. If $(g_i)_{i \in \mathbb{N}}$ generates the British number system $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ and if $(g'_i)_{i \in \mathbb{N}}$ generates the British number system $\mathcal{A}' = (A'_i)_{i \in \mathbb{N}}$, then $\mathcal{A} = \mathcal{A}'$ if and only if $g_i = g'_i$ for all $i \in \mathbb{N}$.

Thus, there is a one-to-one correspondence between British number systems and integer sequences $(g_i)_{i \in \mathbb{N}}$ satisfying $g_i \ge 2$ for all $i \in \mathbb{N}$.

Proof. If $A_1 = A'_1$, then $[0, g_1) = [0, g'_1)$ and so $g_1 = g'_1$. If $r \ge 2$ and $g_i = g'_i$ for all $i \le r - 1$, then

$$G_{r-1} = \prod_{i \in [1, r-1]} g_i = \prod_{i \in [1, r-1]} g'_i = G'_{r-1}.$$

If $A_r = A'_r$, then

$$G_{r-1} * [0, g_r) = G'_{r-1} * [0, g'_r) = G_{r-1} * [0, g'_r)$$

and so $g_r = g'_r$. If $\mathcal{A} = \mathcal{A}'$, then it follows by induction that $g_i = g'_i$ for all $i \in \mathbb{N}$. \Box

de Bruijn's theorem is that *every* additive system is a contraction of a British number system. The proof depends on the following fundamental lemma.

Lemma 7. Let $\mathcal{A} = (A_i)_{i \in I}$ be an additive system. If $|I| \ge 2$, then there exist $i_1 \in I$, an integer $g \ge 2$, and a family of sets $\mathcal{B} = (B_i)_{i \in I}$ such that

$$A_{i_1} = [0,g) \oplus g * B_{i_1}$$

and, for all $i \in I \setminus \{i_1\}$,

$$A_i = q * B_i.$$

If $B_{i_1} = \{0\}$, then $\mathcal{B} = (B_i)_{i \in I \setminus \{i_1\}}$ is an additive system, and \mathcal{A} is the dilation of the additive system \mathcal{B} by the integer g. If $B_{i_1} \neq \{0\}$, then $\mathcal{B} = (B_i)_{i \in I}$ is an additive system and \mathcal{A} is a contraction of the additive system \mathcal{B} dilated by g. *Proof.* The inequality $|I| \geq 2$ implies that $A_i \neq \mathbf{N}_0$ for all $i \in I$. Because $1 \in \sum_{i \in I} A_i$, it follows that $1 \in A_{i_1}$ for some $i_1 \in I$. Because $A_{i_1} \neq \mathbf{N}_0$, there is a smallest positive integer g such that $g \notin A_{i_1}$. Then $g \geq 2$ and $[0,g) \subseteq A_{i_1}$. The sets in the family $(A_i \setminus \{0\})_{i \in I}$ are pairwise disjoint, and so $\{1, \ldots, g-1\} \cap A_i = \emptyset$ for all $i \in I \setminus \{i_1\}$.

We have $g = \sum_{i \in I} a_i \in \sum_{i \in I} A_i$, with $0 \le a_i \le g$ for all $i \in I$. If $1 \le a_{i_1} \le g-1$, then there must exist $j \in I \setminus \{i_1\}$ such that $1 \le a_j \le g-1$, which is absurd. Therefore, $a_{i_2} = g$ for some $i_2 \in I \setminus \{i_1\}$ and $a_i = 0$ for all $i \in I \setminus \{i_2\}$.

Let $r \in \{1, 2, \dots, g - 1\}$. Then

$$r+g \in A_{i_1} + A_{i_2} \subseteq \sum_{i \in I} A_i.$$

Because the representation of an integer in $\sum_{i \in I} A_i$ is unique, it follows that $r + g \notin A_i$ for all $i \in I$.

We shall prove that for every nonnegative integer k the following holds:

- (i) $[kg+1, (k+1)g) \cap \bigcup_{i \in I \setminus \{i_1\}} A_i = \emptyset$,
- (ii) If $[kg, (k+1)g) \cap A_{i_1} \neq \emptyset$, then $[kg, (k+1)g) \subseteq A_{i_1}$.

The proof is by induction on k. Statements (i) and (ii) already been verified for k = 0 and k = 1. Let $k \ge 2$ and assume that statements (i) and (ii) are true for all nonnegative integers k' < k.

For each $i \in I$ there exists $a_i \in A_i$ such that $0 \le a_i \le kg$ and

$$kg = \sum_{i \in I} a_i = a_{i_1} + \sum_{i \in I \setminus \{i_1\}} a_i.$$

By the induction hypothesis, $k'g + r \notin \bigcup_{i \in I \setminus \{i_1\}} A_i$ for all $k' \in [0, k)$ and $r \in [1, g)$. Therefore, $a_i \equiv 0 \pmod{g}$ for all $i \in I \setminus \{i_1\}$, and so $a_{i_1} \equiv 0 \pmod{g}$.

There are two cases. In the first case we have $kg \notin A_{i_1}$, and so $a_{i_1} = k'g$ for some nonnegative integer k' < k. By the induction hypothesis, $a_{i_1} + r = k'g + r \in A_{i_1}$ for all $r \in [1, g)$, and so

$$kg+r=(a_{i_1}+r)+\sum_{i\in I\backslash\{i_1\}}a_i\in \sum_{i\in I}A_i.$$

Because the integer kg + r has a unique representation in the sumset $\sum_{i \in I} A_i$, it follows that $kg + r \notin \bigcup_{i \in I} A_i$ for all $r \in [1, g)$.

In the second case we have $kg \in A_{i_1}$. Because $g \in A_{i_2}$, we have

$$(k+1)g = kg + g \in A_{i_1} + A_{i_2} \subseteq \sum_{i \in I} A_i.$$

Let $r \in \{1, 2, \dots, g-1\}$. Because $\{1, 2, \dots, g-1\} \subseteq A_{i_1}$, it follows that $g-r \in A_{i_1}$. If $kg + r \in A_{i_3}$ for some $r \in \{1, 2, \dots, g-1\}$ and $i_3 \neq i_1$, then

$$(k+1)g = (g-r) + (kg+r) \in A_{i_1} + A_{i_3} \subseteq \sum_{i \in I} A_i.$$

This gives two distinct representations of (k + 1)g in $\sum_{i \in I} A_i$, which is absurd. Therefore, $kg + r \notin A_i$ for all $i \in I \setminus \{i_1\}$. Thus, if $a_i \in A_i$ for $i \in I \setminus \{i_1\}$ and $a_i < (k + 1)g$, then $a_i \equiv 0 \pmod{g}$. Writing

$$kg+r=a_{i_1}+\sum_{i\in I\backslash\{i_1\}}a_i\in \sum_{i\in I}A_i$$

we conclude that there exists a nonnegative integer $\ell \leq k$ such that

$$a_{i_1} = \ell g + r \in A_{i_1}$$

and

$$\sum_{\in I \setminus \{i_1\}} a_i = (k - \ell)g.$$

If $\ell < k$, then the induction hypothesis implies that $\ell g \in A_{i_1}$ and so

 $i \in$

$$\ell g + \sum_{i \in I \setminus \{i_1\}} a_i = kg,$$

which is impossible since $kg \in A_{i_1}$. Therefore, $\ell = k$ and $kg + r \in A_{i_1}$ for all $r \in [0, g)$ This completes the induction.

For each $i \in I$, let $B_i = \{k \in \mathbf{N}_0 : kg \in A_i\}$. Then

(3)
$$A_{i_1} = [0,g) \oplus g * B_{i_1}$$

and, for every $i \in I \setminus \{i_1\}$,

$$A_i = g * B_i.$$

Let $n \in N_0$. There is a unique sequence of integers $(b_i)_{i \in I}$ with $b_i \in B_i$ for all $i \in I$ such that

$$1 + gn = (1 + gb_{i_1}) + \sum_{i \in I \setminus \{i_1\}} gb_i \in \sum_{i \in I} A_1.$$

It follows that $n = \sum_{i \in I} b_i \in \sum_{i \in I} B_i$. If $B_{i_1} = \{0\}$, then $A_{i_1} = [0, g - 1)$ and $\mathcal{B} = (b_i)_{i \in I \setminus \{i_1\}}$ is an additive system. Thus, \mathcal{A} is the dilation of the additive system \mathcal{B} by the integer g.

If $B_{i_1} \neq \{0\}$, then $\mathcal{B} = (b_i)_{i \in I}$ is an additive system and the decomposition (3) shows that \mathcal{A} is a contraction of the additive system \mathcal{B} dilated by the integer g. This completes the proof.

We can now prove de Bruijn's theorem.

Theorem 3. Every additive system is a British number system or a proper contraction of a British number system.

Proof. Let $\mathcal{A} = (A_i)_{i \in I}$ be an additive system, where, as usual, we assume that $I \cap \mathbf{N} = \emptyset$. If |I| = 1, then the additive system \mathcal{A} consists of the single set \mathbf{N}_0 , and \mathbf{N}_0 is a proper contraction of every British number system.

Let $\mathcal{A} = \mathcal{A}_0$. If $|I| \geq 2$, then Lemma 7 produces an additive system $\mathcal{A}_1 = (A_{i,1})_{i \in I_1}$, with $I_1 \subseteq I$, and an integer $g_1 \geq 2$, such that \mathcal{A}_0 is a contraction of \mathcal{A}_1 dilated by g_1 .

Let $r \geq 1$, and suppose that we have constructed a sequence $(g_i)_{i \in [1,r]}$ of integers $g_i \geq 2$ and a sequence of additive systems $(\mathcal{A}_i)_{i \in [0,r]}$ such that \mathcal{A}_{i-1} is a contraction of \mathcal{A}_i dilated by g_i for all $i \in [1,r]$. If $\mathcal{A}_r = (\mathcal{A}_{i,r})_{i \in I_r}$ and $|I_r| \geq 2$, then there is an additive system $\mathcal{A}_{r+1} = (\mathcal{A}_{i,1})_{i \in I_{r+1}}$, with $I_{r+1} \subseteq I_r \subseteq I$, and an integer $g_{r+1} \geq 2$ such that \mathcal{A}_r is a contraction of \mathcal{A}_{r+1} dilated by g_{r+1} .

There are two cases. In the first case, the process of constructing \mathcal{A}_{r+1} from \mathcal{A}_r terminates after *n* steps. This means that, after constructing the finite sequence of additive systems $(\mathcal{A}_i)_{i \in [0,n]}$, we obtain $\mathcal{A}_n = (\mathcal{A}_{i,n})_{i \in I_n}$ with $|I_n| = 1$, that is, \mathcal{A}_n is the additive system consisting only of the set \mathbf{N}_0 . By Lemma 5, \mathcal{A} is a contraction of a dilation of (\mathbf{N}_0) by the sequence $(g_i)_{i \in [1,n]}$. Because (\mathbf{N}_0) is a contraction of every British number system, it follows that \mathcal{A} is also a contraction of a British number system.

In the second case, the process of constructing \mathcal{A}_{r+1} from \mathcal{A}_r never terminates, and we obtain an infinite sequence $(\mathcal{A}_i)_{i\in\mathbb{N}}$ of additive systems and an infinite sequence $(g_i)_{i\in\mathbb{N}}$ of integers $g_i \geq 2$ such that \mathcal{A}_{i-1} is a contraction of the dilation of \mathcal{A}_i by g_i for all $i \in \mathbb{N}$. By Lemma 5, we know that, for every positive integer n, the additive system \mathcal{A} is a contraction of \mathcal{A}_n dilated by the sequence $(g_i)_{i\in[1,n]}$.

Recall that the additive system $\mathcal{A}_n = (A_{i,n})_{i \in I_n}$ dilated by the sequence $(g_i)_{i \in [1,n]}$ consists of the sets $G_{i-1} * [0, g_i)$ for $i \in [1, n]$ and $G_n * A_{i,n}$ for $i \in I$.

Let \mathcal{A}^{\flat} be the British number system generated by the infinite sequence $(g_i)_{i \in \mathbb{N}}$. We must prove that \mathcal{A} is a contraction of \mathcal{A}^{\flat} . Equivalently, we must construct a partition $(L_i)_{i \in I}$ of \mathbb{N} into pairwise disjoint nonempty sets such that

(4)
$$A_{i} = \sum_{n \in L_{i}} G_{n-1} * [0, g_{n})$$

for all $i \in I$. Let

$$L_i = \{ n \in \mathbf{N} : G_{n-1} \in A_i \}$$

and

$$I^{\flat} = \{ i \in I : L_i \neq \emptyset \}.$$

Let $n \in \mathbf{N}$. The additive system \mathcal{A} is a contraction of the additive system \mathcal{A}_n , and $G_{n-1} * [0, g_n)$ is a set in \mathcal{A}_n . Therefore, the set $G_{n-1} * [0, g_n)$ is a summand in some set A_i in \mathcal{A} . Because

$$G_{n-1} \in G_{n-1} * [0, g_n) \subseteq A_i$$

it follows that $n \in L_i$ and so $\mathbf{N} = \bigcup_{i \in I^{\flat}} L_i$. Because the sets $(A_i)_{i \in I}$ are pairwise disjoint, it follows that there is a unique $i \in I^{\flat}$ such that $G_{n-1} \in A_i$ and $n \in L_i$, and so $(L_i)_{i \in I^{\flat}}$ is a partition of \mathbf{N} into nonempty, pairwise disjoint sets.

Let $i \in I$ and $x \in A_i \setminus \{0\}$. Then $1 \leq x < G_N$ for some $N \in \mathbb{N}$. Because \mathcal{A} is a contraction of \mathcal{A}_N , the set A_i is a sum of sets of the form $G_{n-1} * [0, g_n)$ with $n \in [1, N]$ and sets all of whose positive elements are greater than or equal to G_N . It follows that there is a nonempty subset J of [1, N] such that $x = \sum_{n \in J} G_{n-1} x_n$, with $x_n \in [1, g_n)$ and $G_{n-1} x_n \in A_i$ for all $n \in J$. This is possible only if $G_{n-1} \in G_{n-1} * [0, g_n) \subseteq A_i$ for all $n \in J$, and so $J \subseteq L_i$ and $x \in \sum_{n \in L_i} G_{n-1} * [0, g_n)$, that is,

(5)
$$A_i \subseteq \sum_{n \in L_i} G_{n-1} * [0, g_n).$$

Moreover, $L_i \neq \emptyset$ implies that $i \in I^{\flat}$ and so $I^{\flat} = I$. Conversely, if $i \in I$ and $n \in L_i$, then $G_{n-1} * [0, g_n) \subseteq A_i$ and so

(6)
$$\sum_{n \in L_i} G_{n-1} * [0, g_n) \subseteq A_i.$$

The set inclusions (5) and (6) imply (4). This proves that \mathcal{A} is a contraction of the British number system \mathcal{A}^{\flat} .

4. Remarks and open problems

Remark 1: The set A of integers is *decomposable* if there exist sets B and C such that $|B| \ge 2$, $|C| \ge 2$, and $A = B \oplus C$. An *indecomposable set* is a set that does not decompose. An *indecomposable additive system* is an additive system $\mathcal{A} = (A_i)_{i \in I}$ in which every set A_i is indecomposable. Equivalently, an indecomposable additive system is an additive system that is not a proper contraction of another additive system. The following result classifies indecomposable additive systems.

Theorem 4 (Nathanson [14]). Every infinite sequence of prime numbers generates an indecomposable British number system, and every indecomposable additive system is a British number system generated by an infinite sequence of prime numbers. There is a one-to-one correspondence between infinite sequences of prime numbers and indecomposable British number systems. Moreover, every additive system is either indecomposable or a contraction of an indecomposable system.

Remark 2: Let X be a nonempty set. The *free monoid* on X is the set $\mathcal{M}(X)$ consisting of all finite sequences of elements of X, and also an element e (the "empty sequence"), with the binary operation of *concatenation*. We define the product of the nonempty sequences $(g_i)_{i \in [1,r]}$ and $(g'_i)_{j \in [1,s]}$ as follows:

$$(g_i)_{i \in [1,r]} * (g'_j)_{j \in [1,s]} = (g''_k)_{k \in [1,r+s]}$$

where

$$g_k'' = \begin{cases} g_k & \text{if } k \in [1, r] \\ g_{k-r}' & \text{if } k \in [r+1, r+s] \end{cases}$$

and we define ee = e and

$$(g_i)_{i \in [1,r]} e = e(g_i)_{i \in [1,r]} = (g_i)_{i \in [1,r]}.$$

The isomorphism class of the free monoid $\mathcal{M}(X)$ depends only on the cardinality of X. Lemma 1 states that the free monoid on the set $\mathbf{N} \setminus \{1\}$ acts by dilation on the set of additive systems.

Remark 3: Additive systems for the nonnegative integers are part of the general study of sumsets. If A and B are sets of integers, then their sumset is the set $A + B = \{a + b : a \in A \text{ and } b \in B\}$. It is, in general, difficult to determine if a set of integers is a sumset or "almost" a sumset, or to determine if a set is decomposable. Here are some open problems: Let C be a nonempty finite or infinite set of integers.

- (1) Do there exist sets A and B with $|A| \ge 2$, $|B| \ge 2$, and $A \oplus B = C$?
- (2) Do there exist sets A and B with $|A| \ge 2$, $|B| \ge 2$, and A + B = C?
- (3) Do there exist sets A and B with $|A| \ge 2$ and $|B| \ge 2$ such that $A + B \subseteq C$ and $C \setminus (A + B)$ is "small"?
- (4) Do there exist sets A and B with $|A| \ge 2$ and $|B| \ge 2$ such that $A + B \supseteq C$ and $(A + B) \setminus C$ is "small"?
- (5) Does there exist a set A with $|A| \ge 2$ and A + A = C?
- (6) Does there exist a set A with $|A| \ge 2$ such that $A + A \subseteq C$ and $C \setminus (A + A)$ is "small"?
- (7) Does there exist a set A with $|A| \ge 2$ such that $A + A \supseteq C$ and $(A + A) \setminus C$ is "small"?

These problems are related to Freiman's theorem [8] and other inverse problems in additive number theory (cf. Nathanson [11] and Tao and Vu [18]).

Remark 4: It is natural to investigate additive systems for the additive group \mathbf{Z} of integers, that is, sequences $(A_i)_{i \in I}$ of sets of integers such that $0 \in A_i$ and $|A_i| \geq 2$ for all $i \in I$, and $\mathbf{Z} = \bigoplus_{i \in I} A_i$. For example, if $a_i = \varepsilon_i 2^{i-1}$ with $\varepsilon_i \in \{1, -1\}$ for all $i \in \mathbf{N}$, then $(\{0, a_i\})_{i \in \mathbf{N}}$ is an additive system for \mathbf{Z} if and only if $\varepsilon_i = 1$ for infinitely many i and $\varepsilon_i = -1$ for infinitely many i. The classification problem for additive systems for the integers is unsolved. Even the special case $A_i = \{0, a_i\}$ for all i is difficult. de Bruijn [3] proved the following conjecture of T. Szele: If $(a_i)_{i \in \mathbf{N}}$ is an infinite sequence of nonzero integers such that $(\{0, a_i\})_{i \in \mathbf{N}}$ is an additive system for \mathbf{Z} , then there is a sequence $(d_i)_{i \in \mathbf{N}}$ of odd integers such that, after rearrangement, $a_i = 2^{i-1}d_i$ for all $i \in \mathbf{N}$.

There are many interesting recent results about additive systems for \mathbf{Z} , for example, [1, 5, 6, 7, 10, 17, 19]. However, de Bruijn's remark at the end of his 1956 paper on \mathbf{N}_0 still accurately describes the current state of the problem: "Some years ago the author [3] discussed various aspects of the analogous problem for number systems representing uniquely all integers (without restriction to nonnegative ones). That problem is much more difficult than the one dealt with above [additive systems for \mathbf{N}_0], and it is still far from a complete solution."

Remark 5: The interval identity [0, mn) = [0, m) + m * [0, n), basic to the problem of additive systems for \mathbf{N}_0 , also led to the study of multiplication rules for quantum integers (cf. [2, 9, 12, 13, 15]).

Remark 6: de Bruijn's paper [4] fills less than three pages. He uses but does not explicitly state Lemma 4, which is technically the most difficult step in the proof of the main result (Theorem 3). After proving Lemma 7, de Bruijn writes, "[Theorem 3] easily follows by repeated application of the ...lemma." R. A. Rankin [16] repeated this in his report on de Bruijn's paper in *Mathematical Reviews*: "[Theorem 3] follows from repeated applications of [the] lemma...." Mathematicians, from the humblest graduate student to the grandest Fields medalist, often don't bother to write out justifications for statements that are "obvious" or that "easily follow" from previously proved results. But what is obvious to an author is not necessarily obvious to a reader (and sometimes the "obvious" is false). I prefer not to overindulge the virtue of brevity.

Appendix A. Proof of Lemma 4

Proof. For $k \in [1, s]$, we define $G'_k = \prod_{j=1}^k g'_j = \prod_{i=r+1}^{r+k} g_i$ and for $k \in [1, r+s]$, we define $G_k = \prod_{i=1}^k g_i$. If $k \in [1, s]$, then $G_r G'_k = G_{r+k}$. Let $G_0 = G'_0 = 1$.

Let \mathcal{C}' be the additive system $\mathcal{C} = (C_k)_{k \in K}$ dilated by $(g'_k)_{k \in [1,s]}$. We can assume that $K \cap \mathbf{N} = \emptyset$. From the definition of dilation, we have $\mathcal{C}' = (C'_k)_{k \in K'}$, where $K' = [1, s] \cup K$ and

$$C'_{k} = \begin{cases} G'_{k-1} * [0, g'_{k}) & \text{if } k \in [1, s] \\ G'_{s} * C_{k} & \text{if } k \in K. \end{cases}$$

Let $\mathcal{B} = (B_j)_{j \in J}$ be a contraction of \mathcal{C}' , where $J \cap \mathbf{N} = \emptyset$. This means that there is a partition $(K'_j)_{j \in J}$ of K' such that $K'_j \neq \emptyset$ and

$$B_j = \bigoplus_{k \in K'_j} C'_j$$

for all $j \in J$.

Let \mathcal{B}' be the additive system \mathcal{B} dilated by $(g_i)_{i \in [1,r]}$. Then $\mathcal{B}' = (B'_j)_{j \in J'}$, where $J' = [1,r] \cup J$ and

$$B'_{j} = \begin{cases} G_{j-1} * [0, g_{j}) & \text{if } j \in [1, r] \\ G_{r} * B_{j} & \text{if } j \in J. \end{cases}$$

Because $\mathcal{A} = (A_i)_{i \in I}$ is a contraction of $\mathcal{B}' = (B'_j)_{j \in J'}$, there is a partition $(J'_i)_{i \in I}$ of J' such that, for all $i \in I$, we have $J'_i \neq \emptyset$ and

$$A_{i} = \bigoplus_{j \in J'_{i}} B'_{j} = \left(\bigoplus_{j \in J'_{i} \cap [1,r]} G_{j-1} * [0,g_{j})\right) \oplus \left(\bigoplus_{j \in J'_{i} \setminus [1,r]} G_{r} * B_{j}\right)$$
$$= \left(\bigoplus_{j \in J'_{i} \cap [1,r]} G_{j-1} * [0,g_{j})\right) \oplus \left(\bigoplus_{j \in J'_{i} \setminus [1,r]} \bigoplus_{k \in K'_{j}} G_{r} * C'_{k}\right).$$

Note that $J'_i \setminus [1, r] \subseteq J$. For $j \in J'_i \setminus [1, r]$, we have

$$\bigoplus_{k \in K'_j} G_r * C'_k = \left(\bigoplus_{k \in K'_j \cap [1,s]} G_r * G'_{k-1} * [0,g'_k) \right) \oplus \left(\bigoplus_{k \in K'_j \setminus [1,s]} G_r * G'_s * C_k \right)$$
$$= \left(\bigoplus_{k \in K'_j \cap [1,s]} G_{r+k-1} * [0,g_{r+k}) \right) \oplus \left(\bigoplus_{k \in K'_j \setminus [1,s]} G_{r+s} * C_k \right).$$

It follows that

$$A_{i} = \left(\bigoplus_{j \in J_{i}^{\prime} \cap [1,r]} G_{j-1} * [0,g_{j})\right) \oplus \left(\bigoplus_{j \in J_{i}^{\prime} \setminus [1,r]} \bigoplus_{k \in K_{j}^{\prime} \cap [1,s]} G_{r+k-1} * [0,g_{r+k})\right)$$
$$\oplus \left(\bigoplus_{j \in J_{i}^{\prime} \setminus [1,r]} \bigoplus_{k \in K_{j}^{\prime} \setminus [1,s]} G_{r+s} * C_{k}\right).$$

This is a decomposition of A_i into a sum of sets. We call these sets the *summands* of A_i . The summands of A_i are pairwise distinct sets.

We must prove that $\mathcal{A} = (A_i)_{i \in I}$ is a contraction of the additive system \mathcal{C} dilated by $(g_i)_{i=1}^{r+s}$. This dilated additive system can be written in the form $\mathcal{A}^{\sharp} = (A_k^{\sharp})_{k \in K^{\sharp}}$, where $K^{\sharp} = [1, r+s] \cup K$ and

$$A_{k}^{\sharp} = \begin{cases} G_{k-1} * [0, g_{k}) & \text{if } k \in [1, r+s] \\ G_{r+s} * C_{k} & \text{if } k \in K. \end{cases}$$

Every summand in A_i is equal to A_k^{\sharp} for some $k \in K^{\sharp}$. Thus, it suffices to show that for every $k \in K^{\sharp}$ there is a unique $i \in I$ such that A_k^{\sharp} is a summand in A_i .

The sets in the family $(J'_i)_{i \in I}$ partition $J' = [1, r] \cup J$. Thus, for every $j \in [1, r]$ there is a unique $i \in I$ such that $j \in J'_i \cap [1, r]$, and so there is a unique $i \in I$ such that $G_{j-1} * [0, g_j)$ is a summand in A_i .

Because the sets in the family $(K'_j)_{j\in J}$ partition $K' = [1, s] \cup K$, for every $k \in [1, s]$ there is a unique $j \in J$ such that $k \in K'_j \cap [1, s]$. The sets $(J'_i \setminus [1, r])_{i\in I}$ partition J, and so there is a unique $i \in I$ such that $j \in J'_i \setminus [1, r]$. It follows that there is a unique $i \in I$ such that $G_{r+k-1} * [0, g_{r+k})$ is a summand in A_i .

Let $k \in K$. There is a unique $j \in J$ such that $k \in K'_j \setminus [1, s]$, and there is a unique $i \in I$ such that $j \in J'_i \setminus [1, r]$. It follows that there is a unique $i \in I$ such that $G_{r+s} * C_k$ is a summand in A_i . This proves that \mathcal{A} is a contraction of the additive system \mathcal{A}^{\sharp} . Indeed, defining

$$K_i^{\sharp} = (J_i' \cap [1, r]) \cup \left(\bigcup_{j \in J_i' \setminus [1, r]} \left(r + (K_j' \cap [1, s]) \right) \right) \cup \left(\bigcup_{j \in J_i' \setminus [1, r]} K_j' \setminus [1, s] \right)$$

we obtain a partition $\left(K_{i}^{\sharp}\right)_{i\in[1,r+s]\cup I}$ of K such that

$$A_i = \bigoplus_{k \in K_i^{\sharp}} A_k^{\sharp}$$

for all $i \in I$. This completes the proof.

References

- A. Biró, Divisibility of integer polynomials and tilings of the integers, Acta Arith. 118 (2005), no. 2, 117–127.
- [2] A. Borisov, M. B. Nathanson, and Y. Wang, Quantum integers and cyclotomy, J. Number Theory 109 (2004), no. 1, 120–135.
- [3] N. G. de Bruijn, On bases for the set of integers, Publ. Math. Debrecen 1 (1950), 232–242.
- [4] _____, On number systems, Nieuw Arch. Wisk. (3) 4 (1956), 15–17.
- [5] _____, Some direct decompositions of the set of integers, Math. Comp. 18 (1964), 537–546.
- [6] S. Eigen and A. Hajian, Hereditary tiling sets of the integers, Integers 8 (2008), A54, 9.
- [7] S. J. Eigen, Y. Ito, and V. S. Prasad, Universally bad integers and the 2-adics, J. Number Theory 107 (2004), no. 2, 322–334.
- [8] G. A. Freiman, Foundations of a Structural Theory of Set Addition, Translations of Mathematical Monographs, vol. 37, American Mathematical Society, Providence, R.I., 1973.
- [9] A. V. Kontorovich and M. B. Nathanson, Quadratic addition rules for quantum integers, J. Number Theory 117 (2006), no. 1, 1–13.
- [10] Ž. Ljujić, Periodicity of complementing multisets, Funct. Approx. Comment. Math. 46 (2012), no. part 2, 161–175.
- [11] M. B. Nathanson, Additive Number Theory: Inverse Problems and the Geometry of Sumsets, Springer-Verlag, New York, 1996.
- [12] _____, A functional equation arising from multiplication of quantum integers, J. Number Theory 103 (2003), no. 2, 214–233.
- [13] _____, Semidirect products and functional equations for quantum multiplication, J. Algebra Appl. 10 (2011), no. 5, 827–834.
- [14] _____, Decomposition and limits of additive systems, preprint, 2013.
- [15] L. Nguyen, On the solutions of a functional equation arising from multiplication of quantum integers, J. Number Theory 130 (2010), no. 6, 1292–1347.
- [16] R. A. Rankin, Review of [4], (MathSciNet identifier: MR0077556 (17,1056d)), Mathematical Reviews, Amer. Math. Soc., Providence.
- [17] W. M. Schmidt and D. M. Tuller, Covering and packing in \mathbb{Z}^n and \mathbb{R}^n . I, Monatsh. Math. **153** (2008), no. 3, 265–281.
- [18] T.Tao and V. Vu, Additive Combinatorics, Cambridge Univ. Press, Cambridge, 2006.
- [19] R. Tijdeman, Decomposition of the integers as a direct sum of two subsets, Number Theory (Paris, 1992–1993), Cambridge Univ. Press, Cambridge, 1995, pp. 261–276.

LEHMAN COLLEGE (CUNY), BRONX, NEW YORK 10468 *E-mail address*: melvyn.nathanson@lehman.cuny.edu

12