ON THE CHARACTERIZATION OF GALOIS EXTENSIONS

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ABSTRACT. We present a shortcut to the familiar characterizations of finite Galois extensions, based on an idea from an earlier Monthly note by Sonn and Zassenhaus.

Let $L \supseteq K$ be a field extension and $G := \operatorname{Aut}(L, K)$. We assume throughout that $[L : K] < \infty$. One easily sees that then automatically $|G| < \infty$. The following result is an essential part of the usual development (and teaching) of Galois theory.

Theorem 1. The following conditions are equivalent.

- (a) |G| = [L:K].
- (b) L is a splitting field for a polynomial $0 \neq f \in K[X]$ which does not have multiple roots in L.
- (c) $K = \{y \in L \mid \sigma(y) = y \text{ for all } \sigma \in G\}.$

If these conditions hold, then $L \supseteq K$ is called a *Galois extension*. For example, condition (b) yields a simple criterion for an extension to be a Galois extension, and condition (c) is crucial for many applications of Galois theory. Combining all three immediately shows that, if $L \supseteq K$ is a Galois extension and M is an intermediate field, then $L \supseteq M$ also is a Galois extension and so

 $M = \{ y \in L \mid \sigma(y) = y \text{ for all } \sigma \in H \} \quad \text{where} \quad H := \operatorname{Aut}(L, M) \subseteq G,$

which is a significant part of the "Main Theorem of Galois Theory".

Proofs of Theorem 1 often rely on Dedekind's Lemma on group characters and Artin's Theorem (see [1, Chap. II, \S F]), and some results on normal and separable extensions. Some textbooks (e.g. [2], [4]) use the "Theorem on primitive elements" at an early stage to obtain a shortcut. It is the purpose of this note to point out that there is a different shortcut which avoids using the existence of primitive elements and actually establishes this existence as a by-product (at least for Galois extensions). This only relies on a few basic results about fields (e.g., the degree formula and the uniqueness of splitting fields); no assumptions on the characteristic are required. The starting point is the following observation.

Lemma 2. If $K \subsetneq L$, then L is not the union of finitely many fields M such that $K \subseteq M \subsetneq L$.

Proof. If K is infinite, then each M as above is a proper subspace of the K-vector space L, and it is well-known and easy to prove that a finite-dimensional vector space over an infinite field is not the union of finitely many proper subspaces. Now assume that K is finite. Then L is also finite and so $|L| = p^n$ for some prime p. In this case, it is not enough just to argue with the vector space structure. One could use the fact that the multiplicative group of L is cyclic. Or one can argue as follows. Again, it is well-known and easy to prove that, for every $m \leq n$, there is at most one subfield $M \subseteq L$ such that $|M| = p^m$. (The elements of such a subfield are roots of the polynomial $X^{p^m} - X \in L[X]$.) So the total number of elements in L which lie in proper subfields is at most $1 + p + \cdots + p^{n-1} < p^n = |L|$, as desired.

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Corollary 3. There exists an element $z \in L$ such that $\operatorname{Stab}_G(z) = \{\operatorname{id}\}$.

Proof. If $G = \{id\}$, there is nothing to prove. Now assume that $G \neq \{id\}$. For $id \neq \sigma \in G$ we set $M_{\sigma} := \{y \in L \mid \sigma(y) = y\}$. Then M_{σ} is a field such that $K \subseteq M_{\sigma} \subsetneq L$. Now apply Lemma 2. \Box

Corollary 4. We always have $|G| \leq [L:K]$. If equality holds, then there is some $z \in L$ such that L = K(z) and the minimal polynomial $\mu_z \in K[X]$ has only simple roots in L; furthermore, L is a splitting field for μ_z .

Proof. Let $z \in L$ be as in Corollary 3. Let $G = \{\sigma_1, \ldots, \sigma_m\}$. Then $\{\sigma_i(z) \mid 1 \leq i \leq m\}$ has precisely *m* elements. Now $[L:K] \geq [K(z):K] = \deg(\mu_z)$. Since μ_z has coefficients in *K*, we have $\mu_z(\sigma_i(z)) = \sigma_i(\mu_z(z)) = 0$ for all *i*. So μ_z has at least *m* distinct roots; in particular, $\deg(\mu_z) \geq m = |G|$. This shows that $[L:K] \geq |G|$. If [L:K] = |G|, then all of the above inequalities must be equalities. This yields L = K(z) and $\deg(\mu_z) = m$; in particular, $\mu_z = \prod_{i=1}^m (X - \sigma_i(z))$ has only simple roots and *L* is a splitting field for μ_z .

Corollary 4 shows the implication "(a) \Rightarrow (b)" in Theorem 1 and also establishes the existence of a primitive element. Then the remaining implications in Theorem 1 are proved by standard arguments, which we briefly sketch:

Proof of "(a) \Rightarrow (c)": Let $M := \{y \in L \mid \sigma(y) = y \text{ for all } \sigma \in G\}$. Then M is a field such that $K \subseteq M \subseteq L$ and it is clear from the definitions that $G = \operatorname{Aut}(L, M)$. Hence, Corollary 4 shows that $|G| \leq [L:M] \leq [L:K]$. Since (a) holds, this implies that [L:M] = [L:K] and so M = K.

Proof of "(c) \Rightarrow (b)": Let $L = K(z_1, \ldots, z_m)$ and form the set $B := \{\sigma(z_i) \mid 1 \le i \le m, \sigma \in G\}$. Since (c) holds, we have $f := \prod_{z \in B} (X - z) \in K[X]$; furthermore, L is a splitting field for f, and f has no multiple roots.

Proof of "(b) \Rightarrow (a)": This relies on a standard result on extending field isomorphisms (which is also used to prove that any two splitting fields of a polynomial are isomorphic; see, e.g., [1, Theorem 10]). Using this result and induction on [L:K], it is a simple matter of book-keeping (no further theory required) to construct [L:K] distinct elements of G; the details can be found, for example, in [2, Chap. 14, (5.4)]. This shows that $|G| \ge [L:K]$, and Corollary 4 yields equality.

Once Theorem 1 and Corollary 4 are established, the "Main Theorem of Galois Theory" now follows rather quickly; see [2, Chap. 14, §5] or [4, §9.3].

Remark 5. The idea of looking at elements of L which do not lie in proper subfields is taken from [3].

References

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