# ON THE CHARACTERIZATION OF GALOIS EXTENSIONS 

MEINOLF GECK


#### Abstract

We present a shortcut to the familiar characterizations of finite Galois extensions, based on an idea from an earlier Monthly note by Sonn and Zassenhaus.


Let $L \supseteq K$ be a field extension and $G:=\operatorname{Aut}(L, K)$. We assume throughout that $[L: K]<\infty$. One easily sees that then automatically $|G|<\infty$. The following result is an essential part of the usual development (and teaching) of Galois theory.
Theorem 1. The following conditions are equivalent.
(a) $|G|=[L: K]$.
(b) $L$ is a splitting field for a polynomial $0 \neq f \in K[X]$ which does not have multiple roots in $L$.
(c) $K=\{y \in L \mid \sigma(y)=y$ for all $\sigma \in G\}$.

If these conditions hold, then $L \supseteq K$ is called a Galois extension. For example, condition (b) yields a simple criterion for an extension to be a Galois extension, and condition (c) is crucial for many applications of Galois theory. Combining all three immediately shows that, if $L \supseteq K$ is a Galois extension and $M$ is an intermediate field, then $L \supseteq M$ also is a Galois extension and so

$$
M=\{y \in L \mid \sigma(y)=y \text { for all } \sigma \in H\} \quad \text { where } \quad H:=\operatorname{Aut}(L, M) \subseteq G
$$

which is a significant part of the "Main Theorem of Galois Theory".
Proofs of Theorem 1 often rely on Dedekind's Lemma on group characters and Artin's Theorem (see [1, Chap. II, $\S \mathrm{F}]$ ), and some results on normal and separable extensions. Some textbooks (e.g. [2], 4]) use the "Theorem on primitive elements" at an early stage to obtain a shortcut. It is the purpose of this note to point out that there is a different shortcut which avoids using the existence of primitive elements and actually establishes this existence as a by-product (at least for Galois extensions). This only relies on a few basic results about fields (e.g., the degree formula and the uniqueness of splitting fields); no assumptions on the characteristic are required. The starting point is the following observation.
Lemma 2. If $K \varsubsetneqq L$, then $L$ is not the union of finitely many fields $M$ such that $K \subseteq M \varsubsetneqq L$.
Proof. If $K$ is infinite, then each $M$ as above is a proper subspace of the $K$-vector space $L$, and it is well-known and easy to prove that a finite-dimensional vector space over an infinite field is not the union of finitely many proper subspaces. Now assume that $K$ is finite. Then $L$ is also finite and so $|L|=p^{n}$ for some prime $p$. In this case, it is not enough just to argue with the vector space structure. One could use the fact that the multiplicative group of $L$ is cyclic. Or one can argue as follows. Again, it is well-known and easy to prove that, for every $m \leq n$, there is at most one subfield $M \subseteq L$ such that $|M|=p^{m}$. (The elements of such a subfield are roots of the polynomial $X^{p^{m}}-X \in L[X]$.) So the total number of elements in $L$ which lie in proper subfields is at most $1+p+\cdots+p^{n-1}<p^{n}=|L|$, as desired.

Corollary 3. There exists an element $z \in L$ such that $\operatorname{Stab}_{G}(z)=\{\mathrm{id}\}$.
Proof. If $G=\{\mathrm{id}\}$, there is nothing to prove. Now assume that $G \neq\{\mathrm{id}\}$. For id $\neq \sigma \in G$ we set $M_{\sigma}:=\{y \in L \mid \sigma(y)=y\}$. Then $M_{\sigma}$ is a field such that $K \subseteq M_{\sigma} \varsubsetneqq L$. Now apply Lemma 2,
Corollary 4. We always have $|G| \leq[L: K]$. If equality holds, then there is some $z \in L$ such that $L=K(z)$ and the minimal polynomial $\mu_{z} \in K[X]$ has only simple roots in $L$; furthermore, $L$ is a splitting field for $\mu_{z}$.

Proof. Let $z \in L$ be as in Corollary 3, Let $G=\left\{\sigma_{1}, \ldots, \sigma_{m}\right\}$. Then $\left\{\sigma_{i}(z) \mid 1 \leq i \leq m\right\}$ has precisely $m$ elements. Now $[L: K] \geq[K(z): K]=\operatorname{deg}\left(\mu_{z}\right)$. Since $\mu_{z}$ has coefficients in $K$, we have $\mu_{z}\left(\sigma_{i}(z)\right)=\sigma_{i}\left(\mu_{z}(z)\right)=0$ for all $i$. So $\mu_{z}$ has at least $m$ distinct roots; in particular, $\operatorname{deg}\left(\mu_{z}\right) \geq$ $m=|G|$. This shows that $[L: K] \geq|G|$. If $[L: K]=|G|$, then all of the above inequalities must be equalities. This yields $L=K(z)$ and $\operatorname{deg}\left(\mu_{z}\right)=m$; in particular, $\mu_{z}=\prod_{i=1}^{m}\left(X-\sigma_{i}(z)\right)$ has only simple roots and $L$ is a splitting field for $\mu_{z}$.

Corollary 4 shows the implication "(a) $\Rightarrow$ (b)" in Theorem 1 and also establishes the existence of a primitive element. Then the remaining implications in Theorem 1 are proved by standard arguments, which we briefly sketch:
Proof of "(a) $\Rightarrow \mathbf{( c ) " : ~ L e t ~} M:=\{y \in L \mid \sigma(y)=y$ for all $\sigma \in G\}$. Then $M$ is a field such that $K \subseteq M \subseteq L$ and it is clear from the definitions that $G=\operatorname{Aut}(L, M)$. Hence, Corollary 4 shows that $|G| \leq[L: M] \leq[L: K]$. Since (a) holds, this implies that $[L: M]=[L: K]$ and so $M=K$.
Proof of "(c) $\Rightarrow \mathbf{( b ) " : ~ L e t ~} L=K\left(z_{1}, \ldots, z_{m}\right)$ and form the set $B:=\left\{\sigma\left(z_{i}\right) \mid 1 \leq i \leq m, \sigma \in G\right\}$. Since (c) holds, we have $f:=\prod_{z \in B}(X-z) \in K[X]$; furthermore, $L$ is a splitting field for $f$, and $f$ has no multiple roots.

Proof of "(b) $\Rightarrow$ (a)": This relies on a standard result on extending field isomorphisms (which is also used to prove that any two splitting fields of a polynomial are isomorphic; see, e.g., [1, Theorem 10]). Using this result and induction on $[L: K]$, it is a simple matter of book-keeping (no further theory required) to construct $[L: K]$ distinct elements of $G$; the details can be found, for example, in [2, Chap. 14, (5.4)]. This shows that $|G| \geq[L: K]$, and Corollary 4 yields equality.

Once Theorem 1 and Corollary 4 are established, the "Main Theorem of Galois Theory" now follows rather quickly; see [2, Chap. 14, §5] or [4, §9.3].

Remark 5. The idea of looking at elements of $L$ which do not lie in proper subfields is taken from 3].

## References

[^0]IAZ - Lehrstuhl für Algebra, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany meinolf.geck@mathematik.uni-stuttgart.de


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