

Deleting edges from Ramsey minimal examples - a paper written leaving the “scaffolding in place”

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1. Introduction. A famous remark attributed to Carl F. Gauss is “no self-respecting architect leaves the scaffolding in place after completing the building.” This style of presenting mathematical research is surely elegant and economical; however it gives the reader almost no insight into how the results were actually obtained. In this note, I will describe in some detail how I was led to make a conjecture and the process that finally led me to a proof of the conjecture. The result is in an area of mathematics called Ramsey Theory. Ramsey theory is a flourishing area of graph theory with an enormous number of difficult open problems (see [2]). I am certainly not an expert in Ramsey Theory, never having done any research in this area before, and was not intending to do any original research when I began this project. Rather, I wanted to use some problems from Ramsey Theory to illustrate using Boolean computation for a paper that I was writing for the *Mathematica* Journal[1]. Most of my research involves logic, set theory and some graph theory (see my webpage[2]). The idea for the *Mathematica* paper was to take some combinatorial problems, in this case, some standard problems from Ramsey Theory, translate them into *Mathematica*'s logical language and use its “industrial strength” Boolean computational capability to “solve” these problems. I felt that *Mathematica*'s Boolean capability was under-utilized and wanted to encourage others to make use of it.

2. Ramsey Theory Preliminaries. If n is a positive integer, K_n , denotes the complete graph on n vertices; that is, the graph with n vertices that contains every possible edge between these vertices. If s, t are positive integers, the Ramsey number, $r(s, t)$ is the smallest integer p , such that if the edges of K_n are colored either red or blue, there must be either a red K_s or a blue K_t . Interchanging colors implies, $r(s, t) = r(t, s)$. It is a well-known theorem, due to Frank Ramsey, that p exists, for each s and t (see, for example, [5]). If $r(s, t) = p$, we shall call K_p a *Ramsey Minimal Example*. Thus if K_p is a Ramsey Minimal Example, there is a red/blue edge coloring of K_{p-1} without a red K_s or a blue K_t . Thus, it is natural to ask: how many edges, e , must be removed from K_p before we can red/blue color the edges without getting a red K_s or a blue K_t ? Since K_{p-1} has such a coloring and K_p can be obtained from K_{p-1} by adding a new vertex and $p - 1$ edges, $1 \leq e \leq p - 1$. It is well known that $r(3,3) = 6$ (see [3],[4],[5]) and Figure 1 shows that $e = 1$, when $p = 6$, since neither the red or blue subgraph contains a triangle.

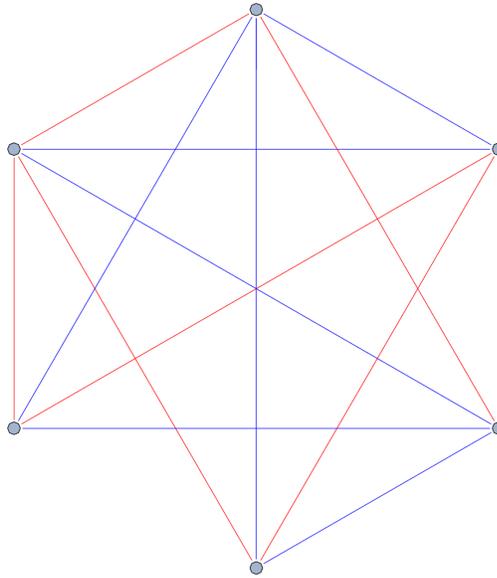


Figure 1

Utilizing Mathematica's Boolean capability I was able to show that in several other cases the answer also was $e=1$ (some of these examples appear in [1]). I began to conjecture that this might always be true, but didn't have any ideas on how to prove or disprove it nor did I know if anyone had already solved this problem. I tried a "Google search" for this or related results. After several attempts, I entered the search terms: "removing edges ramsey theory" and the second entry displayed by the search was [3]. On this site I found a result from a paper by S. Golomb [4], attributed to his student, Herbert Taylor, that was a special case of the result I wanted to prove. The special case was when K_s and K_t are both triangles; however, the proof given in [4] was easily adapted to prove my conjecture.

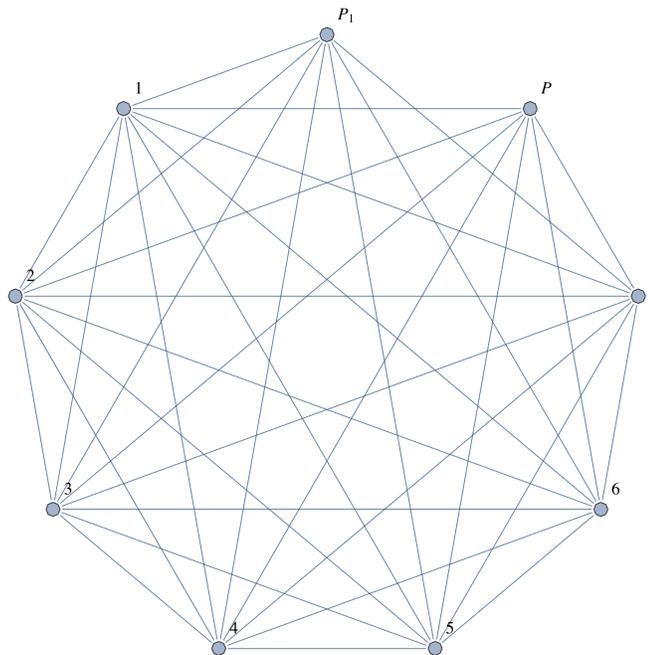


Figure 2

Theorem. Suppose $r(s, t) = p$. If G results from K_p by deleting a single edge, then G has a red/blue edge coloring and with respect to this coloring there is neither a red K_s or a blue K_t .

Proof. Let P be a vertex of K_{p-1} and let P_1 be a new vertex. Connect P_1 to each vertex Q of K_{p-1} , except P , by a new edge, P_1Q . Let us denote by L_{p-1} , the graph that results from K_{p-1} by replacing P by P_1 and each edge PQ , $Q \neq P$, by P_1Q . Then L_{p-1} is isomorphic to K_{p-1} . (In Figure 2, $p = 9$, so K_{p-1} and L_{p-1} are isomorphic to K_8) Also, G is isomorphic to $K_{p-1} \cup L_{p-1}$, the graph whose vertex set is the union of the vertex sets of K_{p-1} and L_{p-1} and whose edge set is the union of the edge sets of K_{p-1} and L_{p-1} . We construct an edge coloring, σ , of G as follows. Choose a red/blue edge coloring, γ for K_{p-1} that does not have either a red K_s or a blue K_t . Extend this coloring to a coloring σ of $K_{p-1} \cup L_{p-1}$, by coloring the additional edges, P_1Q , the same color that PQ received under γ . Then the subgraph L_{p-1} also has the property that it does not contain a red K_s or a blue K_t with respect to the coloring σ . Any complete subgraph of G cannot contain both P and P_1 , since there is no edge, PP_1 in G . Therefore, any K_s that is a subgraph of G , is either a subgraph of K_{p-1} (if it does not contain P_1) or a subgraph of L_{p-1} (if it does not contain P). In either case K_s is not colored red by σ . Similarly K_t is not colored blue by σ .

Conclusion. I have found, in my own research, asking questions and doing computer “experiments,” often leads to new theorems. Gauss, it should be remembered was a calculating prodigy and might have discovered many of his results this way as well. Unfortunately, by “removing the scaffolding,” he also hid his methods of discovery.

References.

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