# On the Greatest Common Divisor of Binomial Coefficients $\binom{n}{q},\binom{n}{2 q},\binom{n}{3 q}, \ldots$ 

Carl McTague


#### Abstract

Every binomial coefficient aficionado ${ }^{1}$ knows that the greatest common divisor of the binomial coefficients $\binom{n}{1},\binom{n}{2}, \ldots,\binom{n}{n-1}$ equals $p$ if $n=p^{i}$ for some $i>0$ and equals 1 otherwise. It is less well known that the greatest common divisor of the binomial coefficients $\binom{2 n}{2},\binom{2 n}{4}, \ldots,\binom{2 n}{2 n-2}$ equals (a certain power of 2 times) the product of all odd primes $p$ such that $2 n=p^{i}+p^{j}$ for some $i \leq j$. This note gives a concise proof of a tidy generalization of these facts.


THEOREM 1 ([|Ram09]). For any integer $n>1$ :

$$
\underset{0<k<n}{\mathrm{GCD}}\binom{n}{k}= \begin{cases}p & \text { if } n=p^{i} \text { for some prime } p \text { and some integer } i>0 \\ 1 & \text { otherwise }\end{cases}
$$

THEOREM 2 (Lemma 12 of [McT14a]). For any integer $n>1$ and any prime $p>2$ :

$$
\operatorname{ord}_{p}\left[\underset{0<k<n}{\operatorname{GCD}}\binom{2 n}{2 k}\right]= \begin{cases}1 & \text { if } 2 n=p^{i}+p^{j} \text { for some integers } 0 \leq i \leq j \\ 0 & \text { otherwise }\end{cases}
$$

where $\operatorname{ord}_{p}(m)$ is the highest power of $p$ dividing an integer $m$.
REMARK. For a given integer $n>1$, at most one prime $p$ divides the GCD in Theorem 1. But more than one prime can divide the GCD in Theorem 2, which is why ord ${ }_{p}$ is used to state it. For example, if $n=3$ then $2 n=3^{1}+3^{1}=5^{0}+5^{1}$ and indeed $\mathrm{GCD}_{0<k<3}\binom{6}{2 k}=15=3 \cdot 5$. In fact, more than two primes can divide: if $n=15$ then $2 n=3^{1}+3^{3}=5^{1}+5^{2}=29^{0}+29^{1}$ and indeed $\mathrm{GCD}_{0<k<15}\binom{30}{2 k}=435=3 \cdot 5 \cdot 29$.

These theorems are special cases of a (new) more general result:
THEOREM $Q$. For any integers $n>q>0$, and for any prime $p$ congruent to 1 modulo $q$ :

$$
\operatorname{ord}_{p}\left[\underset{0<k<n / q}{\mathrm{GCD}}\binom{n}{q k}\right]= \begin{cases}1 & \text { if } \alpha_{p}(n) \leq q \\ 0 & \text { otherwise }\end{cases}
$$

where $\alpha_{p}(n)$ is the sum of the digits of the base- $p$ expansion of $n$, equivalently the smallest integer $r$ such that $n=p^{i_{1}}+\cdots+p^{i_{r}}$ for integers $0 \leq i_{1} \leq \cdots \leq i_{r}$.

REMARK. Since $p$ is congruent to 1 modulo $q$, the inequality $\alpha_{p}(n) \leq q$ is equivalent to the equality $\alpha_{p}(n)=s$, where $s$ is the unique integer in the range $0<s \leq q$ congruent to $n$ modulo $q$. (Indeed, since $p$ is congruent to 1 modulo $q$, so is each power $p^{i}$, so $\alpha_{p}(n)$ is congruent to $n$ modulo $q$.) For example, for $n>1$ :

$$
\operatorname{ord}_{p}\left[\underset{0<k<n}{\mathrm{GCD}}\binom{q n}{q k}\right]= \begin{cases}1 & \text { if } \alpha_{p}(q n)=q \\ 0 & \text { otherwise }\end{cases}
$$

[^0]while:
\[

\operatorname{ord}_{p}\left[\underset{0<k \leq n}{\operatorname{GCD}}\binom{q n+1}{q k}\right]= $$
\begin{cases}1 & \text { if } \alpha_{p}(q n+1)=1 \\ 0 & \text { otherwise }\end{cases}
$$
\]

When $q=2$, the former is Theorem 2, while the latter a priori extends Theorem 2. However, due to the symmetry of Pascal's triangle $\binom{n}{k}=\binom{n}{n-k}$, this extension can already be deduced from Theorem 1.

REMARK. The hypothesis that $p$ is congruent to 1 modulo $q$ was chosen for its balance of simplicity and generality, and is used in two different ways in the proof of Theorem Q (below). It can be weakened, for example, to $p>q$ being relatively prime and $p^{i_{1}} \equiv \cdots \equiv p^{i_{r}}$ modulo $q$. (In the last paragraph of the proof, replace $(p-1) p^{i_{r}-1}$ with $q p^{i_{r}-1}$ when $p^{i_{1}} \equiv \cdots \equiv p^{i_{r}} \not \equiv 1$ modulo $q$.) But it cannot be eliminated altogether since, for example, $\operatorname{ord}_{2}\left[\mathrm{GCD}_{0<k<2}\binom{6}{3 k}\right]=\operatorname{ord}_{2}(20)=2$.

REMARK. A different generalization of Theorem 1 is obtained in [JOS85] by determining the greatest common divisor of $\binom{n}{r},\binom{n}{r+1}, \ldots,\binom{n}{s}$ for any $r \leq s \leq n$.

The proof of Theorem Q relies on:
KUMMER'S THEOREM ([Kum52], cf [Gra97, §1]). For any integers $0 \leq k \leq n$ and any prime $p$ :

$$
\operatorname{ord}_{p}\left[\binom{n}{k}\right]=\#\{\text { carries when adding } k \text { to } n-k \text { in base } p\}
$$

In particular, it relies on the following consequence of Kummer's theorem:
LEMMA 3. Given two integers $0 \leq k \leq n$, write their base-p expansions in the form:

$$
k=p^{j_{1}}+\cdots+p^{j_{s}} \quad n=p^{i_{1}}+\cdots+p^{i_{r}}
$$

with $r$ and $s$ minimal, $i_{1} \leq \cdots \leq i_{r}$ and $j_{1} \leq \cdots \leq j_{s}$. Then $\operatorname{ord}_{p}\left[\binom{n}{k}\right]=0$ if and only if $\left(j_{1}, \ldots, j_{s}\right)$ is a subsequence of $\left(i_{1}, \ldots, i_{r}\right)$.

Proof of Lemman3. By Kummer's theorem, $\left.\operatorname{ord}_{p}\left[\begin{array}{c}n \\ k\end{array}\right)\right]=0$ if and only if there are no carries when adding $k$ to $n-k$ in base $p$. This happens if and only if each base- $p$ digit of $k$ is $\leq$ the corresponding base $-p$ digit of $n$. And this in turn is equivalent to $\left(j_{1}, \ldots, j_{s}\right)$ being a subsequence of $\left(i_{1}, \ldots, i_{r}\right)$.

Proof of Theorem Q . To begin, note that for any set $S$ of integers:

$$
\operatorname{ord}_{p}\left[\mathrm{GCD}_{m \in S} m\right]=\min _{m \in S} \operatorname{ord}_{p}(m)
$$

So this order equals 0 if there is an integer $m$ in $S$ with $\operatorname{ord}_{p}(m)=0$. Similarly, this order equals 1 if (a) for every integer $m$ in $S, \operatorname{ord}_{p}(m)>0$ and (b) there is an integer $m$ in $S$ with $\operatorname{ord}_{p}(m)=1$.

Now, write the base- $p$ expansion of $n$ in the form:

$$
n=p^{i_{1}}+\cdots+p^{i_{r}}
$$

with $r$ minimal and $i_{1} \leq \cdots \leq i_{r}$.
If $r>q$ then by Lemma 3

$$
\operatorname{ord}_{p}\left[\binom{p^{i_{1}}+\cdots \cdots+p^{i_{r}}}{p^{i_{1}}+\cdots+p^{i_{q}}}\right]=0
$$

Since $p$ is congruent to 1 modulo $q$, so is each power $p^{i}$, so $p^{i_{1}}+\cdots+p^{i_{q}}$ is divisible by $q$, and it follows that $\operatorname{ord}_{p}\left[\mathrm{GCD}_{0<k<n / q}\binom{n}{q k}\right]=0$.

If $r \leq q$ then $p^{j_{1}}+\cdots+p^{j_{s}}$ is not divisible by $q$ for any nonempty proper subsequence $\left(j_{1}, \ldots, j_{s}\right)$ of $\left(i_{1}, \ldots, i_{q}\right)$. Therefore, by Lemma3 3 , ord $p\binom{n}{q k}>0$ for any $k$ with $0<q k<n$. So ord ${ }_{p}\left[\mathrm{GCD}_{0<k<n / q}\binom{n}{q k}\right]>0$.

The largest exponent $i_{r}$ must be $>0$ since otherwise $n=p^{0}+\cdots+p^{0}=r \leq q$, and by assumption $n>q$. Since $r$ is minimal, it equals the sum $\alpha_{p}(n)$ of the base- $p$ digits of $n$, so this sum is by assumption $\leq q$. And $q<p$ since $p$ is prime and congruent to 1 modulo $q$. It follows that the ( $i_{r}-1$ )st base- $p$ digit of $n$ is less than $p-1$. So there is exactly one carry when adding $(p-1) p^{i_{r}-1}$ to $n-(p-1) p^{i_{r}-1}$. By Kummer's theorem then:

$$
\operatorname{ord}_{p}\left[\binom{p^{i_{1}}+\cdots+p^{i_{r}}}{(p-1) p^{i_{r}-1}}\right]=1
$$

Since $p$ is congruent to 1 modulo $q,(p-1) p^{i_{r}-1}$ is divisible by $q$, and it follows that $\operatorname{ord}_{p}\left[\operatorname{GCD}_{0<k<n / q}\binom{n}{q k}\right]=$ 1.

Thanks to Doug Ravenel, David Gepner and Marcus Zibrowius for helpful conversations. Thanks to the referee who suggested generalizing an earlier version of Theorem $Q$. Thanks to Günter Ziegler for pointing out [Ram09].
Thanks to the villains [McT14b] who haunt the Hopkins mathematics department for helping inspire this work.

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E-mail address: carl.mctague@rochester.edu
URL: www.mctague.org/carl
Mathematics Department, University of Rochester, Rochester, NY 14627, USA


[^0]:    ${ }^{1}$ The author regards himself less aficionado than espontáneo, cf Bur01 p. 52].

