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C*-compactness in L-Fuzzy Topological Spaces

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Abstract

In this paper we introduce stronger form of the notion of cover so-called p-cover which is more appropriate. According to this cover we introduce and study another type of compactness in L-fuzzy topology so-called C*-compact and study some of its properties with some interrelation.

Key Words: L-Fuzzy topology, quasi-coincident, p-cover, C-compact, C*-compact,

0. Introduction

The notion of compactness is one of the most important concepts in general topology, but it seems that the research about this notion in fuzzy setting is not effective. Therefore, the problem of generalization of the classical compactness to fuzzy topological spaces has been intensively discussed over the past four Decades. Chang [4] was first introduced the notion of fuzzy compactness, but this notion not seems very natural in fuzzy setting, because it has many deviations and a fuzzy topology with one point fail to be compact (see[25]). So various kinds of fuzzy compactness have been presented and studied to avoid these deviations such as [1,3, 5-9,16, 21, 23].

The fuzzy compactness in L-fuzzy topological spaces was introduced by Gantner etc.al [8]. Hutton[12] introduced another notion of compactness in L-fuzzy topological spaces. Zhao[27] generalized the N-compactness [23] to L-fuzzy topological spaces. Warner and McLean [21] introduced another notion of L-fuzzy compactness, after that Kudri [13] proposed an extension to arbitrary L-fuzzy sets of compactness defined in [21], after that many authors introduced some investigations of L-fuzzy compactness such as [2,19,20].

In this paper we introduce another type of compactness in L-fuzzy topological space so-called C*-compactness which is extension, development of Chang's notion [4] and Hutton's notion of compactness [12] and is avoid many deviations also many interesting properties are discussed.

1. Preliminaries

Throughout this paper, $(L,\leq,')$ denotes a fuzzy lattice, i.e. a completely distributive lattice with order-reversing evaluative operator $a \mapsto a'$, and its smallest element and largest element are 0,1 respectively ($0 \neq 1$). If $A \subset X$, then χ_A denotes the characteristic function of A. A mapping from X into L is called an L-Fuzzy sets on X (L-FSs, for short). The collection of all L-FSs on X, denoted by L^X can be naturally seen as a fuzzy lattice $(L,\leq,\vee,\wedge,')$.

The smallest element and largest element of L^X are $\underline{0}$, $\underline{1}$, respectively, Where $\underline{0}(x) = x$, $\underline{1}(x) = x \; \forall x \in X$.

We denote by $\underline{\alpha}$ an L-FS which takes the constant value $\alpha \in L \quad \forall x \in X$. An L-Fuzzy topological space (L-FTS, for short), is a pair (L^X, δ) where δ is called an L-Fuzzy topology on X and, $\delta' = \{A' : A \in \delta\}$. We use the notation $S(A) = \{x \in X : A(x) > 0\}$ to denote the support of A. An L-fuzzy set B is called finite if $S(A)$ is finite. The set of all L-Fuzzy points in L^X is denoted by $FP(L^X)$. We say that x_α is a member of $A \in L^X$ denoted by $x_\alpha \in A$ if and only if $\alpha \leq A(x)$. For any $A \in L^X$,

define $A_\alpha = \{x \in X : A(x) \not\leq \alpha\}, \alpha \in L_1$ and $A_{\overline{\alpha}} = \{x \in X : A(x) \geq \alpha\}, \alpha \in L_0$, where $L_1 = L \setminus \{1\}$ and $L_0 = L \setminus \{0\}$. An L-fuzzy open set $O_{x_\alpha} \in \delta$ contai-ning x_α is called a neighborhood (nbd, for short) of x_α. The set of all neighborhoods of x_α will be denoted by $N(x_\alpha)$.

Definition 1.1: [15] Let $A, B \in L^X$, Then A is called quasi-coincident with B, denoted by $A \,\mathrm{q}\, B$, iff there exists an element $x \in X$ such that $A(x) \not\leq B'(x)$ and, A is not quasi-coincident with B denoted by $A \,\not{\mathrm{q}}\, B$ iff, $\forall x \in X,\ A(x) \leq B'(x)$.

Proposition 1.2: [15] Let $A, B, C \in L^X$ $\{A_i : i \in J\}$ and $x_\alpha, y_\beta \in FP(L^X)$. Then:

1) $A \,\not{\mathrm{q}}\, B \Leftrightarrow A \subseteq B'$,
2) $A \cap B = \underline{0} \Rightarrow A \,\not{\mathrm{q}}\, B$,

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3) $A \,\bar{q}\, B$, $C \subseteq B \Rightarrow A \,\bar{q}\, C$,

4) $x_\alpha q(\bigcup_{i\in J} A_i) \Leftrightarrow \exists\, i_0 \in J$ such that $x_\alpha \, q \, A_{i_0}$,

5) $x_\alpha \, q(\bigcap_{i\in J} A_i) \Rightarrow x_\alpha \, q A_i \quad \forall i \in J$

6) $x_\alpha \,\bar{q}\, y_\beta \Leftrightarrow x \neq y \; or \; (x = y \; and \; \alpha \leq \beta')$.

Theorem 1.3: [15] (a) Let (X,τ) be a topological space. Then:

i) $\omega_L(\tau) = \{A \in L^X : A_\alpha \in \tau, \alpha \in L\}$,

ii) $\delta_{01} = \{\chi_A : A \in \tau\}$ and

iii) $\delta_\tau = \{A \in L^X : S(A) \in \tau\}$ are L-Fuzzy topologies on X induced by τ.

(b) Let (L^X, δ) be an L-FTS. Then:

i) $\tau_\delta = \{S(A) : A \in \delta\}$ and

ii) $[\delta] = \{A \subset X : \chi_A \in \delta\}$

are ordinary topologies on X induced by δ.

Lemma 1.4: [15] Let (X,τ) be a topological space $A \in L^X$ and $B \subset X$. Then:

i) $\underline{\alpha} \in \delta_\tau \quad \forall \alpha \in L$.

ii) $U \in \tau \Rightarrow \alpha\chi_U \in \delta_\tau \quad \forall \alpha \in L$, in particular, $\delta_{01} \leq \delta_\tau$.

iii) $A \in \omega_L(\tau)$ iff ($A_\alpha \in \tau \; \forall \alpha \in L_1$),

iv) $B \in \tau$ iff $\chi_B \in \omega_L(\tau)$.

Proposition 1.5: Let X be an infinite set and a_t be any fixed L-fuzzy point in X. Then the following families:

i) $\delta_{a_t} = \{A \in L^X : a_t \,\bar{q}\, A\} \cup \{1\}$

ii) $\delta_\infty = \{A \in L^X : S(A') \, is \; finite\} \cup \{\underline{0}\}$

iii) $\delta_\infty^{a_t} = \delta_\infty \cup \delta_{a_t}$, are L-fuzzy topologies on X.

δ_{a_t} is called excluded L-fuzzy topology, δ_∞ is called cofinite L-fuzzy topology and $\delta_\infty^{a_t}$ is called the fort L-fuzzy topology.

Proof. Straightforward

Definition 1.6: [15] An L-FTS (L^X, δ) is said to be:

1) L-FT$_1$ iff $\forall x_\alpha, y_\beta \in FP(L^X)$ with $x_\alpha \,\bar{q}\, y_\beta$ implies $x_\alpha \,\bar{q}\, \bar{y}_\beta$ and $\bar{x}_\alpha \,\bar{q}\, y_\beta$

2) L-FT$_2$ iff $\forall x_\alpha, y_\beta \in FP(L^X)$ with $x_\alpha \,\bar{q}\, y_\beta$ implies there exist O_{x_α} and $O_{y_\beta} \in \delta$ such that $O_{x_\alpha} \,\bar{q}\, O_{y_\beta}$

3) L-FR$_0$ iff $\forall x_\alpha, y_\beta \in FP(L^X)$ with $x_\alpha \,\bar{q}\, \bar{y}_\beta$ implies $\bar{x}_\alpha \,\bar{q}\, y_\beta$,

4) L-FR$_1$ iff $\forall x_\alpha, y_\beta \in FP(L^X)$ with $x_\alpha \,\bar{q}\, \bar{y}_\beta$ implies there exist O_{x_α} and $O_{y_\beta} \in \delta$ such that $O_{x_\alpha} \,\bar{q}\, O_{y_\beta}$.

5) L-FR$_2$ (L-fuzzy regular) iff for every ($x_\alpha \in FP(L^X)$, $A \in \delta'$) with $x_\alpha \,\bar{q}\, A$ implies there exist O_{x_α} and $O_A \in \delta$ such that $O_{x_\alpha} \,\bar{q}\, O_A$.

6) L-FR$_3$ (L-fuzzy normal) iff for every $A, B \in \delta'$ with $A \,\bar{q}\, B$ implies there exist O_A and $O_B \in \delta$ such that $O_A \,\bar{q}\, O_B$.

7) L-FT$_3$ iff it is L-FR$_2$ and L-FT$_1$.

8) L-FT$_4$ iff it is L-FR$_3$ and L-FT$_1$.

Definition 1.7: [8] Let (X,δ) be an L-FTS and $\alpha \in L$. A family $\eta \subseteq \delta$ is called an open α^*-shading of X if for every $x \in X$ there exists $U \in \eta$ with $U(x) \geq \alpha$. The L-FTS (X,δ) is called α^*-compact iff each α^*-shading of X has a finite α^*-subshading of X.

Definitions 1.8: [14]

(1) The relation $r : L^X \times L^X \to \{0,1\}$ is called an L-fuzzy proximity on X if it satisfies the following axioms:

P1) $r(A,B) = r(B,A) \quad \forall A,B \in L^X$

P2) $r(A,B\cup C) = r(A,B) r(A,C) \; \forall A,B,C \in L^X$

P3) $r(A,0) = 1 \quad \forall A \in L^X$

P4) $A \, qB \Rightarrow r(A,B) = 0 \quad \forall A,B \in L^X$

P5) $r(A,B) = 1 \Rightarrow \exists\, D \in L^X$ such that $r(A,D) = r(B,D') = 1$.

The pair (X,δ) is called L-fuzzy proximity space (for short,L-FPS).

(2) The L-FPS (X,r) is called separated if it satisfies the axiom, P4') $r(x_\alpha, y_\beta) = 0 \Rightarrow x_\alpha q \, y_\beta$ for all $x_\alpha, y_\beta \in FP(L^X)$.

(3) In the L-FPS (X,r) we shall write $A >> B$ iff $r(B, A') = 1$ and we say A is r-nbd of B.

Propositions 1.9: [14]

(1) Every L-fuzzy proximity space (X,r) induces an L-fuzzy topology δ_r on X given by the following closure operator $c : L^X \to L^X$ where, $C(A) = \bar{A} = \cap\{B \in L^X : r(A,B') = 1\} \; \forall A \in L^X$

(2) For every L-FT$_4$ space (L^X, δ), there exists an L-fuzzy proximity r on X given by, $r(A,B)=1 \Leftrightarrow \overline{A}$ $\not q$ $\overline{B}$ such that $\delta_r = \delta$.

(3) Let (X, r) be an L-FPS. Then we have the following:

(i) $r(x_\alpha, A) = 0 \Leftrightarrow x_\alpha \ q \ \overline{A}$,

(ii) $r(A,B) = r(\overline{A}, \overline{B}) \forall A, B \in L^X$.

(iii) If $x_\alpha \in FP(L^X)$, for every O_{x_α} there exists $O^*_{x_\alpha}$ such that, $O_{x_\alpha} >> O^*_{x_\alpha} >> x_\alpha$.

2. C-Compact L-Fuzzy topological spaces

Definition 2.1: Let (L^X, δ) be an L-FTS and $A \in L^X$. Then the family $\gamma \subseteq \delta$ is called an open p-cover of A iff $\forall\, x_t \in A$ there exists $U \in \gamma$ such that $x_t \in U$. A subfamily β of an open p-cover γ of A, which is also an open p-cover of A is called an open p-subcover of A.

Clearly, every open p-cover is an open cover. The converse may not be true in general as it can be shown by simple examples.

Definition 2.2: Let (L^X, δ) be an L-FTS and $A \in L^X$. Then A is called a C-compact iff every open p-cover of A has finite p-subcover

Definition 2.3:

An L-FTS (L^X, δ) is called a C*-compact iff every closed L-fuzzy subset of X is a C-compact space.

Proposition 2.4:

If (L^X, δ) is C*-compact, then (L^X, δ) is C-compact.

The converse of the above proposition may not be true in general as shown by the following example.

Example 2.5: Let X be an infinite set, $L=[0,1]$ and $\delta = \{A \in L^X : A \subseteq \underline{0.5}\} \cup \{\underline{1}\}$ Then it is easy to see that (L^X, δ) is a C-compact but it is not a C*-compact. Indeed the set $A = \underline{0.5}$ is closed in, (L^X, δ) and the family, $\gamma = \{x_{0.5} : x \in X\}$ is an open p-cover of A which has no finite subcover. Hence (L^X, δ) is not C*-compact.

Examples 2.6:

1) Let (L^X, δ) be any L-FTS. Then every finite L fuzzyset is a C-compact set. And so if X is a finite, then (L^X, δ) is a C*-compact space.

2) The discrete L-FTS (L^X, δ) is a C*-compact iff X is a finite set.

3) Then L-FTS (L^X, δ_C), where $\delta_C = \{\underline{\alpha} : \alpha \in L\}$ is C*-compact.

Examples 2.7:

1) Let (L^X, δ_∞) be the cofinite L-FT defined in proposition (1.5). Then (L^X, δ_∞) is C*-compact.

Proof. Let F be a closed subset in (L^X, δ_∞). Then F is a finite or equal to X.

If F is a finite, then it is a C-compact from (i) of example (2.6).

If $F = X$, suppose η be an open p-cover of X. Then choose, $x_1 \in FP(L^X)$, there is $U_k \in \eta$ such that $x_1 \in U_k$. Since $U_k \in \delta_\infty$, then U'_k is finite. Now take,

$\omega = \{y_1^i : y \in S(U'_k), i = 1,2,\ldots,n\}$ which is finite, thus $\forall\, y_1^i \in \omega\ \exists U_i$ such that $y_1^i \in U_i$, $i = 1,2,\ldots,n$ and so the family, $\{U_i : i = 1,2,\ldots,n\} \cup \{U_k\}$ is a finite open p-subcover of X, so X is C-compact. Hence (L^X, δ_∞) is a C*-compact space.

2) Let $(L^X, \delta_\infty^{a_t})$ be the L-FTS defined in proposition (1.5). Then $(L^X, \delta_\infty^{a_t})$ is a C*-compact space.

Proof. Let $G \in \delta'$. Then either G is a finite or $a_t \, q\, G$. Then we have two cases:

1) If G is a finite, then G is a C-compact from (i) of example (2.6).

2) If $a_t \, q\, G$. Let η be an open p-cover of G. Then such that $a_{G(a)} \in V_k \Rightarrow a_{G(a)} \not q\, V'_k$. Since V'_k is closed, then $S(V'_k)$ is finite. Now put,

$\omega = \{x^i_{G(x^i)} : x^i \in S(V'_k), i = 1,2,\ldots,n\}$.

Thus for every $x^i_{G(x^i)} \in \omega \exists U_i \in \eta$ such that $x^i_{G(x^i)} \in U_i, i = 1,2,\ldots,n$,

and so the family $\{U_i : i = 1,2,\ldots,n\} \cup \{V_k\}$ is a finite open p-subcover of G, so G is a C-compact set. Hence $(L^X, \delta_\infty^{a_t})$ is C*-compact.

Definition 2.8:

Let (L^X, δ) be any L-FTS and $\eta = \{A_i : i \in J\} \subseteq L^X$ Then:

i) η is said to be have q-intersection with respect to (w.r.t., for short) $V \in L^X$ iff there exists $x_t \in V$ such that $x_t q\ A_i$ for all $i \in J$.

ii) η is called has the finite intersection property (FIP, for short) w.r.t. $U \in L^X$ iff every finite subfamily of η has a q-intersection w.r.t. U.

Theorem 2.9:

Let (L^X, δ) be an L-FTS. An L-FS $A \in L^X$ is a C-compact iff every family of closed L-FSs in X having the FIP w.r.t. A has q-intersection w.r.t. A.

Proof.

Suppose $A \in L^X$ is a C-compact. Let $\eta = \{F_i : i \in J\} \subseteq \delta'$ be a family of closed L-fuzzy sets in X which has the FIP w.r.t. A. Now suppose η has no a q-intersection w.r.t. A. Then $\forall\, x_t \in A\, \exists\, i \in J$ such that $x_t \not q\, F_i$ and hence $\eta' = \{F_i' : i \in J\} \subseteq \delta$ is an open p-cover of A. Since A is a C-compact, so there exists a finite p-subcover of η', say, $\{F_{i_s}' : s = 1,2,\ldots,n\} \subset \eta'$. Therefore, $\{F_{i_s} : s = 1,2,\ldots,n\}$ has no q-intersec- tion w.r.t. A, contradiction that η has the FIP w.r.t. A.

Conversely, let $\gamma = \{U_i : i \in J\} \subseteq \delta$ be an open p-cover of A. Then, $\gamma' = \{U_i' : i \in J\} \subseteq \delta'$ has no q-intersection w.r.t A. Thus γ' has not the FIP w.r.t. A. So there exist $i_1, i_2, \ldots, i_n \in J$ such that, $\{U_{i_s}' : s = 1,2,\ldots,n\}$ has no q-intersection w.r.t. A. Then $\{U_{i_s} : s = 1,2,\ldots,n\}$ is a finite open p-subcover of A. Hence A is C-compact.

Theorem 2.10:

Every closed subspace of a C*-compact space is a C*-compact.

Proof. Straightforward.

Lemma 2.11:

The continuous image of C-compact set is C-compact.

Proof. Let $\{B_i : i \in J\} \subseteq \delta^*$ be an open p-cover of $f(A)$ in Y. Then the family $\{f^{-1}(B_i) : i \in J\} \subseteq \delta$ is an open p-cover of A in X. Thus there exists a finite p-subcover of A say,

$\{f^{-1}(B_{i_1}), f^{-1}(B_{i_2}), \ldots f^{-1}(B_{i_n})\}$ and so,

$\{(B_{i_1}), (B_{i_2}), \ldots (B_{i_n})\}$ is a finite p-subcover of $f(A)$. Hence $f(A)$ is a C-compact.

Theorem 2.12:

The continuous image of a C*-compact space is a C*-compact.

Proof. Follows immediately from the above lemma.

Theorem 2.13: Let (X, τ) be a topological space. If $(L^X, \omega_L(\tau))$ is C-compact, then (X, τ) is compact.

Proof. Let Γ be an open cover of X. Then $\{\chi_U : U \in \Gamma\}$ is an open p-cover of X in $(L^X, \omega_L(\tau))$. Indeed, $\forall\, x_t \in X, x \in X$ and so there exists, $U \in \Gamma$ such that $x \in U$, then $x_t \in \chi_U$. Since X is a C-compact, then there is a finite open p-subcover say, $\{\chi_{U_i} : i = 1,2,\ldots,n\}$ and so the family $\{U_i : i = 1,2,\ldots,n\}$ is a finite open subcover of X. Hence (X, τ) is compact.

The converse of the above theorem may not be true in general. This can be shown by the following example.

Example 2.14:

Let X be an infinite set, $L = [0,1]$ and τ_∞ be the cofinite topology on X, then $(L^X, \omega_L(\tau_\infty))$ is not C-compact.

Proof. Take the family $\{A_n : n \in N\} \subset P(X)$ such that A_n cover of X but no finite subfamily does and such that $\forall\, n \in N$, A_n is a countable complement. For all $\forall\, n \in N$. Put, $A_n' = \{x_1^n, x_2^n, \ldots.\}$ and let $\eta = \{V_n : n \in N\}$ be a family of fuzzy sets in X defined by:

$$V_n(x) = 1 \qquad , \quad \forall\, x \in A_n,$$

$$V_n(x_J^n) = 1 - \frac{1}{J} \qquad , J = 1,2,\ldots,$$

then $\forall\, n \in N$, $V_n \in \omega_L(\tau_\infty)$. In fact for each $n \in N$ there exists $J_0 \in N$ such that if $J > J_0$, then, $1 - \frac{1}{J} > 1 - \frac{1}{J_0} > 0$ and so,

$S(V_n)' = \{x_1^n, x_2^n, \ldots, x_{J_0}^n\}$ is finite. Hence for all $n \in N$ $V_n \in \omega_L(\tau_\infty)$. Next for every $x \in X$ there exists $n \in N$ such that $x \in A_n$ and so, $V_n(x) = 1$ that is, $\forall\, x_1 \in FP(L^X)$ there exists $n \in N$ such that $x_1 \in V_n$. This shows that η is an open p-cover of X which has no finite p-subcover. Hence $(L^X, \omega_L(\tau_\infty))$ is not a C-compact space.

Theorem 2.15: Let (X, τ) be a topological space. Then (L^X, δ_{01}) is C*-compact if and only if (X, τ) is compact, where $\delta_{01} = \{\chi_U : U \in \tau\}$.

Proof.

Let (X, τ) be a compact and let $G \in \delta_{01}'$, then

$G=\chi_A$, $A\in\tau'\Rightarrow A$ is compact and so $\chi_A=G$ is C-compact. Hence (L^X,δ_{01}) is a C*-compact space.

Conversely, is similar to the proof of theorem (2.13).

That is, a C*-compactness is good extension in the sense of Gottwald [10].

Theorem 2.16: Let (L^X,δ) be an L-FTS. If (L^X,δ) is a C-compact space, then $(X,[\delta])$ is compact.

Proof. The proof is similar to the proof of theorem (2.13).
The converse of the above theorem may not be true in general. This is can be shown by example (2.14) where, $[\omega_L(\tau_\infty)]=\tau_\infty$.

Theorem 2.17: Let (X,τ) be a topological space. If (L^X,δ_τ) is a C-compact, then (X,τ) is compact.

Proof. It is similar to the proof of the above theorem (2.13).
The following example shows that the converse of the above theorem is not true in general.

Example 2.18:

Let $X=[0,1], L=[0,1]$ and τ be a usual topology on X. We know that (X,τ) is compact. Now for each $a\in(0,1)$, define $V_a\in L^X$ by:

$$V_a(x)=\frac{1}{a}x,\ \ if\ \ x\in(0,a],$$
$$V_0(x)=1-x,\ \ \forall x\in X,$$
$$V_a(x)=\frac{x-1}{a-1},\ \ if\ \ x\in(0,1],$$
$$V_1(x)=x\ \ ,\ \ \forall x\in X.$$

Then $V_a\in\delta_\tau$ $\forall\ a\in(0,1)$ and $V_0, V_1\in\delta_\tau$. Clearly the family, $\gamma=\{V_0,V_1\}\cup\{V_a : a\in(0,1)\}$ is an open p-cover of X which has no a finite open p-subcover. Hence X is not a C-compact space.

One can easily shows the following lemma.

Lemma 2.19: Let (L^X,δ) be an L-FTS. Then $\gamma\subseteq\delta$ is an open p-cover of X if and only if γ is an open 1^*-shading of X.

Proposition 2.20:

Let (L^X,δ) be an L-FTS . Then (L^X,δ) is C-compact if and only if (L^X,δ) is 1^*-compact.

Proof. Follows directly from the above lemma.

Definition 2.21: [8,11]

The L-fuzzy unit interval $I(L)$ is the set of all monotone decreasing maps $\lambda:\mathfrak{R}\to L$ satisfying:

$i)\ \lambda(t)=1\quad for\ \ t<0\,,t\in\mathfrak{R},$

$ii)\ \lambda(t)=0\quad for\ \ t>1\,,t\in\mathfrak{R},$ after identification of $\lambda,\mu\in L^{\mathfrak{R}}$ iff

$\lambda(t-)=\mu(t-)\ \ and\ \ \lambda(t+)=\mu(t+)$

$\forall\ t\in\mathfrak{R}$, where $\lambda(t-)=\inf_{s<t}\lambda(s),$

$\lambda(t+)=\sup_{s>t}\lambda(s)$.

The usual L-fuzzy topology on $I(L)$ is generated from the subbase, $\{L_t,R_t : t\in\mathfrak{R}\}$, where $L_t[\lambda]=\lambda(t-)'$ and $R_t[\lambda]=\lambda(t+)$. It is follows that $\{R_a,L_b,R_a\wedge L_b : a,b\in\mathfrak{R}\}$ is a base or usual L-fuzzy topology on $I(L)$ also, $\{R_a\wedge L_b : a,b\in\mathfrak{R}\}$ is another base for usual L-fuzzy topology on $I(L)$.

Gunter, etc. al [8] showed that $I(L)$ is 1^*-compact, hence from proposition (2.20) we have the following result.

Theorem 2.22: The L-fuzzy unit interval $I(L)$ is a C-compact.

Theorem 2.23: If the Cartesian product $(\prod_{s\in S}X_s\ ,\ \prod_{s\in S}\delta_s)$,where $X_s\neq\phi$ for every $s\in S$ is a C*-compact, then all L-FTSs (X_s,δ_s) are C*-compact.

Proof.

The proof follows from theorem (2.12) and from fact that all projections $P_s:\prod_{s\in S}X_s\to X_s$ are continuous.

Definition 2.24: [17]

Let $\{(L^{X_s},\delta_s) : s\in S\}$ be a family of pairwise disjoint L-FTSs and let $X=\bigcup_{s\in S}X_s$. Define the sum topology $\delta=\bigoplus_{s\in S}\delta_s$ of $\{\delta_s : s\in S\}$ on L^X , as follows

$$\delta=\{V\in L^X : V\cap X_s\in\delta_s\ \forall s\in S\}.$$

The pair (L^X,δ) is called the sum space of $\{(L^{X_s},\delta_s) : s\in S\}$ and denoted by $\bigoplus_{s\in S}(L^{X_s},\delta_s)$.

Theorem 2.25: The sum space (L^X,δ) of the family $\{(L^{X_s},\delta_s) : s\in S,\quad where\ \ S\ is\ finite\}$ of pairwise disjoint L-FTSs is C*-compact iff (L^{X_s},δ_s) is C*-compact $\forall\ s\in S.$.

Proof. Straightforward.

3. Separation axioms and C*-compactness

Theorem 3.1: Let (L^X,δ) be an L-F T_3 space and $A\in L^X$ be a C-compact, then for every closed subset B

such that $B \not q A$ $\exists O_A$, $O_B \in \delta$ such that $O_A \not q$ O_B.

Proof. Since (L^X, δ) is L-FT3 , then for every $x_t \in A$ there exist O_{x_t} , $(O_B)^{x_t} \in \delta$ such that $\exists O_A \not q$ $(O_B)^{x_t}$. Clearly $\{O_{x_t} : x_t \in A\}$ is an open p-cover of A. Since A is C-compact, then there exists a finite p-subcover, say , $\{O^i_{x_t} : i = 1,2,\ldots,n\}$. One readily verifies that $O_A = \bigcup_{i=1}^{n} O^i_{x_t}$ and, $O_B = \bigcap_{i=1}^{n} (O_B)^{x^i_t}$ have the required property.

Theorem 3.2: Let (L^X, δ) be an L-F T_2 space, $x_t \in FP(L^X)$ and let A be C-compact such that $x_t \not q A$, then there exists O_{x_t} , $O_A \in \delta$ such that $O_{x_t} \not q$ O_A. Moreover, if A, B are C-compact such that $A \not q B$ then there exists O_A and $O_B \in \delta$ such that $O_A \not q$ O_B.

Proof.

The proof is similar to the above proof.

Theorem 3.3:

Every a C–compact set of L-FT_2 space is closed.

Proof.

Let A be a C-compact set in L-FT$_2$ space (L^X, δ). Then from the above theorem we have for every $x_t \not q A$ there exists $O_{x_t} \in \delta$ such that $O_{x_t} \not q A$, that is, for every $x_t \subseteq A'$ there exists $O_{x_t} \in \delta$ such that $O_{x_t} \subseteq A'$. So A' is open in X. Hence A is closed.

Theorem 3.4:

Let (L^X, δ) , (L^Y, δ^*) be L-FTSs and let $f : X \to Y$ be a continuous map of a C*-compact space (L^X, δ) to an L-FT$_2$ (L^Y, δ^*). Then $f(\overline{A}) = \overline{f(A)}$.

Theorem 3.5: Every continuous map of a C*-compact space to an L-FT$_2$ space is closed.

Proof. Follows immediately from the above theorem.

Corollary 3.6: Every continuous one-to-one map of a C*-compact space onto L-FT_2 space is a homeomorphism.

Theorem 3.7: Every C*-compact, *L-F*T_2 space is an *L-F* T_4 space.

Proof.

Let (L^X, δ) be a C*-compact, L-F T_2 space and let $A_1, A_2 \in \delta'$ such that $A_1 \not q$ A_2. Since (L^X, δ) is a C*-compact, then A_1, A_2 are C-compact (by theorem 2.10) and so from theorem (3.2) there exist $O_{A_1}, O_{A_2} \in \delta$ such that $O_{A_1} \not q$ O_{A_2}. Hence (L^X, δ) is *L-F*T_4 .

Theorem 3.8: Let (L^X, δ) be *L-F*R_1 space. Then (L^X, δ) is *L-F*T_2 space iff every C-compact set is closed.

Theorem 3.9:

Every C*-compact, *L-F*R_1 space is L-fuzzy regular (*L-F*R_2).

Proof. Let (L^X, δ) be a C*-compact, *L-F*R_1 space and let $F \in \delta'$ with, $x_t \not q$ F. Then for all L-FP $y_r \in F$ we have $x_t \not q$ $\overline{y_r}$. Since (L^X, δ) is *L-F*R_1, then there exist O_{x_t} , $O_{y_r} \in \delta$ such that, O_{x_t} $\not q$ O_{y_r} . Then $\{O_{y_r} : y_r \in F\}$ is an open p-cover of F . Since (L^X, δ) is a C*-compact, then F is a C-compact, hence there exists a finite p-subcover, say, $\{O^i_{y_r} : i = 1,2,\ldots,n\}$ of F . Now take $U = \bigcap_{i=1}^{n} (O_{x_t})^{y^i_r}$ and $U = \bigcup_{i=1}^{n} O^i_{y_r}$ then $U, V \in \delta$ and $x_t \in U$, $F \subseteq V$ and $U \not q$ V . Hence (L^X, δ) is L-fuzzy regular (L-FR_2).

Theorem 3.10:

Every C*-compact, L-FR$_1$ space is L-fuzzy normal (L-FR_3).

Proof.

The proof is similar of the proof of the above theorem.

Now from theorems (3.9),(3.10) we have the following result.

Theorem 3.11:

Let (L^X, δ) be C*-compact. Then the following statements are equivalent:

i) (L^X, δ) is L-FR_1,

ii) (L^X, δ) is L-FR_2,

iii) (L^X, δ) is L-FR_0 and L-fuzzy normal space (L-FR_3).

Theorem 3.12:

Let (L^X, δ) be an L-FT$_2$ and a C*-compact space. Then there exists a unique compatible separated L-proximity relation r, given by:

$$r(A, B) = 1 \Leftrightarrow \overline{A} \not q \overline{B}.$$

Proof.

It is follows from (2) of proposition (1.9) and theorem (3.7) that r defines an L-fuzzy proximity on X . Now let r^* be an another L-fuzzy proximity on X . Then from (3ii) of proposition (1.9) and from (P4) of the definition (1.8) it follows that if $r(A,B)=0$, then $r^*(A,B)=0$. Now let $r(A,B)=1$ we have to prove that $r^*(A,B)=1$. Since $\overline{A} \not q \overline{B}$, then for every $x_t \in \overline{A}$, $\overline{B}'$ is an open L-FS contains x_t, so for some O_{x_t}, $\overline{B}' >> O_{x_t} >> x_t$. The family $\gamma = \{O_{x_t} : x_t \in \overline{A}\}$ is an open p-cover of $\overline{A}$. Since $\overline{A}$ is closed subset of C*-compact (L^X, δ), then $\overline{A}$ is C-compact and so γ has a finite p-subcover say, $\{O^i_{x_t} : i = 1,2,...,n\}$. Now

$\overline{B}' >> O^i_{x_t}$, for $i = 1,2,...,n$.

So, $\overline{B}' >> \vee_i O^i_{x_t} >> \overline{A}$ and hence $\overline{B}' >> \overline{A}$. Thus $r^*(\overline{A}, \overline{B}) = 1 = r^*(A,B)$. Then the result.

References

[1] M. Alimohammady, M. Roohi, "Compactness in fuzzy minimal spaces", *Chaos, Solitons and Fractals*, vol. 28, pp. 906-912, 2006.

[2] H. Aygün, S. R. T. Kudri, M. W. Warner, "Relative compactness in L-fuzzy topological spaces", *Fuzzy Sets Sys.* vol. 116, pp. 311-316, 2000.

[3] J.J. Chadwick, "A generalized form of compactness in fuzzy topological spaces", *J. Math. Anal. Appl.*, vol. 162, pp. 92–110, 1991.

[4] C.L. Chang, "Fuzzy topological spaces", *J. Math. Ana. App.*, vol. 24, pp. 182-190, 1968.

[5] Erdal Ekici, "On fuzzy covering properties", *Periodica Math. Hungarica*, vol. 54, no. 1, pp. 77-84, 2007.

[6] Erdal Ekici, "More on θ-compact fuzzy topological spaces", *Chaos, Solitons and Fractals,* vol. 2864, pp. 1157-1161, 2006.

[7] R. Erturk, "Some results on fuzzy compact spaces", *Fuzzy Sets Sys.*, vol. 70, no. 1, pp. 107–112, 1995.

[8] T. E. Gantner, R. C. Steinlage, R.H. Warren, "Compactness in fuzzy topological spaces", *J. Math. Anal. Appl.*, vol. 56, pp. 547–562, 1978.

[9] D. N. Georgiou, B. Papadopoulos, "On fuzzy compactness", vol. 233, pp. 86-101, 1999.

[10] S. Gottwald, "Fuzzy points and local properties of fuzzy topological spaces", *Fuzzy Sets and Sys.*, vol. 5, pp. 199-201, 1981.

[11] B. Hutton, "Normality in fuzzy topological spaces", *J. Math. Anal. Appl.*, vol. 50, pp. 74–79, 1975.

[12] B. Hutton, "Products of fuzzy topological spaces", *Topology Appl.*, vol. 11, pp. 59-67, 1980.

[13] S.R.T. Kudri, "Compactness in L-fuzzy topological spaces", *Fuzzy Sets and Sys.*, vol. 67, pp. 329-336, 1994.

[14] A. Kandil , A. M. Abd-Allah, "Fuzzy Q-proximity spaces", *Bull. Fac. Sci. Assiut Univ.*, vol. 14, pp. 107-120, 1985.

[15] A. Kandil, O. Tantawy, M. Yakout, S. Saleh, "Regularity and separation axioms on L-fuzzy topological spaces", *J. Egy. Math. Soc.*, to appear.

[16] R. Lowen, "A comparison of different compactness notions in fuzzy topological spaces", *J. Math. Anal. Appl.*, vol. 64, pp. 446–454, 1978.

[17] Y-M. Liu, L. Mao-Kang, "Fuzzy topology". *World Scientific Publishing, Singapore*, 1997.

[18] S. Nanda, "Strongly compact fuzzy topological spaces", *Fuzzy Sets and Sys.*, vol. 42, pp. 259–62, 1991.

[19] Fu-Gui Shi, "A new definition of fuzzy compactness", *Fuzzy Sets and Sys.*, vol. 158, pp. 1486-1495, 2007.

[20] Fu-Gui Shi, "a new notion of fuzzy compactness in L topology-ical spaces", *Information Sciences*, vol. 173, pp. 35-48, 2005.

[21] S.S. Thakur, R.K. Saraf, "θ-compact fuzzy topological spaces", *Mathematica Bohemica 120*, vol. 3, pp. 299-303, 1995.

[22] M.W. Warner, R.G. McLean, "On compact Hausdorff L-fuzzy spaces", *Fuzzy Sets and Sys.*, vol. 56, pp. 103-110, 1993.

[23] G.-J. Wang, "A new fuzzy compactness defined by fuzzy nets", *J. Math. Anal. Appl.*, vol. 94, pp. 1–23, 1983.

[24] G.-J.Wang, *Theory of L-fuzzy topological spaces*, Shanxi Normal Univ. Press, Xi'an, 1988.

[25] M. D. Weiss, "Fixed point, separation and induced topologies for fuzzy sets", *J. Math. Anal. Appl.*, vol. 50, pp. 142–150, 1975.

[26] D.S. Zhao, "The N- compact-ness in L-fuzzy topological spaces", *J. Math. Anal. Appl.*, vol. 128, pp. 64–70, 1987.

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