5-Regular Graphs are 3-Colorable
with Positive Probability*

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Abstract. We show that uniformly random 5-regular graphs of n vertices are 3-colorable with probability that is positive independently of n.

1   Introduction

The problem of finding the chromatic number of a graph has been a cornerstone in the field of discrete mathematics and theoretical computer science. Recall that the chromatic number of a graph is the minimum number of colors needed to legally color the vertices of the graph, where a coloring is *legal* if no two adjacent vertices share the same color. The problem is trivial for two colors but became difficult for three or more colors. Due to the difficulty of the problem, a large effort has been made in looking into structural properties of the problem, in the hope of finding more efficient procedures which apply to larger families of graphs.

An active line of research has been the characterization of classes of graphs due to their chromatic number. In particular, intense effort has been devoted to

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the study of the chromatic number on random structures. One active approach consists in finding the threshold p_k of k-colorability for the binomial model G_{n,p_k}. An early result in this line is presented in [10], where it is shown that, for any $d \in \mathbb{N}$, $d > 0$, $\exists k \in \mathbb{N}$ such that *asymptotically almost surely* (a.a.s.) the chromatic number of $G_{n,d/n}$ is either k or $k+1$. Recall that a sequence of events $\mathcal{E}_n$ holds a.a.s. if $\lim_{n\to\infty} \mathbf{Pr}\,[\mathcal{E}_n] = 1$. Further results on the particular threshold p_3 for 3-colorability can be found in the introduction of [2].

A related approach is to consider the chromatic number of random d-regular graphs (i.e., graphs for which every vertex has degree d). For a comprehensive review on random regular graphs, the reader is referred to [14].

For $d \geq 6$, a simple counting argument shows that d-regular graphs are not 3-colorable, a.a.s. In fact, Molloy and Reed [12] proved that 6-regular graphs have chromatic number at least 4, a.a.s. Achlioptas and Moore [2] proved that 4-regular graphs have chromatic number 3 *with uniform positive probability* (w.u.p.p.), where a sequence of events $\mathcal{E}_n$ holds w.u.p.p. if $\liminf_{n\to\infty} \mathbf{Pr}\,[\mathcal{E}_n] > 0$. The proof was algorithmic in the sense that a backtracking-free algorithm based on Brelaz' heuristic was designed and shown to produce a 3-coloring w.u.p.p. Subsequently, Achlioptas and Moore [3] showed that a.a.s. the chromatic number of a d-regular graph ($d \geq 3$) is k or $k+1$ or $k+2$, where k is the smallest integer such that $d < 2k \log k$. They also showed that if furthermore $d > (2k-1) \log k$, then a.a.s. the chromatic number is either $k+1$ or $k+2$. This result however gives no information for the chromatic number of either 4-regular or 5-regular graphs, apart from the known fact that a.a.s. the former have chromatic number either 3 or 4 and, the latter either 3 or 4 or 5. Shi and Wormald [13] showed that a.a.s. the chromatic number of a 4-regular graph is 3, that a.a.s. the chromatic number of a 6-regular graph is 4 and that a.a.s. the chromatic number of a 5-regular graph is either 3 or 4. They also showed that a.a.s. the chromatic number of a d-regular graph, for all other d up to 10, is restricted to a range of two integers.

The above results leave open the question of whether the chromatic number of a 5-regular graph can take the value 3 w.u.p.p., or perhaps even a.a.s.

On the other hand, building on a statistical mechanics analysis of the space of truth assignments of the 3-SAT problem, which has not been shown yet to be mathematically rigorous, and Survey Propagation (SP) algorithm for 3-SAT inspired by this analysis (see e.g. [11] and the references therein), Krząkała et al. [9] provided strong evidence that 5-regular graphs are a.a.s. 3-colorable by a SP algorithm. They also showed that the space of assignments of three colors to the vertices (legal or not) consists of clusters of *legal* color assignments inside of which one can move from point to point by steps of small Hamming distance. However, to go from one cluster to another by such small steps, it is necessary to go through assignments of colors that grossly violate the requirement of legality. Moreover, the number of clusters that contain points with energy that is a local, but not global, minimum is exponentially large. As a result, local search algorithms are easily trapped into such local minima. These considerations left as the only

plausible alternative to try to prove that 5-regular graphs are 3-colorable w.u.p.p. in an analytic way.

In this paper, we solve in the positive the question, showing that 5-regular graphs are 3-colorable w.u.p.p. The technique used is the Second Moment Method: Let X be a non-negative random variable (r.v.), then

$$\mathbf{Pr}\,[X > 0] \geq \frac{(\mathbf{E}\,(X))^2}{\mathbf{E}\,(X^2)}. \tag{1}$$

Thus, if this ratio is $\Theta(1)$, then we have that $X > 0$ w.u.p.p. This well known technique (see Remark 3.1 in [8]) was used by Achiloptas and Naor [4] to solve the long standing open problem of computing the two possible values of the chromatic number of a random graph.

To apply the Second Moment Method, we work under the *configuration model* (see [6]) and consider the r.v. X that counts a special type of colorings, which we call the *stable balanced 3-colorings*. In Section 2, we give exact expressions for the first and second moments of X, as sums of terms where each term consists of the product of polynomial (in n) and exponential terms. In Section 3 we compute the asymptotic values of those exact expressions, which turned out to be a non-trivial task. We show that $E(X^2) = O((E(X))^2)$, so that the probability in (1) is uniformly positive. Finally, the result is transferred from configurations to random 5-regular graphs.

An important remaining open question that we are working on is the extension of our result to a.a.s. It is plausible that an affirmative answer to the question can be obtained by concentration results similar to those in [3].

2   Exact Expressions for the First and Second Moments

Everywhere in this paper n will denote a positive integer divisible by 6. Asymptotics are always in terms of n.

In the sense of the configuration model (see, e.g., [6] or [14]), let $\mathcal{C}_{n,5}$ be the probability space of 5-regular *configurations*, obtained by considering a set of n vertices, labelled $1, \ldots, n$, for each of these vertices a set of 5 *semi-edges*, labelled $1, \ldots, 5$, and a uniform random perfect matching of all $5n$ semi-edges. Each pair of semi-edges according to the matching defines one edge of the configuration.

Definition 1. *A 3-coloring of a configuration $G \in \mathcal{C}_{n,5}$ is called stable if for every vertex v and every color $i = 0, 1, 2$, either v itself or one of its neighbors are colored by i. Equivalently, for no single vertex v can we change its color without the appearance of an edge with the same color at its endpoints. A 3-coloring is called balanced if for each $i = 0, 1, 2$, the number of vertices with color i is $n/3$.*

Given a configuration $G \in \mathcal{C}_{n,5}$, let $\mathcal{S}_G$ be the class of balanced stable 3-colorings of G. Let X be the random variable that counts the number of balanced stable 3-colorings of $\mathcal{C}_{n,5}$. Then, the following equations can be easily shown:

$$|\{G \mid G \in \mathcal{C}_{n,5}\}| = \frac{(5n)!}{2^{5n/2}(5n/2)!}, \quad \mathbf{E}\,(X) = \frac{|\{(G, C) \mid G \in \mathcal{C}_{n,5}, C \in \mathcal{S}_G\}|}{|\{G \mid G \in \mathcal{C}_{n,5}\}|}, \tag{2}$$

$$\mathbf{E}\left(X^2\right) = \frac{|\{(G, C_1, C_2) \mid G \in \mathcal{C}_{n,5}, C_1, C_2 \in \mathcal{S}_G\}|}{|\{G \mid G \in \mathcal{C}_{n,5}\}|}. \tag{3}$$

2.1   First Moment

Below we assume we are given a configuration G and a balanced stable 3-coloring C on G. The arithmetic in the indices is modulo 3. We start by giving some useful terminology and notation:

Definition 2. *A* 1-*spectrum s is an ordered pair of non-negative integers, $s = (s_{-1}, s_1)$, such that $s_{-1} + s_1 = 5$ and $s_{-1}, s_1 > 0$.*

Notice that there are four 1-spectra. They are intended to express the distribution of the five edges stemming from a given vertex according to the color of their other endpoint. Formally, a vertex v of color i is said to have 1-spectrum $s = (s_{-1}, s_1)$, if s_{-1} out of its five edges are incident on vertices of color $i - 1$ and the remaining s_1 edges are incident on vertices of color $i + 1$. The condition $s_{-1}, s_1 > 0$ expresses the fact that the 3-coloring is a stable one.

For each $i = 0, 1, 2$ and 1-spectrum s, we denote by $d(i; s)$ the scaled (with respect to n) number of vertices of G which are colored by i and have 1-spectrum s. Then, $\sum_s d(i; s) = 1/3$ and therefore $\sum_{i,s} d(i; s) = 1$.

Let $N_1 = |\{(G, C) \mid G \in \mathcal{C}_{n,5}, C \in \mathcal{S}_G\}|$. Given any two colors i and j, observe that there are exactly $5n/6$ edges connecting vertices with color i and j, respectively.

Given a fixed sequence $(d(i; s)n)_{i,s}$ that corresponds to a balanced stable 3-coloring, let us denote by $\binom{n}{(d(i;s)n)_{i,s}}$ the multinomial coefficient that counts the number of ways to distribute the n vertices into classes of cardinality $d(i; s)n$ for all possible values of i and s. Let also $\binom{5}{s}$ stand for $\binom{5}{s_{-1}} = \binom{5}{s_1}$.

By a counting argument, we have that

$$N_1 = \sum_{d(i;s)_{i,s}} \binom{n}{(d(i;s)n)_{i,s}} \left(\prod_{i,s} \binom{5}{s}^{d(i;s)n}\right) \left(\frac{5n}{6}!\right)^3, \tag{4}$$

where the summation above is over all possible sequences $(d(i; s))_{i,s}$ that correspond to balanced stable 3-colorings.

2.2   Second Moment

Below we assume we are given a configuration G and two balanced stable 3-colorings C_1 and C_2 on G. For $i, j = 0, 1, 2$, let V_i^j be the set of vertices colored with i and j with respect to colorings C_1 and C_2, respectively. Let $n_i^j = |V_i^j|/n$, and let E_i^j be the set of semi-edges whose starting vertex is in V_i^j. Also, for $r, t \in \{-1, 1\}$, let $E_{i,r}^{j,t}$ be the set of semi-edges in E_i^j which are matched with one in E_{i+r}^{j+t}. Let $m_{i,r}^{j,t} = |E_{i,r}^{j,t}|/n$. We have that $\sum_{r,t} m_{i,r}^{j,t} = 5n_i^j$, $\sum_{i,j} n_i^j = 1$, and therefore $\sum_{i,j,r,t} m_{i,r}^{j,t} = 5$. And, since matching sets of semi-edges should have equal cardinalities, we also have that $m_{i,r}^{j,t} = m_{i+r,-r}^{j+t,-t}$.

Definition 3. *A* 2-*spectrum* s *is an ordered quadruple of non-negative integers,* $s = (s_{-1}^{-1}, s_{-1}^1, s_1^{-1}, s_1^1)$, *such that* $s_{-1}^{-1} + s_{-1}^1 + s_1^{-1} + s_1^1 = 5$ *and* $(s_{-1}^{-1} + s_{-1}^1)(s_1^{-1} + s_1^1)(s_{-1}^{-1} + s_1^{-1})(s_{-1}^1 + s_1^1) > 0$.

Notice that the number of 2-spectra is 36. Let v be a vertex in V_i^j. Vertex v is said to have 2-spectrum $s = (s_{-1}^{-1}, s_{-1}^1, s_1^{-1}, s_1^1)$ if s_r^t out of its five edges, $r, t \in \{-1, 1\}$, are incident on vertices in V_{i+r}^{j+t}. The condition $(s_{-1}^{-1} + s_{-1}^1)(s_1^{-1} + s_1^1)(s_{-1}^{-1} + s_1^{-1})(s_{-1}^1 + s_1^1) > 0$ expresses the fact that both C_1 and C_2 are stable.

For each $i, j = 0, 1, 2$ and 2-spectrum s, we denote by $d(i, j; s)$ the scaled number of vertices which belong to V_i^j and have 2-spectrum s. We have:

$$\sum_s s_r^t\, d(i, j; s) = m_{i,r}^{j,t}, \quad \sum_s d(i, j; s) = n_i^j \quad \text{and therefore} \quad \sum_{i,j,s} d(i, j; s) = 1.$$

Throughout this paper we refer to the set of the nine numbers n_i^j as the set of the *overlap matrix variables*. We also refer to the set of the thirty-six numbers $m_{i,r}^{j,t}$ as the set of the *matching variables*. Finally, we refer to the 9×36 numbers $d(i, j; s)$ as the *spectral variables*.

Let $N_2 = |\{(G, C_1, C_2) \mid G \in \mathcal{C}_{n,5}, C_1, C_2 \in \mathcal{S}_G\}|$. Given a fixed sequence $(d(i, j; s)n)_{i,j,s}$ that corresponds to a pair of balanced stable 3-colorings, let us denote by $\binom{n}{(d(i,j;s)n)_{i,j,s}}$ the multinomial coefficient that counts the number of ways to distribute the n vertices into classes of cardinality $d(i, j; s)n$ for all possible values of i, j and s. Let also $\binom{5}{s}$ stand for $\binom{5}{s_{-1}^{-1}, s_{-1}^1, s_1^{-1}, s_1^1}$ (the distinction from a similar notation for 1-spectra will be obvious from the context). Now, by an easy counting argument, we have:

$$N_2 = \sum_{d(i,j;s)_{i,j,s}} \left\{ \binom{n}{(d(i,j;s)n)_{i,j,s}} \left(\prod_{i,j,s} \binom{5}{s} \right)^{d(i,j;s)n} \left(\prod_{i,j,r,t} ((m_{i,r}^{j,t}n)!)^{\frac{1}{2}} \right) \right\}, \quad (5)$$

where the summation above is over all possible sequences $(d(i, j; s))_{i,j,s}$ that correspond to pairs of balanced stable 3-colorings.

3   Asymptotics

In this section we will show that $E(X^2) = O((E(X))^2)$. An immediate consequence of this is that 5-regular configurations have a balanced stable 3-coloring and hence a generic 3-coloring w.u.p.p.

By applying Stirling approximation to formulae (2), (4) and (5), we get:

$$\mathbf{E}(X) \sim \sum_{d(i;s)_{i,s}} f_1(n, d(i; s)_{i,s}) \left(6^{-\frac{5}{2}} \prod_{i,s} \left(\frac{\binom{5}{s}}{d(i; s)} \right)^{d(i;s)} \right)^n, \quad (6)$$

$$\mathbf{E}(X^2) \sim \sum_{d(i,j;s)_{i,j,s}} f_2(n, d(i, j; s)_{i,j,s}) \left[5^{-\frac{5}{2}} \prod_{i,j,s} \left(\frac{\binom{5}{s}}{d(i, j; s)} \right)^{d(i,j;s)} \left(\prod_{i,j,r,t} (m_{i,r}^{j,t})^{\frac{1}{2}m_{i,r}^{j,t}} \right) \right]^n, \quad (7)$$

where f_1 and f_2 are functions that are sub-exponential in n and also depend on the sequences $d(i;s)_{i,s}$ and $d(i,j;s)_{i,j,s}$, respectively. By a result similar to Lemma 3 in [3], by using a standard Laplace-type integration technique, we can prove that the Moment Ratio is asymptotically positive if

$$(E(X))^2 \asymp E(X^2), \quad \text{i.e. } \ln((E(X))^2) \sim \ln(E(X^2)). \tag{8}$$

Let M_1 be the maximum base $6^{-5/2} \prod_{i,s} \left(\frac{\binom{5}{s}}{d(i;s)} \right)^{d(i;s)}$ as $d(i;s)_{i,s}$ ranges over all possible sequences that correspond to balanced stable 3-colorings. Analogously, let M_2 be the maximum base

$$5^{-5/2} \left(\prod_{i,j,s} \left(\frac{\binom{5}{s}}{d(i,j;s)} \right)^{d(i,j;s)} \right) \left(\prod_{i,j,r,t} (m_{i,r}^{j,t})^{\frac{1}{2} m_{i,r}^{j,t}} \right).$$

From the equations (6) and (7) one can immediately deduce that the relation (8) is true if $(M_1)^2 = M_2$. We need to compute the exact values of M_1 and M_2.

3.1   First Moment: Computing M_1

Let $f = \prod_{i,s} \left(\frac{\binom{5}{s}}{d(i;s)} \right)^{d(i;s)}$ be a real function of 12 non-negative real variables $d(i;s)$ defined over the polytope $\sum_s d(i;s) = 1/3$, where $i = 0, 1, 2$ and s runs over 1-spectra. The following lemma follows from the application of elementary analysis techniques and the computation of Lagrange multipliers.

Lemma 1. *The function $\ln f$ is strictly convex. Let $\mathcal{D}_1 = \sum_{i,s} \binom{5}{s} = 3 \times 30$. The function f has a maximizer at the point where $d(i;s) = \frac{\binom{5}{s}}{\mathcal{D}_1}, \forall i, s.$*

By direct substitution, we obtain:

Lemma 2. $M_1 = 6^{-5/2} \left(\prod_{i,s} \mathcal{D}_1^{d(i,s)} \right) = \left(\frac{1}{6} \right)^{5/2} \mathcal{D}_1 = \sqrt{\frac{25}{24}}.$

From the above and from (6), we get:

Theorem 1. *The expected number of balanced stable 3-colorings of a random 5-regular configuration approaches infinity as n grows large.*

3.2   Second Moment: Computing M_2

Let
$$F = \left(\prod_{i,j,s} \left(\frac{\binom{5}{s}}{d(i,j;s)} \right)^{d(i,j;s)} \right) \left(\prod_{i,j,r,t} (m_{i,r}^{j,t})^{\frac{1}{2} m_{i,r}^{j,t}} \right) \tag{9}$$

be a real function of non-negative real variables $d(i,j;s)$ (where $i, j = 0, 1, 2$, $r, t = -1, 1$ and s runs over 2-spectra) defined over the polytope determined by:

$$\sum_{j,s} d(i,j;s) = 1/3, \ \forall i; \ \sum_{i,s} d(i,j;s) = 1/3, \ \forall j \ \text{ and } m_{i,r}^{j,t} = m_{i+r,-r}^{j+t,-t}, \tag{10}$$

where $m_{i,r}^{j,t} = \sum_s s_r^t \, d(i,j;s)$. Notice that F is a function of 9×36 variables.

We will maximize F in three phases. In the first one, we will maximize F assuming the matching variables $m_{i,r}^{j,t}$ are fixed constants such that their values are compatible with the polytope over which F is defined. Thus we will get a function F_m of the 36 matching variables $m_{i,r}^{j,t}$. At the second phase we will maximize F_m assuming that the nine overlap matrix variables $5n_i^j = \sum_{r,t} m_{i,r}^{j,t}$ are fixed constants compatible with the domain of the matching variables. Thus we will get a function F_n of the overlap matrix variables. The preceding two maximizations will be done in an analytically exact way. Observe that since we consider balanced 3-colorings, F_n depends only on the values of four n's. We will maximize F_n by going through its 4-dimensional domain over a fine grid. Let us point out that the maximizations above will not be done *ex nihilo*. Actually, we know (see below) the point where we would like the maximizer to occur. Therefore all we do is not find the maximizer but rather prove that it is where we want it to be. The proof of the next lemma is done by direct substitution.

Lemma 3. *Let $\mathcal{D}_2 = \sum_{i,j,s} \binom{5}{s} = 9 \times 900$ and let $d(i,j;s) = \binom{5}{s}/\mathcal{D}_2$, $\forall i,j,s$. Then the value of the base $5^{-5/2}F$ at the above values of $d(i,j;s)_{i,j,s}$ is equal to $(M_1)^2 = 25/24$.*

We call the sequence $d(i,j;s) = \binom{5}{s}/\mathcal{D}_2$ the *barycenter*. Barycenter as well we call the corresponding point in the domain of F_m, i.e. the point $m_{i,r}^{j,t} = 5/36$, $\forall i,j,r,t$. Finally, barycenter also we call the corresponding point in the domain of F_n, i.e. the point $n_i^j = 1/9$, $\forall i,j$. We will see, by direct substitutions, that the functions $5^{-5/2}F_m$ and $5^{-5/2}F_n$ as well take the value $(M_1)^2 = 25/24$ at their corresponding barycenters. Therefore, after computing F_m and F_n, all that will remain to be proved is that F_n has a maximizer at its barycenter $n_i^j = 1/9$, $i,j = 0,1,2$. Below, we compute the functions F_m and F_n and then we show that the barycenter is a maximizer for F_n by sweeping its 4-dimensional domain.

From the Spectral to the Matching Variables. Everywhere below we assume that the 36 matching variables $m_{i,r}^{j,t}$ are non-negative and moreover take only values for which there exist 9×36 spectral non-negative variables $d(i,j,s)$ such that

$$m_{i,r}^{j,t} = \sum_s s_r^t \, d(i,j;s), \ i,j = 0,1,2, \ r,t = -1,+1. \tag{11}$$

and such that the equations in (10) hold. It is not hard to see that the above restrictions on the matching variables are equivalent to assuming that $\forall i,j = 0,1,2$ and $\forall r,t = -1,1$,

$$m_{i,r}^{j,t} = m_{i+r,-r}^{j+t,-t}, \ \sum_{i,r,t} m_{i,r}^{j,t} = 5/3, \ \sum_{j,r,t} m_{i,r}^{j,t} = 5/3 \ \text{and} \tag{12}$$

$$m_{i,r}^{j,t} \geq 0, \quad m_{i,r}^{j,t} + m_{i,r}^{j,-t} \leq 4(m_{i,-r}^{j,t} + m_{i,-r}^{j,-t}), \quad m_{i,r}^{j,t} + m_{i,-r}^{j,t} \leq 4(m_{i,r}^{j,-t} + m_{i,-r}^{j,-t}). \tag{13}$$

Fix such values for the $m_{i,r}^{j,t}$. To maximize the function F given by equation (9) over the polytope described in (10) for the fixed values of the matching variables $m_{i,r}^{j,t}$, $i,j = 0,1,2$, $r,t = \{-1,1\}$, it is sufficient to maximize the function F subject to the 36 constraints in (11). We call this maximum F_m.

Since for different pairs of (i, j), $i, j = 0, 1, 2$, neither the variables $d(i, j; s)$ nor the constraints in (11) have anything in common, and since the matching variables are fixed, it is necessary and sufficient to maximize separately for each $i, j = 0, 1, 2$ the function $F_{i,j} = \prod_s \left(\binom{5}{s}/d(i, j; s)\right)^{d(i,j;s)}$, subject to the four constraints: $\sum_s s_r^t\, d(i, j; s) = m_{i,r}^{j,t}$, $r, t = -1, 1$. We will use Lagrange multipliers to maximize the logarithm of these functions. Notice that the functions $\ln F_{i,j}$ are strictly convex. We define the following function:

$$\Phi(x, y, z, w) = (x+y+z+w)^5 - (x+y)^5 - (x+z)^5 - (y+w)^5 - (z+w)^5 + x^5 + y^5 + z^5 + w^5.$$

Also for each of the nine possible pairs (i, j), $i, j = 0, 1, 2$, consider the 4×4 system:

$$\frac{\partial\, \Phi(\mu_{i,-1}^{j,-1}, \mu_{i,-1}^{j,1}, \mu_{i,1}^{j,-1}, \mu_{i,1}^{j,1})}{\partial \mu_{i,r}^{j,t}}\, \mu_{i,r}^{j,t} = m_{i,r}^{j,t}, \quad r, t = -1, 1, \tag{14}$$

where $\mu_{i,r}^{j,t}$ denote the 36 unknowns of these nine 4×4 systems. Applying the method of the Lagrange multipliers, we get

Lemma 4. *Each of the nine systems in (14) has a unique solution. Moreover in terms of the solutions of these systems*

$$F_m = \prod_{i,j,r,t} \left(\frac{(m_{i,r}^{j,t})^{\frac{1}{2}}}{\mu_{i,r}^{j,t}}\right)^{m_{i,r}^{j,t}}.$$

By the above Lemma, we have computed in an analytically exact way the function F_m. Notice that the function F_m is a function of the 36 matching variables $m_{i,r}^{j,t}$, $i, j = 0, 1, 2$ and $r, t = -1, 1$, over the domain given by (12) and (13). However its value is given through the solutions of the systems in (14), which have a unique solution.

From the Matching to the Overlap Matrix Variables. We assume now that we fix nine non-negative overlap matrix variables n_i^j such that $\sum_i n_i^j = 1/3$, $\forall j$ and $\sum_j n_i^j = 1/3$, $\forall i$. Using again multiple Lagrange multipliers, we will find the maximum, call it F_n, of the function F_m given in Lemma 4 under the constraints:

$$\sum_{r,t} m_{i,r}^{j,t} = 5n_i^j, \quad \text{and } m_{i,r}^{j,t} = m_{i+r,-r}^{j+t,-t}, \quad \text{for } i, j = 0, 1, 2 \text{ and } r, t = \{-1, 1\}. \tag{15}$$

assuming in addition that the $m_{i,r}^{j,t}$ satisfy the inequalities in (13). We consider the latter inequality restrictions not as constraints to be dealt with Lagrange multipliers, but as restrictions of the domain of F_m that must be satisfied by the maximizer to be found.

We will need that the function $\ln F_m$ over the polytope determined by the constraints (15) (for fixed values of the variables n_i^j) is strictly convex. To show this it is *sufficient* to fix an arbitrary $i, j = 0, 1, 2$ and show that the 4-variable function $\ln \left(\prod_{r,t} \left((m_{i,r}^{j,t})^{\frac{1}{2}}/\mu_{i,r}^{j,t}\right)^{m_{i,r}^{j,t}}\right)$, subject to the single linear constraint

$\sum_{r,t} m_{i,r}^{j,t} = n_i^j$, is strictly convex. To show the latter, we computed the Hessian and its LPMD's after solving the single linear constraint for one of its variables (thus we obtained a function of three variables). Notice that the value of the function under examination is given through the unique solutions of a 4×4 system. The Hessian and the LPMD's were analytically computed in terms of these solutions by implicit differentiation of the equations of the system. The strict convexity then would follow if we showed that at every point of the domain of this 3-variable function, the LPMD's were non-zero and of alternating sign. We demonstrated this by going over this domain over a fine grid and computing at all its points the LPMD's. The values of the LPMD's that we got were safely away from zero and with the desired algebraic sign. Notice that although to prove the convexity of the function $\ln F_m$, subject to the constraints in (15), we relaxed the constraints $m_{i,r}^{j,t} = m_{i+r,-r}^{j+t,-t}$, the latter ones are essential for correctly computing F_n.

To apply Lagrange multipliers, we have to find the partial derivatives of the function $\ln F_m = \sum_{i,j,r,t} (\frac{1}{2} m_{i,r}^{j,t} \ln m_{i,r}^{j,t} - m_{i,r}^{j,t} \ln \mu_{i,r}^{j,t})$. In fact, after a few manipulations, we obtain:

Lemma 5. $\sum_{r',t'} m_{i,r'}^{j,t'} \dfrac{\partial \ln \mu_{i,r'}^{j,t'}}{\partial m_{i,r}^{j,t}} = \dfrac{1}{5},$ *and thus* $\dfrac{\partial \ln F_m}{\partial m_{i,r}^{j,t}} = \dfrac{3}{10} + \dfrac{1}{2} \ln m_{i,r}^{j,t} - \ln \mu_{i,r}^{j,t}$

By applying now the technique of multiple Lagrange multipliers, we get:

Lemma 6. *Consider the 45×45 system with unknowns $\mu_{i,r}^{j,t}$ and x_i^j:*

$$
\begin{cases}
\dfrac{\partial}{\partial \mu_{i,r}^{j,t}} \Phi(\mu_{i,-1}^{j,-1}, \mu_{i,-1}^{j,1}, \mu_{i,1}^{j,1}, \mu_{i,1}^{j,-1})\ \mu_{i,r}^{j,t} = \mu_{i,r}^{j,t}\ \mu_{i+r,-r}^{j+t,-t}\ x_i^j\ x_{i+r}^{j+t}, & \\
5n_i^j = \sum_{r,t} \left(\mu_{i,r}^{j,t}\ \mu_{i+r,-r}^{j+t,-t}\ x_i^j\ x_{i+r}^{j+t} \right), & \begin{array}{l} i,j = 0,1,2, \\ r,t = -1,1. \end{array}
\end{cases}
$$

This system has a unique solution. Moreover in terms of the solution of this system:

$$ F_n = \prod_{i,j} (x_i^j)^{5n_i^j}. $$

So we have computed in an analytically exact way the function F_n. Since solving the 45×45 system in Lemma 6 when $n_i^j = 1/9$, $i,j = 0,1,2$ is trivial, we get by direct substitution:

Lemma 7. *The value of $5^{-5/2} F_n$ at the barycenter $n_{i,j} = 1/9$, $i,j = 0,1,2$ is equal to $(M_1)^2 = 25/24$. Therefore the value of F_n at the barycenter is > 58.2309.*

Therefore all that it remains to be proved is that the function F_n maximizes at the barycenter.

From the Overlap Matrix Variables to the Conclusion. We have to prove the function F_n maximizes at the barycenter. Since we have assumed that the 3-coloring is balanced, i.e. $\forall i, \sum_j n_i^j = 1/3$ and $\forall j, \sum_i n_i^j = 1/3$, the domain of F_n has four degrees of freedom, all in the range $[0, 1/3]$. We swept over this domain

going over the points of a grid with 200 steps per dimension. The sweeping avoided a thin layer (of width 1/1000) around the boundary (the points in the domain where at least one of the $n_i^j = 0$), because at the boundary the derivative of the original function F is infinity, thus no maximum occurs there. Moreover, we have computed the Hessian at the barycenter and proved that it is negative definite so in an neighborhood of the barycenter F_n is convex, and we know that F_n will be smaller than at the barycenter. At all points where we got a value for F_n greater than 58, we made an additional sweep at their neighborhood of step-size 1/7500 (all these points where close the barycenter). Nowhere did we get a value greater than the value at the barycenter. To solve the 45×45 systems efficiently, we designed a fast search algorithm based on an algorithm by Byrd et al. [7]. We also devised a way to select good starting points for each system, whose basic principle was to select for each successive system a starting point that belonged to the convex hull of the solutions to the previous systems. The algorithm was implemented in Fortran and run on the IBM's supercomputer in the Barcelona Supercomputing Center, which consists of 2.268 dual 64-bit processor blade nodes with a total of 4.536 2.2 GHz PPC970FX processors. Therefore,

Theorem 2. *Random 5-regular configurations are 3-colorable with uniformly positive probability.*

In order to transfer this result to random 5-regular graphs, we need to consider the restriction of $\mathcal{C}_{n,5}$ to *simple* configurations (i.e. those without loops and multiple edges). We write $\mathbf{Pr}^*$ and $\mathbf{E}^*$ to denote probability and expectation conditional to the event "$G \in \mathcal{C}_{n,5}$ is simple". By using similar techniques to the ones developed in [5] (see also Theorem 2.6 in [14]), we get:

Lemma 8. *Let C be any fixed balanced 3-coloring of n vertices. Then,*
$\mathbf{Pr}\left[G \text{ is simple} \mid C \text{ is stable coloring of } G\right]$ *is bounded away from 0, independ–ently of C and n.*

From this lemma, we obtain: $\mathbf{E}^*(X) = \Theta(\mathbf{E}(X))$ and $\mathbf{E}^*(X^2) = O\left(\mathbf{E}(X^2)\right)$.
Therefore, $\mathbf{Pr}^*[X > 0] \geq \dfrac{(\mathbf{E}^*(X))^2}{\mathbf{E}^*(X^2)} = \Theta(1)$, and we can conclude:

Theorem 3. *The chromatic number of random 5-regular graphs is 3 with uniformly positive probability.*

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