

Comparison Maps for Relatively Free Resolutions*

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Abstract. Let Λ be a commutative ring, A an augmented differential graded algebra over Λ (briefly, DGA-algebra) and X be a relatively free resolution of Λ over A . The standard bar resolution of Λ over A , denoted by $B(A)$, provides an example of a resolution of this kind. The comparison theorem gives inductive formulae $f: B(A) \rightarrow X$ and $g: X \rightarrow B(A)$ termed comparison maps. In case that $fg = 1_X$ and A is connected, we show that X is endowed a A_∞ -tensor product structure. In case that A is in addition commutative then (X, μ_X) is shown to be a commutative DGA-algebra with the product $\mu_X = f*(g \otimes g)$ ($*$ is the shuffle product in $B(A)$). Furthermore, f and g are algebra maps. We give an example in order to illustrate the main results of this paper.

1 Introduction

Calculations in homological algebra are commonly expressed in terms of resolutions. It is not unusual that these resolutions are embedded in the bar construction (or some other standard resolution) in a special way. When this occurs, they are said to *split off* of the standard resolution (see [14]).

A classic example is the Koszul resolution $K = A \otimes E_A[u_1, \dots, u_n]$ related to the ideal $I = (x_1, \dots, x_n)$ in the polynomial ring $A = \Lambda[x_1, \dots, x_n]$; as A is an augmented algebra over Λ , the *bar resolution* $B(A)$ for Λ over A [5, 16] can be constructed. K is also a resolution of Λ over A and by the comparison theorem [16], there is a chain homotopy equivalence $B(A) \leftarrow K$. In this case, an explicit contraction (special homotopy equivalence) $B(A) \rightrightarrows K$ exists [17]. This contraction makes that the Koszul resolution splits off of the bar resolution. Using this contraction and some homological perturbation tools, perturbations of this resolution can be computed and the perturbed resolutions can be used to make complete effective calculations where previously only partial or indirect results were obtainable. This idea has been exploited in a series of papers by Lambe [12, 13, 14] and provides an algorithm for computing resolutions *which split off of the bar construction*. This algorithm has been extended to a more

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general context in [11]. We point out that the notion of contraction is essential in order to find effective algorithms in homological algebra using the set of techniques provided by homological perturbation theory, since the input data of our algorithm have to be codified in this form.

The well-known comparison theorem in homological algebra states that any two projective resolutions are chain homotopy equivalent. For relatively free resolutions $Y = A \otimes \bar{Y}$ and $X = A \otimes \bar{X}$ of Λ over A with explicit contracting homotopies, there are recursive procedures for obtaining explicit equivalences $f: Y \rightarrow X$ and $g: X \rightarrow Y$ where the explicit contracting homotopies play a principal role. In addition, there are inductive procedures for obtaining explicit chain homotopies of fg and the identity and with gf and the identity. Generally, the maps defined in this way do not form a contraction. Here, assuming that $Y = B(A)$, we give a necessary and sufficient condition for determining when these maps form a contraction, which seems to be new.

In the special case that A is connected, Y is the bar resolution of Λ over A , and the above maps form a contraction from $B(A)$ to X (i.e., X is a resolution which splits off of the bar resolution), we define a degree minus one map $\tau: \bar{X} \rightarrow A$ which is an A_∞ -twisting cochain, so that $(X, \tau, \{\Delta_i\}_{i \geq 0})$ becomes an A_∞ -twisted tensor product where $\{\Delta_i\}_{i \geq 0}$ is the A_∞ -coalgebra structure of \bar{X} transferred from $\bar{B}(A)$ by means of ‘tensor trick’ [8] (see Algorithm 1). This provides an elegant codification of the differential of the complex X in terms of the A_∞ -twisting cochain and the A_∞ -coalgebra structure of \bar{X} . Furthermore, assuming in addition that A is commutative (but not necessarily connected) and that the contracting homotopy of X is a quasi algebra homotopy (see [19]), we prove that the morphism $\mu_x = f * (g \otimes g)$ ($*$ is the shuffle product in $B(A)$) endows X a commutative algebra structure, for which f and g are algebra maps (see Theorem 7) and give a method for computing new resolutions taking advantage of this algebra structure (see Algorithm 2). In the example given in this paper, we compute a resolution \tilde{X} of \mathbf{Z}_p over $\Gamma(w, 2n)$ using an initial resolution which splits off of the standard resolution, $B(A) \Rightarrow X$, and perturbing this contraction. The contraction $B(A) \Rightarrow X$ has been computed by means of the comparison theorem. We prove that X is a DGA-algebra as well as \tilde{X} . A computational advantage is deduced from this fact, since it is only necessary to compute the perturbed differential on the generators of \tilde{X} as an algebra, better than on the whole set of generators as a module. This type of computational advantage is our main motivation for studying the algebra structures underlying the resolutions.

We organize the paper as follows. In section 2 we give the necessary definitions and notations for defining the comparison maps when Y is the bar resolution and X is a contractile relatively free resolution. We also give a necessary and sufficient condition for guaranteeing that a contraction arises. Section 3 is devoted to study the A_∞ -structure inherent in X , when X is a relatively free resolution over a connected DGA-algebra A . In section 4 we analyse the multiplicative behaviour of the comparison maps, assuming that A is a commutative DGA-algebra and that the contracting homotopy of X is a quasi algebra homotopy. Finally, we give an example in order to illustrate the main results of the paper.

2 The Canonical Comparison Contraction – A Necessary and Sufficient Condition

We will quickly review some basic notions of Homological Algebra. More details can be found in [16]. Let A be a commutative ring with $1 \neq 0$, and A an augmented differential graded algebra over A , briefly termed DGA-algebra. The differential, product, augmentation and coaugmentation of A will be denoted respectively by d_A , μ_A , ϵ_A , and η_A . Nevertheless, sometimes, we will write them simply by d , μ , ϵ , and η when no confusion can arise. In what follows, the Koszul sign conventions will be used. A morphism $\rho : A_* \rightarrow A_{*-1}$ is called *derivation* if it is compatible with the algebra structures on A . The degree of an element $a \in A$ is denoted by $|a|$. Let us recall that if B is also a DGA-algebra, then $A \otimes B$ is canonically endowed an algebra structure by means of the morphism $\mu_{A \otimes B} = (\mu_A \otimes \mu_B)(1_A \otimes T \otimes 1_B)$, where $T(b \otimes a) = (-1)^{|b||a|}a \otimes b$. If the DG-algebra A is connected, that is $A_0 = A$ and $d_1 : A_1 \rightarrow A_0$ is zero, then there is a canonical augmentation $\epsilon_A = 1_A : A_0 \rightarrow A$.

Let n be a positive integer. The exterior algebra with one generator u in degree $2n - 1$, the polynomial algebra with one generator v in degree $2n$, and the divided power algebra with one “generator” w in degree $2n$ are denoted by $E(u, 2n - 1)$, $P(v, 2n)$, and $\Gamma(w, 2n)$, respectively.

We need here the *reduced bar construction* $\bar{B}(A)$ of a DGA-algebra A (see [16]). Recall that it is defined as the connected DGA-coalgebra, $\bar{B}(A) = T^c(S(\bar{A}))$, where $T^c(\)$ is the tensor coalgebra, $S(\)$ is the suspension functor, and $\bar{A} = \text{Ker } \epsilon_A$ is the augmentation ideal of A . The element of $\bar{B}_0(A)$ corresponding to the identity element of A is denoted by $[\]$ and the element $S\bar{a}_1 \otimes \cdots \otimes S\bar{a}_n$ of $\bar{B}(A)$ is denoted by $[a_1 | \cdots | a_n]$. The tensor and simplicial degrees of the element $[a_1 | \cdots | a_n]$ are $|[a_1 | \cdots | a_n]|_t = \sum |a_i|$ and $|[a_1 | \cdots | a_n]|_s = n$, respectively; its total degree is the sum of its tensor and simplicial degree. The tensor and simplicial differential are defined by:

$$d_t([a_1 | \cdots | a_n]) = - \sum_i (-1)^{e_i-1} [a_1 | \cdots | d_A(a_i) | \cdots | a_n],$$

and

$$d_s([a_1 | \cdots | a_n]) = \sum_i (-1)^{e_i} [a_1 | \cdots | \mu_A(a_i \otimes a_{i+1}) | \cdots | a_n]$$

where

$$e_i = i + |a_1| + \cdots + |a_i|.$$

If the product of A is commutative, a product $*$ (called shuffle product) can be defined on $\bar{B}(A)$. In this way, the reduced bar construction has a commutative Hopf algebra structure.

We will use here the structure of *twisted tensor product*. Let A be a DG-algebra and C a DG-coalgebra. It is well known that $\tau : C_* \rightarrow A_{*-1}$ is a *twisting cochain* if and only if $d^\tau = d_A \otimes 1 + 1 \otimes d_C + \tau \cap$ is a differential on $A \times C$ (see [4]), where the morphism $\tau \cap$ is defined by:

$$\tau \cap = (\mu_A \otimes 1)(1 \otimes \tau \otimes 1)(1 \otimes \Delta_C). \quad (1)$$

The DG-module $(A \otimes C, d^\tau)$ is called *the twisted tensor product (or TTP) of A and C along τ* . We will also use the notation of $A \otimes_\tau C$ for such DG-module.

A *relatively free resolution of Λ over A* is a pair (X, ϵ) where X is a graded differential A -module of the form $X = A \otimes_A \bar{X}$ with \bar{X} a DG- A -module and $\epsilon: X \rightarrow \Lambda$ a morphism of graded differential A -modules which is a weak equivalence, thereby, the homology of X is zero except in degree 0 where it is Λ . We will call the complex $(\bar{X}, d_{\bar{X}})$ the *reduced complex*, and it is always obtained in the form $(\bar{X}, d_{\bar{X}}) = (\Lambda \otimes_A X, 1_A \otimes_A d_X)$, by means of the the classical ‘neglect’ functor on the category of all A -modules to the category of all Λ -modules. It is standard terminology to call the elements of \bar{X} *reduced elements*. Given a morphism $\psi: X \rightarrow Y$ the notation $\psi|_{\bar{X}}(\bar{x})$ means $\psi(1 \otimes \bar{x})$ where 1 is the unit in A and $\bar{x} \in \bar{X}$. We follow these conventions throughout the paper.

A resolution $\epsilon: (X, d) \rightarrow \Lambda$ is called *contractile* if there exists a ‘*contracting homotopy*’, i.e., a family of Λ -module morphisms, $h_{-1}: \Lambda \rightarrow X_0$, $h_n: X_n \rightarrow X_{n+1}$, such that $1 = d_{n+1}h_n + h_{n-1}d_n$, $\forall n \geq 0$, where $d_0 = \epsilon$ and $h_{-1} = \eta$. Besides, it may always be assumed to hold that $h^2 = 0$ (see [3]).

Throughout this paper, (X, h, d) will denote a contractile relatively free resolution (X, d) with contracting homotopy h .

An important example of relatively free and contractile resolution of Λ over A is the *bar resolution* $(B(A), s, d)$ (or $B(A)$) [16, 13]. More specifically, $B(A)$ is the *twisted tensor product* $A \otimes_\theta \bar{B}(A)$, where the twisting cochain θ is given by

$$\theta([a_1 | \cdots | a_n]) = \begin{cases} a_1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where the weak equivalence $\epsilon_{B(A)}: B(A) \rightarrow \Lambda$ is the canonical augmentation of $B(A)$ (in fact, it is a homotopy equivalence) and the contracting homotopy s is given by

$$s: B(A) \rightarrow B(A) \quad \text{where} \quad s(a \otimes [a_1 | \cdots | a_n]) = [a | a_1 | \cdots | a_n].$$

From now on, we will use s for denoting the above homotopy.

A *contraction* (see [6], [9]) is a data set $c: \{N, M, f, g, \phi\}$ where $f: N \rightarrow M$ and $g: M \rightarrow N$ are morphisms of DG-modules (called, respectively, *projection* and *inclusion*) and $\phi: N \rightarrow N$ is a morphism of graded modules of degree +1 (called *homotopy operator*). These data are required to satisfy the rules: **(c1)** $fg = 1_M$, **(c2)** $\phi d_N + d_N \phi + gf = 1_N$ **(c3)** $\phi \phi = 0$, **(c4)** $\phi g = 0$ and **(c5)** $f \phi = 0$. These three last are called *side conditions* [15]. In fact, these may always be assumed to hold, since the homotopy ϕ can be altered to satisfy these conditions [7, 14]. We will also denote a contraction c by $(f, g, \phi): N \Rightarrow M$.

For instance, the bar resolution $B(A)$ of a DG-algebra A gives the following contraction:

$$C_{B(A)}: \{B(A), A, \epsilon_{B(A)}, \eta_{B(A)}, s\} \quad (3)$$

where $\eta_{B(A)}: A \rightarrow B(A)$ is the canonical coaugmentation of $B(A)$.

By the comparison theorem for resolutions [16], given any relatively free resolution $X = (A \otimes \bar{X}, d_X) \xrightarrow{\epsilon} A$ of A over A , there is an A -linear morphism (a *comparison map*) $g: X \rightarrow B(A)$ inductively defined:

$$g_0|_{\bar{x}_0} = \eta_{B(A)}\epsilon_X|_{\bar{x}_0}, \quad g_{n+1}|_{\bar{x}_{n+1}} = sg_n d_{n+1}|_{\bar{x}_{n+1}}. \quad (4)$$

This map is a homotopy equivalence between $B(A)$ and X . Moreover, if the resolution X is contractile with contracting homotopy t , (X, t, d) , then comparison theorem provides an analogous inductive definition for the A -linear morphism $f: B(A) \rightarrow X$:

$$f_0|_{B(A)_0} = \eta_X \epsilon_{B(A)}|_{B(A)_0}, \quad f_{n+1}|_{B(A)_{n+1}} = t f_n d_{n+1}|_{B(A)_{n+1}}. \quad (5)$$

Both of the compositions fg , gf of these comparison maps are homotopic to the corresponding identity maps. Inductive formulae for the associated homotopies are also available:

$$\phi: B(A)_* \rightarrow B(A)_{*+1}$$

defined on reduced elements and then extended A -linearly,

$$\phi|_{B(A)} = (-sgf - s\phi d)|_{B(A)}, \quad (6)$$

and

$$\kappa: X_* \rightarrow X_{*+1}$$

where

$$\kappa = t(1 - gf).$$

Let us observe that in general κ is not A -linear. These formulae are crucial for the work in [13, 14]. Let us observe that the morphisms g and ϕ satisfy

$$g(\bar{X}) \subseteq \bar{B}(A), \quad \phi(\bar{B}(A)) \subseteq \bar{B}(A), \quad (7)$$

but f and κ do not satisfy the analogous condition.

Generally, fg is different to 1_X , but sometimes a contraction arises, which we call ‘*the canonical comparison contraction*’. A necessary condition for guaranteeing that $fg = 1_X$ is given in [11]. We next give a necessary and sufficient condition for this purpose.

Theorem 1. *The data set $\mathcal{C}: \{B(A), (X, t, d), f, g, \phi\}$ is a contraction if and only if $dt|_{\bar{x}} = 0$.*

PROOF. First we assume that $dt|_{\bar{x}} = 0$. Taking into account that t is a contracting homotopy of X to A , it holds that

$$1|_{\bar{x}} = (dt + td)|_{\bar{x}} = td|_{\bar{x}}$$

Now we will show that $fg = 1_X$, the proof is by induction. We have $f_0g_0 = 1$ by construction and for $n > 0$, on reduced elements,

$$f_n g_n = t f_{n-1} d_n g_n = t f_{n-1} g_{n-1} d_n = t d_n = 1$$

where the first and third equality comes from (7) and induction hypothesis on t , respectively. By A -linearity the proof is extended to elements of X . It is readily checked that the side conditions hold.

Reciprocally, now, let us assume that $fg = 1_X$. Working in a similar way as above, we have, on reduced elements for $n \geq 0$,

$$1 = f_n g_n = t f_{n-1} d_n g_n = t f_{n-1} g_{n-1} d_n = t d_n$$

Hence, $td|_X = 1|_X$ and $dt|_X = 0$.

□

Remark 1. For the remainder of this paper, we will assume that \mathcal{C} is a contraction. In this situation, \mathcal{C} is called *the canonical comparison contraction* for (X, t, d) .

A resolution X *splits off of the bar construction* (see [14]) if there is a contraction (called comparison contraction) from $B(A)$ to X . Note that this contraction can be different from the canonical one.

With this definition at hand, we can state the following proposition:

Proposition 1. *Let (X, t, d) be a contractile relatively free resolution. If $dt|_X = 0$, then X splits off of the bar construction.*

In the sequel proposition we analyze the contracting homotopy t of (X, t, d) .

Proposition 2. *Let (X, t, d) be a contractile relatively free resolution which splits off of the bar construction by means of the canonical comparison contraction. Then $t = fsg$.*

PROOF. First, due to the fact that $s: B(A) \rightarrow \bar{B}(A)$, we can use the inductive definition of f in this composition fs , thus

$$fsg = (tfd)sg$$

since $sd + ds = 1$,

$$tf(ds)g = tf(1 - sd)g = tfg - tfsdg = tfg.$$

The last identity results from the fact that $tfs = t(tfd)s = 0$ (because $t^2 = 0$). Finally,

$$tfg = t$$

since $fg = 1_X$.

□

Remark 2. If we consider $\epsilon_Y: (Y, t_Y, d) \rightarrow A$ any contractile relatively free resolution of A over A , instead of $B(A)$, the comparison theorem for resolutions provides similar formulae (comparison maps):

$$f: Y \rightarrow X, \quad g: X \rightarrow Y, \quad \phi: Y_* \rightarrow Y_{*+1}, \quad \kappa: X_* \rightarrow X_{*+1}$$

where under the hypothesis that $t_Y(Y) \subset \bar{Y}$, it is possible to get the result analogous to Theorem 1.

3 Differential Structures in the Comparison of Resolutions

Up to now little has been said about the nature of the differential d_X in X for an arbitrary contractile relatively free resolution with contracting homotopy t . Here we show that d_X can be rewritten in terms of an A_∞ -twisting cochain $\tau: \bar{X} \rightarrow A$ and the A_∞ -coalgebra structure of \bar{X} transferred from the coalgebra structure of $\bar{B}(A)$. To this end, we prove two previous results (Theorems 4 and 5) which claim that working with resolutions (*à la Cartan*) is equivalent to work with reduced complexes (*à la Eilenberg–Mac Lane*) from a homological point of view. We describe a method for passing from one way to the other.

Now, we recall the concept of a perturbation datum. Let N be a graded module and let $f: N \rightarrow N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all $x \in N$ ($x \neq 0$), a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DG-module* N is a morphism of graded modules $\delta: N \rightarrow N$ of degree -1 , such that $(d_N + \delta)^2 = 0$ and $\delta_1 = 0$. A *perturbation datum of the contraction* $c: \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N verifying that the composition $\phi\delta$ is pointwise nilpotent.

We now introduce the main tool in Homological Perturbation Theory: the Basic Perturbation Lemma ([4, 7, 8, 3, 19]).

Theorem 2. (BPL)

Let $c: \{N, M, f, g, \phi\}$ be a contraction and $\delta: N \rightarrow N$ be a perturbation datum of c . Then, a new contraction

$$c_\delta: \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\}$$

is defined by the formulas: $d_\delta = f\delta\Sigma_c^\delta g$; $f_\delta = f(1 - \delta\Sigma_c^\delta\phi)$; $g_\delta = \Sigma_c^\delta g$; $\phi_\delta = \Sigma_c^\delta\phi$; where

$$\Sigma_c^\delta = \sum_{i \geq 0} (-1)^i (\phi\delta)^i = 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i (\phi\delta)^i + \cdots.$$

Let us note that $\Sigma_c^\delta(x)$ is a finite sum for each $x \in N$ because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DG-module (M, d_M) .

The transference of the algebra structure up to homology equivalence has been considered in [8, 9, 19]. Next, we review several notions.

Definition 1. [19] Let A and A' be two DG-algebras and $c: \{A, A', f, g, \phi\}$ be a contraction. The projection f is a *quasi algebra projection* if the following conditions hold:

$$f\mu_A(\phi \otimes \phi) = 0, \quad f\mu_A(\phi \otimes g) = 0, \quad f\mu_A(g \otimes \phi) = 0.$$

The homotopy operator ϕ is a *quasi algebra homotopy* if the following conditions hold:

$$\phi\mu_A(\phi \otimes \phi) = 0, \quad \phi\mu_A(\phi \otimes g) = 0, \quad \phi\mu_A(g \otimes \phi) = 0.$$

Definition 2. [8] Let A and A' be two DG-algebras and $c : \{A, A', f, g, \phi\}$ be a contraction. The homotopy operator ϕ is said to be *an algebra homotopy* if

$$\phi\mu_A = \mu_A(1_A \otimes \phi + \phi \otimes gf).$$

Definition 3. [19] Let A and A' be two DGA-algebras and $r : \{A, A', f, g, \phi\}$ a contraction. We say that r is

- a *semi-full algebra contraction* if f is a quasi algebra projection, g is a morphism of DGA-algebras and ϕ is a quasi algebra homotopy.
- an *almost-full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is a quasi algebra homotopy.
- a *full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is an algebra homotopy.

Obviously, full and almost-full algebra contractions are, in particular, semi-full algebra contractions. It is not difficult to prove that both sets of semi-full and almost-full algebra contractions are closed by composition and tensor product of contractions.

If A is a commutative DGA-algebra, the contraction (3) is an example of an almost-full algebra contraction.

Definition 4. [7] Let A and A' be two DG-algebras and $c : \{A, A', f, g, \phi\}$ a contraction. An *algebra perturbation datum* δ of c is a perturbation datum of this contraction which is also a derivation.

The following result tells us that the set of semi-full algebra contractions is closed by homological perturbation. This theorem is used in the proof of some theorems of this paper.

Theorem 3 (SF-APL). ([19])

Taking as data a semi-full algebra contraction r and an algebra perturbation datum δ of r , the perturbed contraction r_δ is an algebra contraction of the same type, where the product on A'_δ is the original product $\mu_{A'}$.

3.1 From Resolutions to Reduced Complexes

Throughout this subsection, A will denote a connected DGA-algebra.

The goal of this subsection is to establish a contraction from $\bar{B}(A)$ to \bar{X} ('reduced complexes') by means of the canonical comparison contraction between the contractile relatively free resolutions $B(A)$ and (X, t, d) :

$$\mathcal{C}: \{B(A), (X, t, d), f, g, \phi\}.$$

To this end, we will apply the classical 'neglect' functor on the category of all A -modules to the category of all Λ -modules, $\Lambda \otimes_A -$ and $1 \otimes_A -$, on the complexes and morphisms involved in the above contraction, respectively.

The following properties will play an important role in what follows.

1. $X \cong (\text{Ker } \epsilon_A \otimes \bar{X}) \oplus (\Lambda \otimes \bar{X})$.
 2. $\Lambda \otimes_A X \cong \Lambda \otimes \bar{X}$ and $1 \otimes_A d_X = (\epsilon_A \otimes 1_{\bar{X}}) d_X$.
 3. $1 \otimes_A g = g|_{\bar{X}}$, $1 \otimes_A \phi = \phi|_{\bar{B}(A)}$.
 4. $1 \otimes_A f = (\epsilon_A \otimes 1_{\bar{X}}) f$.
 5. $f(\bar{b}) = 0 \Rightarrow (1 \otimes_A f)(\bar{b}) = 0$.
 6. $d_X(\bar{x}) = 0 \Rightarrow (1 \otimes_A d_X)(\bar{x}) = 0$.
- (8)

Properties 1, 2, and 4 are deduced from the meaning of tensoring by A . Since g and ϕ are A -linear and satisfy (7), the third is followed. Properties 5 and 6 are consequences of 1 and 2.

By property 3 we have that $1 \otimes_A g$ and $1 \otimes_A \phi$ are DGA-module morphisms. In spite of the fact that f does not satisfy (7), we will prove that $1 \otimes_A f$ is a morphism of DGA-modules as well.

Firstly, note that

$$(1 \otimes_A f)(1 \otimes_A d) = (\epsilon_A \otimes 1) f (\epsilon_A \otimes 1) d = (\epsilon_A \otimes 1) f d = (\epsilon_A \otimes 1) d f,$$

here we have used that f is A -linear and a DGA-module morphism as well.

On the other hand,

$$(1 \otimes_A d)(1 \otimes_A f) = (\epsilon_A \otimes 1) d (\epsilon_A \otimes 1) f,$$

Now, taking into account that the differential d of an A -module X satisfies

$$d(a \otimes \bar{x}) = d_A(a) \otimes \bar{x} + (-1)^{|a|} a \otimes d(\bar{x})$$

it is clear that if $a \otimes \bar{x} \in \text{Ker } A \otimes \bar{X}$ then $d(a \otimes \bar{x}) \in \text{Ker } A \otimes \bar{X}$. Thereby,

$$(\epsilon_A \otimes 1) d f = (\epsilon_A \otimes 1) d (\epsilon_A \otimes 1) f.$$

The properties required for this data set

$$\{\bar{B}(A), \bar{X}, 1 \otimes_A f, 1 \otimes_A g, 1 \otimes_A \phi\}$$

in order to be a contraction are inherited from \mathcal{C} in a straightforward manner. Therefore, we can state:

Theorem 4. *The data set $1 \otimes_A \mathcal{C}: \{\bar{B}(A), \bar{X}, 1 \otimes_A f, 1 \otimes_A g, 1 \otimes_A \phi\}$ is a contraction.*

3.2 From Reduced Complexes to Resolutions

Taking as input a contraction from the reduced bar construction of a connected DGA-algebra A , $\bar{B}(A)$, to a free DGA-module \bar{X} , we describe [2] a method for obtaining resolutions which split off of the bar construction. This process plays an important role in the main result of this section.

Proposition 3. [2] *Let A be a connected DGA-algebra. Given a contraction c from $(\bar{B}(A), d_{\bar{B}(A)})$ to a DGA-module (\bar{X}, \bar{d}) , in which the homotopy operator increases the simplicial degree by one, there is a comparison contraction from the bar resolution $B(A)$ to the resolution X , where the underlying module in X is just the $A \otimes \bar{X}$ and the differential structure is done via perturbation of c .*

Theorem 5. *Assuming that (X, t, d) is a contractile relatively free resolution which splits off of the bar construction of a connected DGA-algebra A , under the canonical comparison contraction, then $\theta \cap$ is a perturbation datum for*

$$\mathcal{C}: \{A \otimes (\Lambda \otimes_A B(A)), A \otimes (\Lambda \otimes_A X), 1 \otimes (1 \otimes_A f), 1 \otimes (1 \otimes_A g), 1 \otimes (1 \otimes_A \phi)\}$$

and the perturbed contraction $\mathcal{C}_{\theta \cap}$ coincides with the canonical comparison contraction.

PROOF. Theorem 4 of subsection 3.1 states that from the canonical comparison contraction $\mathcal{C}: \{B(A), X, f, g, \phi\}$ it is possible to establish a contraction between the reduced complexes

$$\mathcal{C}^{\otimes A}: \{\Lambda \otimes_A B(A), \Lambda \otimes_A X, 1 \otimes_A f, 1 \otimes_A g, 1 \otimes_A \phi\}.$$

Proposition 3 states that if A is connected then $\theta \cap$ is a perturbation datum of the contraction

$$1 \otimes \mathcal{C}^{\otimes A}: \{A \otimes (\Lambda \otimes_A B(A)), A \otimes (\Lambda \otimes_A X), 1 \otimes (1 \otimes_A f), 1 \otimes (1 \otimes_A g), 1 \otimes (1 \otimes_A \phi)\}$$

Now, we prove that the perturbed contraction $(1 \otimes \mathcal{C}^{\otimes A})_{\theta \cap}$ coincides with \mathcal{C} . To this end, it suffices to show that the formulae

$$\begin{aligned} f_{\theta \cap} &= 1 \otimes \bar{f} - (1 \otimes \bar{f})\theta \cap (1 \otimes \bar{\phi}) + (1 \otimes \bar{f})\theta \cap (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{\phi}) - \dots \\ g_{\theta \cap} &= 1 \otimes \bar{g} - (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{g}) + (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{g}) - \dots \\ \phi_{\theta \cap} &= 1 \otimes \bar{\phi} - (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{\phi}) + (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{\phi}) - \dots \\ d_{\theta \cap} &= D + (1 \otimes \bar{f})\theta \cap (1 \otimes \bar{g}) - (1 \otimes \bar{f})\theta \cap (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{g}) + \dots \end{aligned} \quad (9)$$

coming from the BPL are the morphisms integrating \mathcal{C} , where D denotes the usual differential over $A \otimes (\Lambda \otimes_A X)$, and \bar{h} denotes $1 \otimes_A h$.

Let us recall that $1 \otimes_A g = g|_{\bar{X}}$, $1 \otimes_A \phi = \phi|_{\bar{B}(A)}$. Furthermore,

$$(\theta \cap)_s(a \otimes [a_1 | \dots | a_n]) = \begin{cases} a \otimes [a_1 | \dots | a_n] & \text{if } a \otimes [a_1 | \dots | a_n] \in \text{Ker} \epsilon_A \otimes \bar{B}(A) \\ 0 & \text{elsewhere} \end{cases}$$

Taking into account the above identities and the inductive definitions of f , g , and ϕ it follows that

$$(\theta \cap)(1 \otimes \bar{\phi}) = -(\theta \cap)(1 \otimes s\phi d + 1 \otimes sgf)|_{\bar{B}} = -(\mu_A \otimes 1)|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes (\phi d + gf)|_{\bar{B}})$$

and,

$$(\theta \cap)(1 \otimes \bar{g}) = (\theta \cap)(1 \otimes sgd|_{\bar{B}}) = (\mu_A \otimes 1)|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes gd|_{\bar{B}}).$$

In view of the previous identities and using that $\phi\phi = 0$, $\phi g = 0$ and (7), we have that

$$\begin{aligned} (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{\phi}) &= -(1 \otimes \bar{\phi})(\mu_A \otimes 1)|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes (\phi d + gf)|_{\bar{B}}) \\ &= -(\mu_A \otimes 1)|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes \phi(\phi d + gf)|_{\bar{B}}) = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} (1 \otimes \bar{\phi})\theta \cap (1 \otimes \bar{g}) &= (1 \otimes \bar{\phi})(\mu_A \otimes 1)|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes gd|_{\bar{B}}) \\ &= (\mu_A \otimes 1)|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes \phi gd|_{\bar{B}}) = 0. \end{aligned}$$

Thus, the formulae (9) may now be rewritten as

$$\begin{aligned} f_{\theta \cap} &= 1 \otimes \bar{f} - (1 \otimes \bar{f})\theta \cap (1 \otimes \bar{\phi}), & g_{\theta \cap} &= 1 \otimes \bar{g}, \\ \phi_{\theta \cap} &= 1 \otimes \bar{\phi}, & d_{\theta \cap} &= D + (1 \otimes \bar{f})\theta \cap (1 \otimes \bar{g}). \end{aligned}$$

Obviously, $g_{\theta \cap}$ and $\phi_{\theta \cap}$ are the A -linear extensions of \bar{g} and $\bar{\phi}$, hence $g_{\theta \cap} = g$ and $\phi_{\theta \cap} = \phi$. Working out the second summand of $f_{\theta \cap}$, we have that

$$\begin{aligned} -(1 \otimes \bar{f})\theta \cap (1 \otimes \bar{\phi}) &= (\mu_A \otimes \bar{f})|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes (\phi d + gf)|_{\bar{B}}) \\ &= (\mu_A \otimes \bar{f})|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes \phi d|_{\bar{B}}) + (\mu_A \otimes \bar{f})|_{A \otimes Ker \epsilon_A \otimes \bar{B}}(1 \otimes gf|_{\bar{B}}). \end{aligned}$$

By property 5 of (8) the first term is zero, since $f\phi = 0$. The second one, acting over an element $\eta_A(1) \otimes [a_1 | \dots | a_n]$ coincides with the summands of $f([a_1 | \dots | a_n])$ which becomes zero when the functor $1 \otimes_A -$ is applied over it, since $fgf = f$. Hence, $f_{\theta \cap} = f$.

In a similar way, it is proved that $d_{\theta \cap} = d$. □

3.3 A_∞ -Structures and HPT

The notion of an A_∞ -(co)algebra was introduced by J. Stasheff [20], which is ‘‘roughly speaking’’ a differential graded (co)algebra in which the (co)associative law holds up to homotopy. Here we deal with the category of A_∞ -coalgebras. Given a DG-module (M, Δ_1) and a sequence of maps $\{\Delta_i \in \text{Hom}^{i-2}(M, M^{\otimes i})\}_{i \geq 1}$, $(M, \Delta_i)_{n \geq 1}$ is called an A_∞ -coalgebra if the relation

$$\sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} (1^{i-n-k} \otimes \Delta_n \otimes 1^k) \Delta_{i-n+1} = 0$$

holds for each $i \geq 1$.

The problem of transferring (co)algebra structures up to contraction has been largely considered in the literature. Here we will need the following results:

Lemma 1. [8] *Assuming that C is a coalgebra, M a DGA-module and $r: \{C, M, f, g, \phi\}$ a contraction, and using the tensor trick (see [8, 9, 10]), then M becomes an A_∞ -coalgebra.*

Moreover, the maps integrating the A_∞ -coalgebra $(M, \Delta_i)_{i \geq 1}$ are shown in [1] to be explicitly

$$\Delta_i = (-1)^{\frac{(i-1)(i-2)}{2}} f^{\otimes i} \circ \left[\sum_{k_2=1}^2 \sum_{k_3=1}^{k_2+1} \cdots \sum_{k_{i-1}=1}^{k_{i-2}+1} \prod_{j=2}^{i-1} (-1)^{k_j} (1^{\otimes k_j-1} \otimes \Delta_C \phi \otimes 1^{\otimes j-k_j}) \right] \Delta_C g, \quad (10)$$

where $\prod_{j=2}^{i-1} h_j$ denotes the composition $h_{i-1} \circ \cdots \circ h_2$.

Given a DGA-algebra A and an A_∞ -coalgebra C , an A_∞ -twisting cochain (or A_∞ -TTP) $\tau: C \rightarrow A$ is a DG-module morphism of degree -1, such that satisfies the following identity

$$d\tau + \sum_{i=1}^{\infty} \mu^{(i)} \tau^{\otimes i} \Delta_i = 0,$$

where $\mu^{(1)} = 1$, $\mu^{(2)} = \mu$, and in general $\mu^{(k)} = \mu(1 \otimes \mu^{(k-1)})$. Analogously to twisting cochain, $\tau: C \rightarrow A$ is an A_∞ -twisting cochain [18] if and only if $d_\tau = d \otimes 1 + \sum_{i=1}^{\infty} (\mu^{(i)} \otimes 1)(1 \otimes \tau^{\otimes i-1} \otimes 1)(1 \otimes \Delta_i)$ is a differential on $A \otimes C$, which, together with this differential, is denoted by $A \otimes_\tau C$ and is referred to as the A_∞ -twisted tensor product (along τ).

Theorem 6. [1] *Let $t: C \rightarrow A$ be a twisting cochain and $c(f, g, \phi): C \Rightarrow C'$ be a contraction such that c induces on C' an A_∞ -coalgebra structure (see Lemma 1). Additionally, assume that $t\phi = 0$ and $(1 \otimes \phi)t \cap$ is pointwise nilpotent. There is a contraction*

$$A \otimes_t C \Rightarrow A \otimes_{\bar{t}} C',$$

where $\bar{t} = tg$ is an A_∞ -twisting cochain and $A \otimes_{\bar{t}} C'$ is an A_∞ -twisted tensor product.

Remark 3. The hypotheses of Theorem 6 are satisfied when C is a simply connected DGA-coalgebra. For instance, $\bar{B}(A)$ is a simply connected DGA-coalgebra when A is a connected DGA-algebra.

The following result is a straightforward consequence of Theorem 6, and it is a main result of this paper.

Algorithm 1. Computing the A_∞ -twisting tensor product structure

INPUT: A contractile relatively free resolution (X, t, d) of A over a connected DG-algebra A where $d \circ t|_X = 0$.

- Step 1. Form the canonical comparison contraction $(f, g, \phi): B(A) \Rightarrow X$ using the formulas (5), (4) and (6).
- Step 2. Form the contraction $(1 \otimes_A f, 1 \otimes_A g, 1 \otimes_A \phi): \bar{B}(A) \Rightarrow \bar{X}$ as in Theorem 4 .

OUTPUT: The map $\tau = \theta g|_{\bar{X}}$ which is an A_∞ -twisting cochain and the maps Δ_i (given by formula (10) using the morphisms of the contraction of Step 2) integrating the A_∞ -coalgebra structure $(\bar{X}, \Delta_i)_{i \geq 1}$.

CORRECTNESS: Let us emphasize that $1 \otimes_A g = g|_{\bar{X}}$ and $1 \otimes_A \phi = \phi|_{\bar{B}(A)}$ since g and ϕ are A -linear and $g(\bar{X}) \subseteq \bar{B}(A)$, and $\phi(\bar{B}(A)) \subseteq \bar{B}(A)$. An explicit formula for ϕ is given in [13, 14] which increases the simplicial degree in $\bar{B}(A)$ by one and $\phi_0 = 0$. Therefore, $\theta \phi|_{\bar{B}(A)} = 0$ since $\theta: \bar{B}(A) \rightarrow A$ is the universal twisting cochain. Now, applying Theorem 6, we have the following A_∞ -twisting cochain

$$\tau = \theta g|_{\bar{X}}: \bar{X} \rightarrow A$$

The second step is to construct the tensor product contraction

$$A \otimes \bar{B}(A) \Rightarrow A \otimes \bar{X}$$

and to use the Basic Perturbation Lemma with $\theta \cap$ as the perturbation datum (see Theorem 5). Then, it is straightforward to check that $(1 \otimes \phi|_{\bar{B}(A)}) \theta \cap$ is pointwise nilpotent. So we obtain the new contraction,

$$A \otimes_\theta \bar{B}(A) \Rightarrow (A \otimes \bar{X}, d_{\theta \cap})$$

Now, using Theorem 6, we have that

$$(A \otimes \bar{X}, d_{\theta \cap}) = A \otimes_\tau \bar{X}$$

where $A \otimes_\tau \bar{X}$ is an A_∞ -twisted tensor product.

In the proof of the last identity, we use the special properties of the morphisms which take part in the canonical comparison contraction. □

4 Algebra Structures in the Comparison of Resolutions

If A is a commutative DGA-algebra, it is well known that it is possible to define a commutative product $*$ on $\bar{B}(A)$ called shuffle product. Furthermore, $B(A)$ has canonically associated a commutative algebra structure by means of the morphism $\mu_{B(A)} = (\mu_A \otimes *) (1_A \otimes T \otimes 1_{B(A)})$, where $T(\bar{b} \otimes a) = (-1)^{|\bar{b}| |a|} a \otimes \bar{b}$.

Throughout this section, we assume that A is a commutative DGA-algebra, (X, t, d) is a contractile relatively free resolution with t as contracting homotopy which is a quasi algebra homotopy, and there exists $\mathcal{C}: \{B(A), X, f, g, \phi\}$ the canonical comparison contraction.

Before giving the main result of this section we need some preliminary results which are easy to prove:

Lemma 2. [19] *Let M be a DGA-module and $c: \{A, M, f', g', \phi'\}$ be a contraction. If $\phi' \mu_A(g' \otimes g') = 0$, then the morphism $\mu_M = f' \mu_A(g' \otimes g')$ defines a commutative product on M . Furthermore, g' is a DGA-algebra morphism with regard to the products μ_A and μ_M .*

Lemma 3. *Let A' be a DG-algebra, and $c: \{A, A', f', g', \phi'\}$ be a contraction of DG-modules. Then,*

$$\phi' \mu_A - \mu_A \phi'^{[\otimes 2]} = \phi' \mu_A \phi'^{[\otimes 2]} d^{[2]} - d \phi' \mu_A \phi'^{[\otimes 2]} - g' f' \mu_A \phi'^{[\otimes 2]} \quad (11)$$

where $\phi'^{[\otimes 2]}$ and $d^{[2]}$ denote, respectively, $1_A \otimes \phi + \phi \otimes g f$ and $d_A \otimes 1_A + 1_A \otimes d_A$. Assuming that f' is a quasi algebra projection and ϕ' is a quasi algebra homotopy,

$$\mu_A(\phi'(a) \otimes \phi'(b)) = (-1)^{|a|+1} \phi'(\mu_A(\phi'(a) \otimes b)) + \phi'(\mu_A(a \otimes \phi'(b))) \quad (12)$$

where $a, b \in A$.

Lemma 4. *The identity $1 = s d \phi d + s d g f$ holds on reduced elements with degree greater than zero.*

Lemma 5. *If $\phi_n \mu_{B(A)}(g \otimes g)_n(\bar{x}_1 \otimes \bar{x}_2) = 0$, for any a reduced element $\bar{x}_1 \otimes \bar{x}_2$ of degree n then for any element $x_1 \otimes x_2$ of degree n of $X \otimes X$*

$$\phi_n \mu_{B(A)}(g \otimes g)_n(x_1 \otimes x_2) = 0.$$

Now, we state the main result of this section:

Theorem 7. *The A -module (X, d) equipped with the morphism $\mu_X = f^*(g \otimes g)$ becomes a commutative DGA-algebra. Furthermore, \mathcal{C} is an almost-full algebra contraction with regard to the products $\mu_{B(A)}$ and μ_X .*

PROOF. The proof will be divided into two parts.

Firstly, we show that μ_X defines a commutative product on X . To this end, we apply Lemma 2, in order to prove by induction that $\phi \mu_{B(A)}(g \otimes g) = 0$ on reduced elements. Then this relation is extended to $X \otimes X$ by Lemma 5. We have $\phi_0 = 0$ by construction and for $n > 0$, on reduced elements,

$$\phi_n \mu_{B(A)}(g \otimes g)_n = -s \phi_{n-1} d_n \mu_{B(A)}(g \otimes g)_n - s g_n f_n \mu_{B(A)}(g \otimes g)_n.$$

This equality comes from the fact that $\mu_{B(A)}|_{B(A) \otimes B(A)} \subset \bar{B}(A)$ and the property (7) of g . By induction hypothesis

$$s \phi_{n-1} d_n \mu_{B(A)}(g \otimes g)_n = s \phi_{n-1} \mu_{B(A)}(g \otimes g)_{n-1} d_n^{[2]} = 0.$$

Now from Lemma 4

$$\begin{aligned}
sg_n f_n \mu_{B(A)}(g \otimes g)_n &= sg_n f_n (sd_n \phi_{n-1} d_n + sd_n g_n f_n) \mu_{B(A)}(g \otimes g)_n \\
&= sg_n f_n (sd_n \phi_{n-1} + sg_{n-1} f_{n-1}) \mu_{B(A)}(g \otimes g)_{n-1} d_n^{[2]} \\
&= sg_n f_n sd_n \phi_{n-1} \mu_{B(A)}(g \otimes g)_{n-1} d_n^{[2]} + sg_n f_n sg_{n-1} f_{n-1} \mu_{B(A)}(g \otimes g)_{n-1} d_n^{[2]}.
\end{aligned}$$

By induction and Lemma 5 the first summand is null. Let us notice that in the second summand $sg_{n-1} f_{n-1} \mu_{B(A)}(g \otimes g)_{n-1}$ can be applied on non-reduced elements, so in order to prove that it is null we need a more sophisticated argument.

We show by induction that if $sg f \mu_{B(A)}(g \otimes g)$ is null on any element of degree less than n , then it is possible to extend this property to any element in $X \otimes X$ of degree n . The first case of the induction is trivial (for $n = 0$). We take a generic element $(a \otimes \bar{x}) \otimes (a' \otimes \bar{x}')$ in degree n . Thus,

$$sg_n f_n \mu_{B(A)}(g(a \otimes \bar{x}) \otimes g(a' \otimes \bar{x}'))_n = (-1)^{|a'| |\bar{x}|} s(aa' g f \mu_{B(A)}(g(\bar{x}) \otimes g(\bar{x}'))).$$

Now, using again Lemma 4,

$$s(aa' g f \mu_{B(A)}(g(\bar{x}) \otimes g(\bar{x}'))) = s(aa' g f (sd\phi d + sdg f) \mu_{B(A)}(g(\bar{x}) \otimes g(\bar{x}))),$$

let us observe that the summands, $(sd\phi d + sdg f) \mu_{B(A)}(g(\bar{x}) \otimes g(\bar{x}'))$, are zero by induction hypothesis. This fact completes the proof of the first step.

Secondly, we are proving that \mathcal{C} is an almost-full algebra contraction (i.e., f and g are DGA-algebra morphisms and ϕ is a quasi algebra morphism with regard to the products $\mu_{B(A)}$ and μ_X). Lemma 2 guarantees that g is a DGA-algebra morphism. In order to prove that f is a DGA-algebra morphism we need to see that $\mu_X(f \otimes f) = f \mu_{B(A)}$. The proof is by induction on reduced elements and then extended A -linearly in each degree. Obviously, we have $\mu_X(f_0 \otimes f_0) = f_0 \mu_{B(A)}$ and for $n > 0$,

$$\begin{aligned}
\mu_X(f_{n-i}(\bar{b}) \otimes f_i(\bar{b}')) &= \mu_X(tf_{n-i-1}d_{n-i}(\bar{b}) \otimes tf_{i-1}d_i(\bar{b}')) \\
&= (-1)^{|\bar{b}|} t\mu_X(tf_{n-i-1}d_{n-i}(\bar{b}) \otimes f_{i-1}d_i(\bar{b}')) + t\mu_X(f_{n-i-1}d_{n-i}(\bar{b}) \otimes tf_{i-1}d_i(\bar{b}')) \\
&= (-1)^{|\bar{b}|} t\mu_X(f_{n-i}(\bar{b}) \otimes f_{i-1}d_i(\bar{b}')) + t\mu_X(f_{n-i-1}d_{n-i}(\bar{b}) \otimes f_i(\bar{b}')).
\end{aligned}$$

In the second identity above we have taken into account that t is a quasi algebra homotopy with respect to the product μ_X and we have applied (12).

On the other hand,

$$\begin{aligned}
f_n \mu_{B(A)}(\bar{b} \otimes \bar{b}') &= tf_{n-1}d_n \mu_{B(A)}(\bar{b} \otimes \bar{b}') \\
&= tf_{n-1} \mu_{B(A)}(d_{n-i}(\bar{b}) \otimes \bar{b}') + (-1)^{|\bar{b}|} tf_{n-1} \mu_{B(A)}(\bar{b} \otimes d_i(\bar{b}')) \\
&= t\mu_X(f_{n-i-1}d_{n-i}(\bar{b}) \otimes f_i(\bar{b}')) + (-1)^{|\bar{b}|} t\mu_X(f_{n-i}(\bar{b}) \otimes f_{i-1}d_i(\bar{b}')),
\end{aligned}$$

the last identity is obtained from induction hypothesis. So we have actually proved that f is a DGA-algebra morphism.

Finally, we will prove that ϕ is a quasi algebra homotopy, i.e., the conditions

$$\phi\mu_{B(A)}(\phi \otimes g) = 0, \quad \phi\mu_{B(A)}(g \otimes \phi) = 0, \quad \phi\mu_{B(A)}(\phi \otimes \phi) = 0 \quad \text{hold.}$$

The proof is by induction. We have $\phi_0 = 0$ by construction, so the above three identities hold. For $n > 0$, on reduced elements, the proof consists in replacing $\phi\mu_{B(A)}$ by

$$-s\phi d\mu_{B(A)} - sgf\mu_{B(A)} = s\phi\mu_{B(A)}(d \otimes 1 + 1 \otimes d) - sg\mu_X(f \otimes f),$$

then, the summands of the form $sg\mu_X(f \otimes f)$ are all null, since $f\phi = 0$. Moreover, the summands which contain $dg \otimes \phi$ or $\phi \otimes dg$ are zero, since $d_n g_n = g_{n-1} d_n$. So it is possible to apply induction hypothesis. To sum up, we must only study the summands of the form:

$$s\phi\mu_{B(A)}(d\phi \otimes g), \quad s\phi\mu_{B(A)}(g \otimes d\phi), \quad s\phi\mu_{B(A)}(d\phi \otimes \phi), \quad s\phi\mu_{B(A)}(\phi \otimes d\phi).$$

Replacing $d\phi$ by $1 - gf - \phi d$ is immediate to see that all summands are null. By A -linearity the proof is extended to elements of $B(A)$. This completes the proof. \square

Finally, we provide the following algorithm for computing (algebra) resolutions which split off of the bar construction of a commutative DGA-algebra \tilde{A} , taking as input datum a contractile relatively free resolution of Λ over a commutative DGA-algebra A , where A and \tilde{A} coincide as graded module.

Algorithm 2. Computing ‘algebra’ resolutions which split off

INPUT: A contractile relatively free resolution (X, t, d) of Λ over a commutative DG-algebra A where $d \circ t|_{\tilde{X}} = 0$ and t is a quasi algebra homotopy.
 A commutative DGA-algebra \tilde{A} which has the same underlying graded Λ -module structure than A .

- Step 1. Form the canonical comparison contraction $(f, g, \phi): B(A) \rightrightarrows X$ using the formulas (5), (4) and (6).
- Step 2. Construct the bar constructions $(B(\tilde{A}), \partial^+)$ and $(B(A), \partial)$. Define the morphism $\delta = \partial^+ - \partial$.
- Step 3. Perturb the above contraction using δ , (if $\phi\delta$ is nilpotent).

OUTPUT: A semi-full algebra contraction $B(\tilde{A}) \rightrightarrows \tilde{X}$. Hence, \tilde{X} is an algebra resolution of Λ over \tilde{A} , where $\mu_{\tilde{X}} = f * (g \otimes g)$.

We point out that the contraction of Step 1 is almost-full (see Theorem 7). Furthermore, in the case that $\phi\delta$ is pointwise nilpotent, thus δ is an algebra perturbation datum of the contraction of Step 1. Hence, using Theorem 3, we conclude with the desired result. The main computational advantage of the algebra structure in \tilde{X} is that it is only necessary to compute the perturbed differential on the generators of \tilde{X} as an algebra, in spite of, on the whole set of generators as a module. We will clarify this aspect in the following example.

5 An Example

Now, we give an example in order to illustrate the main results of the paper. We work with the resolution $Q_{(p)}(w, 2n) \otimes E(\sigma(w), 2n+1) \otimes \Gamma(\varphi_p(w), 2np+2)$ (see [5]). Making use of the main results of the paper we reach the same results on the A_∞ -structure of this DG-module as Proute in [18]. Furthermore, we prove that this complex is a DGA-algebra. Hence, it is an example of a multiplicative A_∞ -twisted tensor product. Moreover, this resolution can be ‘‘perturbed’’ into a resolution of $\mathbf{Z}_{(p)}$ over $\Gamma(w, 2n)$. Notice that the way for obtaining the resolution above is different from that given in [2].

Following Cartan’s work in [5], we will use in the sequel the suspension ‘ σ ’, p -transpotence ‘ φ_p ’ and k -th divided power ‘ γ_k ’ for terming the generators of the DGA-algebras.

Let p be a prime number and $I = (w^p)$ be the ideal generated by w^p . Then, $Q_{(p)}(w, 2n) = P(w, 2n)/I$ is the truncated polynomial algebra on one generator w of degree $2n$ with zero differential. We consider here the resolution $X = Q_{(p)}(w, 2n) \otimes E(\sigma(w), 2n+1) \otimes \Gamma(\varphi_p(w), 2np+2)$ where the differential is a derivation and is defined by

$$d(\sigma(w)) = w, \quad d(\gamma_i(\varphi_p(w))) = w^{p-1} \sigma(w) \gamma_{i-1}(\varphi_p(w)).$$

The following degree one morphism $t: X \rightarrow X$ linear over Λ (but not over $Q_{(p)}(w, 2n)$) defined as

$$t(1) = 0, \quad t(w^k \gamma_i(\varphi_p(w))) = w^{k-1} \sigma(w) \gamma_i(\varphi_p(w))$$

and

$$t(w^k \sigma(w) \gamma_i(\varphi_p(w))) = \begin{cases} \gamma_{i+1}(\varphi_p(w)) & k = p-1 \\ 0 & k \neq p-1 \end{cases}$$

is a contracting homotopy for X . This explicit formula for t is crucial to many constructions but it is not widely distributed. Moreover, the data set

$$c_X: \{X, \Lambda, \epsilon, \eta, t\} \tag{13}$$

is a contraction, where $\epsilon_0 = 1_\Lambda$, $\epsilon_n = 0$, $n > 0$ and $\eta(\lambda) = \lambda \otimes 1 \otimes 1$. Now, by comparison theorem for resolutions and using the formulae (5), (4) and (6) for comparison maps, we have the following $Q_{(p)}(w, 2n)$ -linear morphisms defined on the reduced complexes by

$$f[w^{r_1} | w^{t_1} | \dots | w^{r_m} | w^{t_m}] = \left\{ \prod_{k=1}^n \delta_{p, r_k + t_k} \right\} \gamma_m(\varphi_p(w)),$$

$$f[w^l | w^{r_1} | w^{t_1} | \dots | w^{r_m} | w^{t_m}] = w^{l-1} \left\{ \prod_{k=1}^n \delta_{p, r_k + t_k} \right\} \sigma(w) \gamma_m(\varphi_p(w)),$$

where the symbols $\delta_{i,j}$ are defined by: $\delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$

$$g(\sigma(w)) = [w], \quad g(\gamma_i(\varphi_p(w))) = [w^{p-1} | w | \overset{i\text{-times}}{\dots} | w^{p-1} | w],$$

$$g(\sigma(w)\gamma_i(\varphi_p(w))) = [w|w^{p-1}| \overset{i\text{-times}}{\dots} |w|w^{p-1}|w]$$

and the homotopy operator ϕ is defined by $-\psi$, where

$$\psi[\] = 0, \quad \psi[w] = 0,$$

$$\psi[w^x] = [w^{x-1}|w], \quad 1 < x < p,$$

$$\psi[w^x|w^y] = [w^x|w^{y-1}|w],$$

$$\psi[z|w^x|w^y] = [z|w^x|w^{y-1}|w] + \delta_{p,x+y}[\psi(z)|w^{p-1}|w]$$

for $z \in \bar{B}(Q_{(p)}(w, 2n))$.

It is a straightforward computation to verify that $dt|_X = 0$. Then by Theorem 1 we can state that the data set

$$\mathcal{C}_{B-X}: \{B(Q_{(p)}(w, 2n)), X, f, g, \phi\}$$

is a contraction: *the canonical comparison contraction between $B(Q_{(p)}(w, 2n))$ and X* . It is immediate to see that t is a quasi algebra homotopy, then by Theorem 7 we can guarantee that \mathcal{C}_{B-X} is an almost-full contraction.

Now, we can apply Algorithm 1 to the resolution X , and define the degree minus one morphism $\tau: E(\sigma(w), 2n + 1) \otimes \Gamma(\varphi_p(w), 2np + 2) \rightarrow Q_{(p)}(w, 2n)$ by

$$\tau(\sigma(w)) = \theta g(\sigma(w)) = w \quad \text{and} \quad \tau = 0 \quad \text{otherwise}$$

which is a A_∞ -twisting cochain.

Working with coefficients in \mathbf{Z}_p , Proute determined in [18] the A_∞ -coalgebra structure of $E(\sigma(w), 2n + 1) \otimes \Gamma(\varphi_p(w), 2np + 2)$ given the following formulae:

$$\Delta_h = 0, \quad h \neq 2, p,$$

$$\Delta_2(\sigma(w)^j \gamma_i(\varphi_p(w))) = \sum_{k=0}^j \sum_{l=0}^i \sigma(w)^k \gamma_l(\varphi_p(w)) \otimes \sigma(w)^{j-k} \gamma_{i-l}(\varphi_p(w)),$$

$$\Delta_p(\sigma(w)^j \gamma_i(\varphi_p(w))) = \sum_{l_1 + \dots + l_p = i-1} \sigma(w)^{j+1} \gamma_{l_1}(\varphi_p(w)) \otimes \dots \otimes \sigma(w)^{j+1} \gamma_{l_p}(\varphi_p(w)).$$

In [18, pp.148-149] it is proved that

$$d = (\mu \otimes 1)(1 \otimes \tau \otimes 1)(1 \otimes \Delta_2) + (\mu^{(p)} \otimes 1)(1 \otimes \tau^{\otimes i-1} \otimes 1)(1 \otimes \Delta_p)$$

as the output of the Algorithm 1 states.

Summing up, $X = Q_{(p)}(w, 2n) \otimes_\tau E(\sigma(w), 2n + 1) \otimes \Gamma(\varphi_p(w), 2np + 2)$ is a multiplicative A_∞ -twisted tensor product, i.e., X is a DGA-algebra and an A_∞ -TTP simultaneously.

For the remainder of this example we have taken $\mathbf{Z}_{(p)}$ (\mathbf{Z} localized at prime p) as the ground ring. In the following, we give the outline of a process for

constructing a resolution of $\mathbf{Z}_{(p)}$ over $\Gamma(w, 2n)$. It is obtained by perturbing a resolution of $\mathbf{Z}_{(p)}$ over $\otimes_{i \geq 0} Q_{(p)}(w_i, 2np^i)$.

Firstly, we use an isomorphism of DGA-algebras [19, Prop. 5.24] between $\Gamma(w, 2n)$ and $\tilde{\otimes}_{i \geq 0} Q_{(p)}(w_i, 2np^i)$. As $\mathbf{Z}_{(p)}$ -module, this last DGA-algebra is equal to the ordinary tensor product $\otimes_{i \geq 0} Q_{(p)}(w_i, 2np^i)$. Its multiplicative law is given by

$$w_i^k w_j^h = \begin{cases} w_i^k \otimes w_j^h & \text{if } i \neq j, \\ w_i^{k+h} & \text{if } i = j \text{ and } h + k < p, \\ -pw_i^t w_{i+1} & \text{if } i = j \text{ and } h + k = p + t \end{cases}$$

From now on, we will identify the generators w_i of the truncated algebras with the elements $\gamma_{p^i}(w)$ of $\Gamma(w, 2n)$; in fact, the image by the isomorphism of w_i coincides with $\gamma_{p^i}(w)$ excluding the coefficient.

Secondly, we give an explicit contracting homotopy t^\otimes for the resolution

$$X^\otimes = \otimes_{i \geq 0} Q_{(p)}(w_i, 2np^i) \otimes (\otimes_{i \geq 0} E(\sigma(w_i), 2np^i + 1) \otimes \Gamma(\varphi_p(w_i), 2np^i + 2)).$$

To this end, we use that the complex above is just the tensor product complex

$$\otimes_{i \geq 0} (Q_{(p)}(w_i, 2np^i) \otimes E(\sigma(w_i), 2np^i + 1) \otimes \Gamma(\varphi_p(w_i), 2np^i + 2))$$

and the formula for t^\otimes is:

$$t \otimes 1 \otimes 1 \otimes \dots + \eta \epsilon \otimes t \otimes 1 \otimes 1 \otimes \dots + \eta \epsilon \otimes \eta \epsilon \otimes t \otimes 1 \otimes 1 \otimes \dots + \dots,$$

thus,

$$t^\otimes(1 \otimes \overset{l\text{-times}}{\otimes} 1 \otimes x \otimes z) = 1 \otimes \overset{l\text{-times}}{\otimes} 1 \otimes t(x) \otimes z$$

where $|x| > 0$, $x \in Q_{(p)}(w_l, 2np^l) \otimes E(\sigma(w_l), 2np^l + 1) \otimes \Gamma(\varphi_p(w_l), 2np^l + 2)$ and $z \in \otimes_{i > l} (Q_{(p)}(w_i, 2np^i) \otimes E(\sigma(w_i), 2np^i + 1) \otimes \Gamma(\varphi_p(w_i), 2np^i + 2))$.

Since the contraction (13) is an almost-full algebra contraction, and the class of almost-full contraction is closed by tensor product [19], it follows that t^\otimes is a quasi algebra homotopy.

Now, by comparison theorem for resolutions and using the formulae (5), (4) and (6) for comparison maps, it is possible to construct three morphisms denoted by f^\otimes , g^\otimes , ϕ^\otimes .

Since $t|_{\bar{x}} = 0$, we have that $dt^\otimes|_{\bar{x}^\otimes} = 0$. Then by Theorem 1 we can state that the data set

$$\{B(\otimes_{i \geq 0} Q_{(p)}(w_i, 2np^i)), X^\otimes, f^\otimes, g^\otimes, \phi^\otimes\} \quad (14)$$

is a contraction. Besides, it is an almost-full algebra contraction thanks to Theorem 7. Hence, in particular, g^\otimes is DGA-algebra morphism, then g^\otimes is completely determined by g , i.e.,

$$g^\otimes(x_0 \otimes \dots \otimes x_n) = g(x_0) * g(x_1) * \dots * g(x_n), \forall n \geq 0;$$

where $x_i \in E(\sigma(w_i), 2np^i + 1) \otimes \Gamma(\varphi_p(w_i), 2np^i + 2)$ and $*$ denotes the well-known shuffle product in the bar construction.

The projection f^\otimes is given by

$$f^\otimes[w_{i_1}^{k_1} \otimes z_1 | w_{i_2}^{k_2} \otimes z_2 | \dots | w_{i_n}^{k_n} \otimes z_n] = \begin{cases} w_{i_1}^{k_1-1} z_1 w_{i_2}^{k_2-1} z_2 \cdots w_{i_n}^{k_n-1} z_n \sigma(w_{i_1}) \sigma(w_{i_2}) \cdots \sigma(w_{i_n}) & i_1 < i_2 < \cdots < i_n, \\ \left\{ \prod_{j=0}^{m-1} \delta_{p, k_{2j+1} + k_{2j+2}} \right\} z_1 \cdots z_n \gamma_m(\varphi_p(w_{i_1})) & i_1 = i_2 = \cdots = i_{n=2m}, \\ \left\{ \prod_{j=1}^m \delta_{p, k_{2j} + k_{2j+1}} \right\} w_{i_1}^{k_1-1} z_1 \cdots z_n \sigma(w_{i_1}) \gamma_m(\varphi_p(w_{i_1})) & i_1 = i_2 = \cdots = i_{n=2m+1}, \\ f^\otimes[a_{i_1} | \dots | a_{i_{i_1}}] \otimes \cdots \otimes f^\otimes[a_{i_h} | \dots | a_{i_n}] & i_1 = \dots = i_{i_1} < \cdots < i_h = \dots = i_n \\ 0 & \text{otherwise,} \end{cases}$$

where $z_j \in \otimes_{i>j} Q_{(p)}(w_i, 2np^i)$ and $a_j = w_{i_j}^{k_j} \otimes z_j$. And the homotopy operator ϕ^\otimes is defined by $-\psi^\otimes$,

$$\psi^\otimes[] = 0,$$

$$\psi^\otimes[w_{i_1}^{k_1} \otimes z_1] = [w_{i_1}^{k_1-1} z_1 | w_{i_1}],$$

$$\psi^\otimes[w_{i_1}^{k_1} z_1 | w_{i_2}^{k_2} z_2] = \begin{cases} \left([w_{i_1}^{k_1} z_1 | w_{i_2}^{k_2-1} z_2 | w_{i_2}] + [w_{i_1}^{k_1-1} z_1 w_{i_2}^{k_2-1} z_2 | w_{i_1} | w_{i_2}] - [w_{i_1}^{k_1-1} z_1 w_{i_2}^{k_2-1} z_2 | w_{i_2} | w_{i_1}] \right) & i_1 < i_2, \\ [w_{i_1}^{k_1} z_1 | w_{i_2}^{k_2-1} z_2 | w_{i_2}] & i_1 \geq i_2. \end{cases}$$

The situation in higher degrees is similar but slightly more complicated and is left to the interested reader.

For the sake of clarity, we will write the DGA-algebras without denoting the degree of the generators.

[19, Prop. 5.24] tells that there is an isomorphism between $\Gamma(w)$ and a tensor product $\tilde{\otimes}_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))$.

$\tilde{\otimes}_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))$ and $\otimes_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))$ have the same underlying graded $\mathbf{Z}_{(p)}$ -module structure and because of this, $B(\tilde{\otimes}_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w)))$ has the same underlying graded $\mathbf{Z}_{(p)}$ -module structure as $B(\otimes_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w)))$. Let B denote this graded $\mathbf{Z}_{(p)}$ -module structure for either case. Thus B supports two different differentials, viz., the bar construction differential ∂^+ for $\tilde{\otimes}_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))$ and the bar construction differential ∂ for $\otimes_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))$. Let $\delta = \partial^+ - \partial$ be the perturbation of the DG-module B . The formula for δ , up to sign, is

$$\begin{aligned} & \delta \left(\otimes_{i \geq 0} (\gamma_{p^i}(w))^{h_{0,i}} \left[\otimes_{i \geq 0} (\gamma_{p^i}(w))^{h_{1,i}} \mid \otimes_{i \geq 0} (\gamma_{p^i}(u))^{h_{2,i}} \mid \dots \right] \right) \\ &= \cdots p(\gamma_{p^t}(w))^t \gamma_{p^{t+1}}(w) \cdots \left[\otimes_{i \geq 0} (\gamma_{p^i}(w))^{h_{2,i}} \mid \dots \right] \\ & \quad + \otimes_{i \geq 0} (\gamma_{p^i}(w))^{h_{0,i}} \left[\cdots p(\gamma_{p^j}(w))^t \gamma_{p^{j+1}}(w) \cdots \mid \dots \right] + \dots \end{aligned}$$

The first summand appears (it is non-zero) if there exists at least one value for i such that $h_{0,i} + h_{1,i} = p + t$ where $0 \leq t \leq p - 2$. The second summand

appears if there exists at least one value for i such that $h_{1,i} + h_{2,i} = p + t$ where $0 \leq t \leq p - 2$. And so on.

It is clear that δ is a derivation and represents the perturbation induced in the differential of $B(\otimes_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w)))$ by the modification produced in the product of the algebra $\otimes_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))$. In this situation, there is a formal process (the Basic Perturbation Lemma) which, taking as the input data the contraction (14) and the perturbation δ , when $\delta\phi^\otimes$ is nilpotent in each degree, it gives a new contraction

$$\begin{aligned} & B(\tilde{\otimes}_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w))) \\ & \quad \downarrow \\ & (\otimes_{i \geq 0} Q_{(p)}(\gamma_{p^i}(w)) \otimes (\otimes_{i \geq 0} E(\sigma\gamma_{p^i}(w)) \otimes \Gamma(\varphi_p\gamma_{p^i}(w))), d + d_\delta) \end{aligned} \quad (15)$$

Our aim here is to verify that $\delta\phi^\otimes$ is nilpotent in each degree. To this end, we take an element of the form

$$\otimes_{i=0}^n (\gamma_{p^i}(w))^{h_{0,i}} [\otimes_{i=0}^n (\gamma_{p^i}(w))^{h_{1,i}} | \dots | \otimes_{i=0}^n (\gamma_{p^i}(w))^{h_{l,i}}] \in B,$$

the number: $\sum_{j=0}^l \sum_{i=0}^n h_{j,i}$ defines a filtration in B .

It is easy to see that ϕ^\otimes does not increase the filtration degree. On the other hand, δ either lowers the filtration degree or is null, $\forall n, \forall l \in \mathbf{N}$. Then, $\delta\phi^\otimes$ either lowers filtration or is null, and this means that this composition is nilpotent in each degree.

Taking into account Theorem 3, the contraction above is a semi-full algebra contraction. Notice that the product on the second complex coincides with the product on X^\otimes . Then, we only need to compute d_δ on the algebra generators, in order to compute d_δ on all module generators. Moreover, we shall show that

$$\phi^\otimes \delta g^\otimes = 0 \quad (16)$$

Let us observe that g^\otimes carries any algebra generator x of the reduced complex into an element y of the form $[\gamma_{p^i}(w)]$ or $[(\gamma_{p^i}(w))^{p-1} | \gamma_{p^i}(w)]$. Now, we study the image of y under δ . It is not difficult to see that this image is zero if $y = [\gamma_{p^i}(w)]$ and $p[\gamma_{p^{i+1}}(w)]$ if $y = [(\gamma_{p^i}(w))^{p-1} | \gamma_{p^i}(w)]$. Since $\phi^\otimes[\gamma_{p^{i+1}}(w)] = 0$, we obtain the desired result.

Consequences of (16) are:

$$(g^\otimes)_\delta = g^\otimes, \quad d_\delta = f^\otimes \delta g^\otimes.$$

Summing up, (15) is a resolution of \mathbf{Z}_p over $\Gamma(w)$ where

$$d_\delta(\sigma\gamma_{p^i}(w)) = f^\otimes \delta g^\otimes(\sigma\gamma_{p^i}(w)) = f^\otimes \delta[\gamma_{p^i}(w)] = f^\otimes(0) = 0,$$

$$d_\delta(\varphi_p\gamma_{p^i}(w)) = f^\otimes \delta g^\otimes(\varphi_p\gamma_{p^i}(w)) = f^\otimes \delta[(\gamma_{p^i}(w))^{p-1} | \gamma_{p^i}(w)]$$

$$= f^\otimes(p[\gamma_{p^{i+1}}(w)]) = p\sigma\gamma_{p^{i+1}},$$

$$d_\delta(\gamma_k\varphi_p\gamma_{p^i}(w)) = f^\otimes \delta g^\otimes(\gamma_k\varphi_p\gamma_{p^i}(w)) = p\sigma\gamma_{p^{i+1}}(w) \gamma_{k-1}\varphi_p\gamma_{p^i}(w).$$

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