ON THE SOLUTION OF A CLASS OF NON LINEAR DIRICHLET PROBLEMS BY A PENALTY-DUALITY METHOD AND FINITE ELEMENTS OF ORDER ONE

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INTRODUCTION.

In this paper, we shall give some results on the approximation and on the numerical solution of some non linear elliptical problems. It is also shown that the iterative method used to solve the approximate problems is also useful for solving other non linear problems arising in mechanics and physics.

1. THE CONTINUOUS PROBLEM.

Let Ω be a bounded open set of $\mathbb{R}^N$, such that its boundary Γ is regular. Let p be such that $1 < p < +\infty$.
We shall denote by V the space $W_o^{1,p}(\Omega)$ whose norm is $\|v\|_1 = (\int_\Omega |\nabla v|^p dx)^{1/p}$.
Let p' be the conjugate of p i.e. $(p-1)(p'-1) = 1$. Let V' be the dual $W^{-1,p'}(\Omega)$ of V and $\|.\|_*$ its norm.
We shall write $\|v\|_s$ instead of $\|v\|_{W^{s,p}(\Omega)}$.

It can be shown (see for example, [1], chapter 2) that the non linear elliptical problem :

(1) $$-\nabla.(|\nabla u|^{p-2}\nabla u) = f, \quad f \in V'$$

(2) $$u = o \text{ on } \Gamma$$

has a unique solution and is equivalent to

(3) $$\int_\Omega |\nabla u|^{p-2}\nabla u . \nabla v \, dx = \langle f,v \rangle \; \forall v \in V, \; u \in V.$$

In (3), $\langle .,. \rangle$ is the bilinear form of the duality between V' and V.

We shall call A the monotonous operator form $V \to V'$ defined by

$$A(v) = -\nabla(|\nabla v|^{p-2}\nabla v) \; ;$$

if $p = 2$, $A = -\Delta$.

2. THE APPROXIMATED PROBLEM.

For the sake of clarity we shall suppose that Ω is a polyhedral of $\mathbb{R}^2$. Let $\mathcal{T}_h$ be a finite triangulation of Ω such that :

(4) $$T \in \bar{\Omega} \quad \forall T \in \mathcal{T}_h, \qquad \bigcup_{T \in \mathcal{T}_h} T = \bar{\Omega}$$

(5) $$\begin{cases} T \text{ and } T' \in \mathcal{T}_h \Rightarrow T \cap T' = \emptyset \quad \text{or} \\ T \text{ and } T' \text{ have only one common vertex or only one common side.} \end{cases}$$

We choose h egual to the length of the greatest side of the $T \in \mathcal{T}_h$ and we approach V by

(6) $$V_h = \{v_h \mid v_h \in C^\circ(\bar{\Omega}),\ v_h = o \text{ on } \Gamma,\ v_h|_T \in P_1 \quad \forall T \in \mathcal{T}_h\}$$

with P_1 = space of polynomials of order $\leqslant 1$; we have $V_h \subset V$ and we approach (3) by the problem in <u>finite dimension</u>

(7) $$\int_\Omega |\nabla u_h|^{p-2} \nabla u_h . \nabla v_h dx = \langle f, v_h \rangle \quad \forall v_h \in V_h, u_h \in V_h.$$

Problem (7) has a unique solution and the following theorem can be shown

<u>THEOREM 2.1</u>

<u>If the angles of $\mathcal{T}_h$ are bounded from below, uniformly in h, by $\theta_o > o$, we have :</u>

(8) $$\lim \|u_h - u\|_1 = o \text{ when } h \to o \text{ where u is the solution of (3).}$$

3. ESTIMATIONS OF THE ERROR OF APPROXIMATION.

<u>LEMMA 3.1</u> : <u>We have</u>, $\forall u, v \in V$

(9) $$< A(v) - A(u), v-u > \geqslant \alpha \|v-u\|_1^p \text{ if } p \geqslant 2$$

(10) $$< A(v) - A(u), v-u > \geqslant \alpha \|v-u\|_1^2 (\|v\|_1 + \|u\|_1)^{p-2} \text{ if } 1 < p \leqslant 2$$

(11) $$\| A(v) - A(u) \|_* \leqslant \beta \|v-u\|_1 (\|v\|_1 + \|u\|_1)^{p-2} \text{ if } p \geqslant 2$$

(12) $$\| A(v) - A(u) \|_* \leqslant \beta \|v-u\|_1^{p-1} \text{ if } 1 < p \leqslant 2$$

<u>with</u> $\alpha, \beta > o$ <u>and independant of u,v.</u>

LEMMA 3.2 : Let u,w be any elements of V. Let u_h, w_h be the solutions of (7) corresponding respectively to $f = A(u)$ and $f = A(w)$. Then :

(13) $$\|u_h\|_1 \leq \|u\|_1$$

(14) $$\|w_h - u_h\|_1 \leq (\frac{\beta}{\alpha})^{p-1} \|w-u\|_1^{\frac{1}{p-1}} (\|w\|_1 + \|u\|_1)^{\frac{p-2}{p-1}} \text{ if } p \geq 2$$

(15) $$\|w_h - u_h\|_1 \leq \frac{\beta}{\alpha} \|w-u\|_1^{p-1} (\|w\|_1 + \|u\|_1)^{2-p} \text{ if } 1 < p \leq 2$$

If Ω is bounded in $\mathbb{R}^2$ and Γ is lipschitz, then $W^{2,p}(\Omega) \subset C^\circ(\bar{\Omega})$ with continuous injection $\forall p$, $1 < p \leq +\infty$; from this property, from the results of [2] on the interpolation of differentiable functions and from the above lemmas, we deduce the following theorem :

THEOREM 3.1

Under the hypothesis of theorem 2.1, we have :

(16) $$\|u_h - u\|_1 \leq C \|u\|_2^{\frac{1}{p-1}} \|u\|_1^{\frac{p-2}{p-1}} h^{\frac{1}{p-1}} \qquad \forall u \in V \cap W^{2,p}(\Omega),\ p \geq 2$$

(17) $$\|u_h - u\|_1 \leq C \|u\|_2^{\frac{1}{3-p}} \|u\|_1^{\frac{2-p}{3-p}} h^{\frac{1}{3-p}} \qquad \forall u \in V \cap W^{2,p}(\Omega),\ 1 < p \leq 2$$

with C independant of h and u.

From L. TARTAR [3] and from the above results, by non linear interpolation between V and $V \cap W^{2,p}(\Omega)$, we prove the following theorem :

THEOREM 3.2

Under the hypothesis of theorem 2.1, we have for $s \in [1,2]$:

(18) $$\|u_h - u\|_1 \leq C \|u\|_1^{\frac{p-2}{p-1}} \|u\|_s^{\frac{1}{p-1}} h^{\frac{s-1}{p-1}} \qquad \forall u \in V \cap W^{s;p}(\Omega),\ p \geq 2$$

(19) $$\|u_h - u\|_1 \leq C \|u\|_1^{\alpha} \|u\|_s^{\beta} h^{\gamma} \qquad \forall u \in V \cap W^{s,p}(\Omega),\ 1 < p \leq 2$$

with C independant of h and u, and in (19) we have

$$\alpha = \frac{(2-p)((2-s)+(s-1)(p-1))}{(2-s) + (s-1)(p-1)(3-p)} \qquad \beta = \frac{p-1}{(2-s)+(s-1)(p-1)(3-p)} \qquad \gamma = (s-1)\beta$$

4. AN ITERATIVE METHOD FOR SOLVING THE APPROXIMATED PROBLEM.

The problems (7) and (20), below, are equivalent.

(20) $$J(u_h) \leqslant J(v_h) \ \forall v_h \in V_h, u_h \in V_h \ ; \ J(v_h) = \frac{1}{p}\|v_h\|_1^p - \langle f, v_h \rangle.$$

The method of non linear surrelaxation described in [4] is almost inefficient if applied to (20) for $p < 1.5$ and $p \geqslant 10$. The method of auxilatory operator of [5] is suitable for (7) only if p is close to 2.

The remedy is to increase the number of variables while simplifying the non linear structure of (20) by taking $z_h = \nabla v_h$.

Then z_h and v_h are decoupled by penalisation and simultanuous dualisation of $z_h - \nabla v_h = o$ (following a principle due to HESTENES [6]).

Indeed, if a penalisation alone is used, it yields a problem different from the initial problem ; all the less different and the most ill conditioned that the parameter of penalty is small ; if duality alone is used, it yields a problem coercive in z_h and linear, therefore non coercive in v_h.

Let χ_T be the caracteristic function of T,

$$L_h = \{z_h \mid z_h = \sum_{T \in \mathcal{T}_h} z_T \chi_T, \ z_T \in \mathbb{R}^2\}$$

$j(v,z) = \frac{1}{p}\int_\Omega |z|^p dx - \langle f, v \rangle$ with $z \in L^p(\Omega) \times L^p(\Omega)$; (20) is equivalent to

(21) $$j(u_h, y_h) \leqslant j(v_h, z_h) \qquad \forall (v_h, z_h) \in V_h \times L_h, \ \nabla v_h - z_h = o$$

with $Y_h = \nabla u_h$.

By penalisation and dualisation of $\nabla v_h - z_h = o$, we are led to introduce (penalisation) $j_\varepsilon = j + \frac{1}{2\varepsilon}\|z - \nabla v\|^2_{L^2(\Omega)}$ with $\varepsilon > o$, then (dualisation) the Lagrangian
$\mathcal{L}(v,z;\mu) = j_\varepsilon(v,z) - \int_\Omega \mu \cdot (z - \nabla v)dx.$

Then, we can show ([7], N°28 can also be used) the following :

<u>PROPOSITION 4.1</u> : <u>$\mathcal{L}$ has a saddle point of the form</u> $(u_h, \nabla u_h ; \lambda_h)$ <u>on</u> $V_h \times L_h \times L_h$; u_h <u>solution of (7)</u>.

From this result, we can use the following algorithm for solving (7) :

(22) $\lambda_h^o \in L_h$, given

λ_h^n known, we compute u_h^n, y_h^n, λ_h^{n+1} by

(23) $\mathscr{L}(u_h^n, y_h^n ; \lambda_h^n) \leqslant \mathscr{L}(v_h, z_h ; \lambda_h^n) \quad \forall v_h \in V_h, z_h \in L_h \; ; \; u_h^n \in V_h, \; y_h^n \in L_h$

(24) $\lambda_h^{n+1} = \lambda_h^n - \rho_n(y_h^n - \nabla u_h^n), \; \rho_n > o$

THEOREM 4.1

If $o < r_o \leqslant \rho_n \leqslant r_1 < \frac{2}{\varepsilon}$, *when* $n \to +\infty$ *and* $\forall \lambda_h^o$, *we have* $u_h^n \to u_h, y_h^n \to \nabla u_h$; u_h *solution of (7)*.

COMMENT 4.1 : For μ fixed, $\mathscr{L}$ is strictly convex in (v,z) and quadratic in v ; it implies that (23) can be solved by a modification of a relaxation type on z_h, surrelaxation on v_h, of the standard surrelaxation method on the Dirichlet problem ; the results of [8] apply to this modification which is easy to implement since, for given v_h, μ_h, the minimization in z_h of $\mathscr{L}$ decomposes into Card($\mathcal{T}_h$), easy problems with two variables (this is one of the justification of algorithm (22)-(24)).

COMMENT 4.2 : In some case, algorithm (22)-(24) applied directly to the continuous problem, converges.

COMMENT 4.3 : For p=2, the above method, applied to solve (7), has little interest, since for $\rho_n = 1/\varepsilon$, the sequence (u_h^n) converges in two iterations.

5. APPLICATIONS TO OTHER NON LINEAR PROBLEMS.

With some minor modifications, we can apply the previous method to the following (continuous or approximated) problems :

Elastic-Plastic torsion of a cylindrical beam :

(25) $\text{Min}\left[\frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv dx\right], \; v \in H_o^1(\Omega), \; |\nabla v| \leqslant 1$ p.p.

Flow of a Plastic-viscous flow in a pipe :

(26) $\text{Min}\left[\frac{1}{2}\int_\Omega |\nabla v|^2 dx + g\int_\Omega |\nabla v| dx - \int_\Omega fv dx\right], \; v \in H_o^1(\Omega)$

Minimal surfaces :

(27) $$\text{Min} \int_\Omega \sqrt{1 + |\nabla v|^2}\,dx, \quad v = g \text{ on } \Gamma .$$

Generally speaking the above method is well adapted to non linear elliptical problems of order 2, when the non linearity is on ∇v ; this is the case of the problem of the Alternator in magneto-static treated in [4] by non linear surrelaxation and of the subsonic flow of a compressible fluid around a profil of R^2, etc...

6. NUMERICAL EXAMPLE.

For $\Omega = \{x \mid x_1^2 + x_2^2 < R^2\}$ and $\langle f,v \rangle = C \int_\Omega v\,dx$, the solution of (3) is given by

$$u(x) = \frac{p-1}{p}\left(\frac{CR}{2}\right)^{\frac{p}{p-1}} R\left(1 - \left(\frac{r}{R}\right)^{\frac{p}{p-1}}\right), \text{ with } r = \sqrt{x_1^2 + x_2^2}.$$

By making use of algorithm (22)-(24), we have been able to extend the field (1) of resolution of (1) to $1.1 \leqslant p \leqslant 50$, with computing time of the order of a minute of CII 10070 and for triangulations of about 250 triangles.

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(1) limited to $1.5 \leqslant p \leqslant 10$ for non linear S.O.R.

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