Motion planning and control problems for underactuated robots

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October 22, 2018

Abstract

Motion planning and control are key problems in a collection of robotic applications including the design of autonomous agile vehicles and of minimalist manipulators. These problems can be accurately formalized within the language of affine connections and of geometric control theory. In this paper we overview recent results on kinematic controllability and on oscillatory controls. Furthermore, we discuss theoretical and practical open problems as well as we suggest control theoretical approaches to them.

1 Motivating problems from a variety of robotic applications

The research in Robotics is continuously exploring the design of novel, more reliable and agile systems that can provide more efficient tools in current applications such as factory automation systems, material handling, and autonomous robotic applications, and can make possible their progressive use in areas such as medical and social assistance applications.

Mobile Robotics, primarily motivated by the development of tasks in unreachable environments, is giving way to new generations of autonomous robots in its search for new and "better adapted" systems of locomotion. For example, traditional wheeled platforms have evolved into articulated devices endowed with various types of wheels
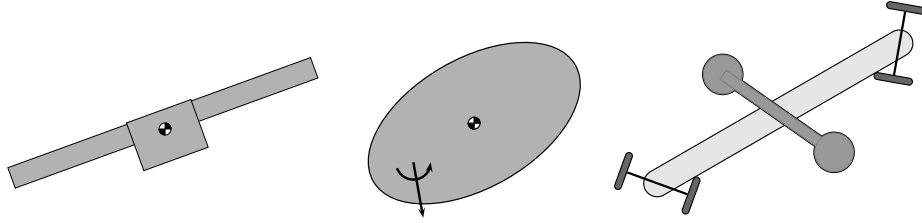
1

Figure 1: Underactuated robots appear in a variety of environments. From left to right, a planar vertical take-off and landing (PVTOL) aircraft model, a horizontal model of a blimp and the snakeboard.

and suspension systems that maximize their traction and the robot's ability to move over rough terrain or even climb obstacles. The types of wheels that are being employed include passive and powered castors, ball-wheels or omni-directional wheels that allow a high accuracy in positioning and yet retain the versatility, flexibility and other properties of wheels. A rich and active literature includes (i) various vehicle designs [39, 42, 45, 47], (ii) the automated guided vehicle "OmniMate" [2], (iii) the roller-walker [15] and other dexterous systems [17] that change their internal shape and constraints in response to the required motion sequence, and (iv) the omni-directional platform in [19].

Other types of remotely controlled autonomous vehicles that are increasingly being employed in space, air and underwater applications include submersibles, blimps, helicopters, and other crafts. More often than not they rely on innovative ideas to affect their motion instead of on classic design ideas. For example, in underwater vehicle applications, innovative propulsion systems such as shape changes, internal masses, and momentum wheels are being investigated. Fault tolerance, agility, and maneuverability in low velocity regimes, as in the previous example systems, are some of the desired capabilities.

A growing field in Mobile Robotics is that of *biomimetics*. The idea of this approach is to obtain some of the robustness and adaptability that biological systems have refined through evolution. In particular, biomimetic locomotion studies the periodic movement patterns or *gaits* that biological systems undergo during locomotion and then takes it as reference for the design of the mechanical counterpart. In other cases, the design of to robots without physical counterpart is inspired by similar principles. Robotic locomotion systems include the classic *bipeds and multi-legged robots* as well as swimming snake-like robots and flying robots. These systems find potential applications in harsh or hazardous environments, such as under deep or shallow water, on rough terrain (with stairs), along vertical walls or pipes and other environments difficult to access for wheeled robots. Specific examples in the literature include hyper-redundant robots [13, 16], the snakeboard [33, 41], the *G*-snakes and roller racer models in [26, 27], fish robots [23, 25], eel robots [21, 37], and passive and hopping robots [18, 36, 43].

All this set of emerging robotic applications have special characteristics that pose new challenges in motion planning. Among them, we highlight:

2

Underactuation. This could be owned to a design choice: nowadays low weight and fewer actuators must perform the task of former more expensive systems. For example, consider a manufacturing environment where robotic devices perform material handling and manipulation tasks: automatic planning algorithms might be able to cope with failures without interrupting the manufacturing process. Another reason why these systems are underactuated is because of an unavoidable limited control authority: in some locomotion systems it is not possible to actuate all the directions of motion. For example, consider a robot operating in a hazardous or remote environment (e.g., aerospace or underwater), an important concern is its ability to operate faced with a component failure, since retrieval or repair is not always possible.

Complex dynamics. In these control systems, the drift plays a key role. Dynamic effects must necessarily be taken into account, since kinematic models are no longer available in a wide range of current applications. Examples include lift and drag effects in underwater vehicles, the generation of momentum by means of the coupling of internal shape changes with the environment in the eel robot and the snakeboard, the dynamic stability properties of walking machines and nonholonomic wheeled platforms, etc.

Current limitations of motion algorithms. Most of the work on motion planning has relied on assumptions that are no longer valid in the present applications. For example, one of these is that (wheeled) robots are kinematic systems and, therefore, controlled by velocity inputs. This type of models allows one to design a control to reach a desired point and then immediately stop by setting the inputs to zero. This is obviously not the case when dealing with complex dynamic models.

Another common assumption is the one of fully actuation that allows to decouple the motion planning problem into path planning (computational geometry) and then tracking. For underactuated systems, this may be not possible because we may be obtaining motions in the path planning stage that the system can not perform in the tracking step because of its dynamic limitations.

Furthermore, motion planning and optimization problems for these systems are nonlinear, non-convex problems with exponential complexity in the dimension of the model. These issues have become increasingly important due to the high dimensionality of many current mechanical systems, including flexible structures, compliant manipulators and multibody systems undergoing reconfiguration in space.

Benefits that would result from better motion planning algorithms for underactuated systems. From a practical perspective, there are at least two advantages to designing controllers for underactuated robotic manipulators and vehicles. First, a fully actuated system requires more control inputs than an underactuated system, which means there will have to be more devices to generate the necessary forces. The additional controlling devices add to the cost and weight of the system. Finding a way to control an underactuated version of the system would improve the overall performance or reduce the cost. The second practical reason for
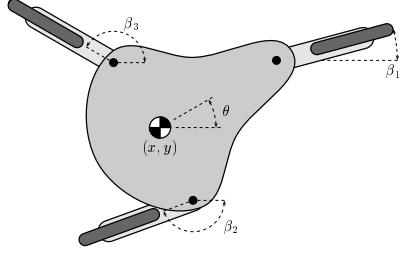
Figure 2: Vertical view of an omni-directional robotic platform with 6 degrees of freedom and 3 nonholonomic constraints [12, 19]. This device is capable of highly accurate positioning, high payloads, and high speed motion. In its fully actuated configuration, the robot is endowed with 6 motors at the three wheels and at the three joints $(\beta_1, \beta_2, \beta_3)$. However, underactuated configurations can arise because of failures or intentional design.

studying underactuated vehicles is that underactuation provides a backup control technique for a fully actuated system. If a fully actuated system is damaged and a controller for an underactuated system is available, then we may be able to recover gracefully from the failure. The underactuated controller may be able to salvage a system that would otherwise be uncontrollable.

2 Mathematical unifying approach to the modeling of robotic systems

Most of the robotic devices we have mentioned so far can be characterized by their special Lagrangian structure. They usually exhibit symmetries and their motion is constrained by the environment where they operate. In the following, we introduce a general modeling language for underactuated robotic systems.

Let $q = (q^1, \ldots, q^n) \in Q$ be the configuration of the mechanical system and consider the control equations:

$$\ddot{q}^i + \Gamma_{jk}^i(q)\dot{q}^j\dot{q}^k = -M^{ij}\frac{\partial V}{\partial q^j} + k_j^i(q)\dot{q}^j + Y_1^i(q)u_1 + \ldots + Y_m^i(q)u_m \,, \qquad (1)$$

where the summation convention is in place for the indices j, k that run from 1 to n, and

(i) $V : Q \to \mathbb{R}$ corresponds to potential energy, and $k_j^i(q)\dot{q}^j$ corresponds to damping forces,

(ii) $\{\Gamma_{jk}^i : i, j, k = 1, \ldots, n\}$ are n^3 Christoffel symbols, derived from $M(q)$, the inertia matrix defining the kinetic energy, according to

$$\Gamma_{ij}^k = \frac{1}{2}M^{mk}\left(\frac{\partial M_{mj}}{\partial q^i} + \frac{\partial M_{mi}}{\partial q^j} - \frac{\partial M_{ij}}{\partial q^m}\right) \,,$$

4

where M^{mk} is the (m, k) component of M^{-1}, and,

(iii) $\{F_a : a = 1, \ldots, m\}$ are the m input co-vector fields, and $\{Y_a = M^{-1}F_a : a = 1, \ldots, m\}$ are the m input vector fields.

Underactuated systems have fewer control actuators, m, than degrees of freedom $n > m$. Other limitations on the control signals u_a might be present, e.g., actuators might have magnitude and rate limits, or they might only generate unilateral or binary signals (e.g., thrusters in satellites).

The notion of affine connection provides a coordinate-free means of describing the dynamics of robotic systems. Given two vector fields X, Y, the *covariant derivative* of Y with respect to X is the third vector field $\nabla_X Y$ defined via

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k. \tag{2}$$

The operator ∇ is called the *affine connection* for the mechanical system in equation (1). We write the Euler-Lagrange equations for a system subject to a conservative force Y_0, a damping force $k(q)\dot{q}$ and m input forces as:

$$\nabla_{\dot{q}}\dot{q} = Y_0(q) + k(q)(\dot{q}) + \sum_{a=1}^{m} Y_a(q)u_a(t). \tag{3}$$

Equation (3) is a coordinate-free version of equation (1). A crucial observation is the fact that systems subject nonholonomic constraints can also be modeled by means of affine connections. In the interest of brevity, we refer to [10, 31] for the exposition of this result and the explicit expression of the Christoffel symbols corresponding to the Lagrange-d'Alembert equations.

The homogeneous structure of mechanical systems.

The fundamental structure of the control system in equation (3) is the polynomial dependence of the various vector fields on the velocity variable $\dot{q}$. This structure affects the Lie bracket computations involving input and drift vector fields. The system (3) is written in first order differential equation form as

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -\Gamma(q, \dot{q}) + Y_0(q) + k(q)(\dot{q}) \end{bmatrix} + \sum_{a=1}^{m}\begin{bmatrix} 0 \\ Y_a \end{bmatrix} u_a(t)$$

where $\Gamma(q, \dot{q})$ is the vector with ith component $\Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k$. Also, if $x = (q, \dot{q})$,

$$Z(x) = \begin{bmatrix} \dot{q} \\ -\Gamma(q, \dot{q}) \end{bmatrix}, \quad Y_a^{\text{lift}}(x) \triangleq \begin{bmatrix} 0 \\ Y_a(q) \end{bmatrix}, \quad \text{and} \quad k^{\text{lift}}(x) \triangleq \begin{bmatrix} 0 \\ k(q)(\dot{q}) \end{bmatrix},$$

the control system is rewritten as

$$\dot{x} = Z(x) + Y_0^{\text{lift}}(x) + k^{\text{lift}}(x) + \sum_{a=1}^{m} Y_a^{\text{lift}}(x)u_a(t).$$

Let $h_i(q, \dot{q})$ be the set of scalar functions on $\mathbb{R}^{2n}$ which are arbitrary functions of q and homogeneous polynomials in $\{\dot{q}^1, \ldots, \dot{q}^n\}$ of degree i. Let $\mathcal{P}_i$ be the set of vector fields on $\mathbb{R}^{2n}$ whose first n components belong to h_i and whose second n components belong to h_{i+1}. We note that these notions can also be defined on a general manifold, see [7].

We are now ready to present two simple ideas. First, all the previous vector fields are homogeneous polynomial vector fields for some specific value of i. Indeed, $Z \in \mathcal{P}_1$, $k^{\text{lift}} \in \mathcal{P}_0$, and $Y_a^{\text{lift}} \in \mathcal{P}_{-1}$. Second, since the Lie bracket between a vector field in $\mathcal{P}_i$ and a vector field in $\mathcal{P}_j$ belongs to $\mathcal{P}_{i+j}$, any Lie bracket of the given relevant vector fields remains a homogeneous polynomial. In other words, the set of homogeneous vector fields is closed under the operation of Lie bracket.

A consequence of this analysis is the definition of symmetric product of vector fields. We define the *symmetric product* between Y_b and Y_a as the vector field $\langle Y_a : Y_b \rangle = \langle Y_b : Y_a \rangle$ given by

$$\langle Y_b : Y_a \rangle^i = \langle Y_a : Y_b \rangle^i = \frac{\partial Y_a^i}{\partial q^j} Y_b^j + \frac{\partial Y_b^i}{\partial q^j} Y_a^j + \Gamma_{jk}^i \left(Y_a^j Y_b^k + Y_a^k Y_b^j \right).$$

Straightforward computations show that $\langle Y_a : Y_b \rangle^{\text{lift}} = [Y_b^{\text{lift}}, [Z_g, Y_a^{\text{lift}}]]$. This operation plays a key role in nearly all the control problems associated with this class of systems: nonlinear controllability [14, 32], optimal control [11, 30], dynamic feedback linearization [44], algorithms for motion planning and stabilization [6, 34, 40], etc.

A series expansion for the forced evolution starting from rest.

The homogeneous structure of the mechanical control system (3), together with the symmetric product, set the basis to establish the following description of the evolution of the system trajectories starting with zero initial velocity [3, 14]. Assume no potential or damping forces are present in the system. Let $Y(q, t) = \sum_{a=1}^{m} Y_a(q) u_a(t)$. Define recursively the vector fields V_k by

$$V_1(q, t) = \int_0^t Y(q, s) ds, \quad V_k(q, t) = -\frac{1}{2} \sum_{j=1}^{k-1} \int_0^t \Big\langle V_j(q, s) : V_{k-j}(q, s) \Big\rangle ds.$$

Then, the solution $q(t)$ of equation (3) satisfies

$$\dot{q}(t) = \sum_{k=1}^{+\infty} V_k(q(t), t), \tag{4}$$

where the series converges absolutely and uniformly in a neighborhood of q_0 and over a fixed time interval $t \in [0, T]$. This series expansion provides a means of describing the open-loop response of the system to any specific forcing. As we will see below, it plays a key role in several motion planning and control strategies for underactuated robots.

3 Existing results on planning for underactuated systems

To design planning algorithms for underactuated robotic systems, we advocate an integrated approach based on modeling, system design, controllability analysis, dexterity, manipulability, and singularities. These analysis concepts are fundamental for robust planning algorithms that do not solely rely on randomization or nonlinear programming. We do not suggest closed-form planning algorithms, rather we envision methods that combine the best features of formal analysis and of numerical algorithms.

For reasons of space, we cannot present a detailed account of all existing results on motion planning for underactuated systems, and not even of the results obtained within the modeling approach proposed in Section 2. Therefore, we focus on two specific control methodologies for motion planning: decoupled planning algorithms for kinematically controllable systems, and approximate inversion algorithms based on oscillatory controls.

Section 3.1 reviews *decoupled planning algorithms* that exploit certain differential geometric properties to reduce the complexity of the motion planning problem (still to be solved via numerical algorithms). The notion of *kinematic controllability* is extremely effective: trajectory planning decouples from being a problem on a $2n$ dimensional space to an n dimensional space. Furthermore, various state constraints can be neglected in the reduced space. For systems that are not kinematically controllable and that require oscillatory controls to locomote, Section 3.2 presents motion planning algorithms based on *approximate inversion*. Both design methods are closely related to recent results on nonlinear controllability [9, 32], power series expansions [3, 14], two time-scales coordinate-free averaging [4, 35], and nonlinear inversion algorithms [6, 34].

The strengths of this methodology are as follows. Both methodologies provide solutions to the corresponding problems, i.e., point to point and trajectory planning. These analytic results do not rely on non-generic assumptions such as feedback linearization, nilpotency or flatness. The results are coordinate-free and hence widely applicable, e.g., to aerospace or underwater robotics settings. Both methods are *consistent*, *complete* and constructive (consistent planners recover the known solutions available for linear and nilpotent systems, and complete planners are guaranteed to find a local solution for any nonlinearly controllable system).

3.1 Kinematic controllability for underactuated robots

The following decoupling methodology was proposed in [9] to reduce the complexity of the motion planning problem. The method is constructive (only quadratic equations and no PDEs are involved) and physically intuitive.

We consider as a motivating example a common pick-&-place manipulator: Fig. 3 shows a vertical view of a three-revolute-joints device. We investigate planning schemes for this system when one of its three motors is either failed or missing. We present a decoupling idea to reduce the complexity of the problem: instead of
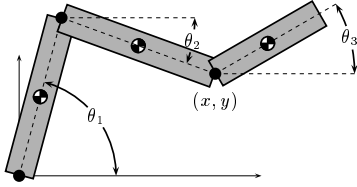
Figure 3: A three-revolute-joints device. It can be proven [9] that any two-actuator configuration of this system is kinematically controllable, i.e., one can always find two decoupling vector fields whose involutive closure is full-rank.

searching for feasible trajectories of a dynamic system in $\mathbb{R}^6$, we show how it suffices to search for paths of a simpler, kinematic (i.e., driftless) system in $\mathbb{R}^3$.

A curve $\gamma : [0, T] \mapsto Q$ is a controlled solution to equation (1) if there exist inputs $u_a : [0, T] \to \mathbb{R}$ for which γ solves (1). To avoid the difficult task of characterizing all controlled solutions of the system (1), we focus on curves satisfying $\dot{\gamma} = \dot{s}(t)X(\gamma)$, where X is a vector field on Q, and where the map $s : [0, T] \to [0, 1]$ is a "time-scaling" parameterization of γ. Such curves are called *kinematic motions*.

We call V a *decoupling vector field* if all curves γ satisfying $\dot{\gamma} = \dot{s}(t)V(\gamma)$ for any time scaling s, are kinematic motions. This definition is useful for three reasons. First, V is decoupling if and only if V and $\nabla_V V$ are linear combinations in $\{Y_1, \ldots, Y_m\}$. Second, decoupling vector fields can be computed by solving $(n - m)$ quadratic equations. Third, if enough decoupling vector fields, say $V_1, \ldots, V_p$, are available to satisfy the LARC, we call the system *kinematically controllable*. In the latter case, we can plan motions for the kinematic system $\dot{q} = \sum_{a=1}^{p} w_a(t)V_a(q)$, and they will automatically be controlled curves for the original system (1).

3.2 Approximate inversion via small amplitude and oscillatory controls

As in the previous section, the objective is to design motion planning and stabilization schemes for underactuated systems. We propose perturbation and inversion methods as widely applicable approaches to solve point to point and trajectory planning problems. Let us regard the flow map Φ of equation (3) over a finite time interval as a map from the input functions $u_i : [0, T] \to \mathbb{R}$ to the target state $x(T)$. The ideal algorithm for point-to-point planning computes an exact (right) inverse Φ^{-1} of Φ. Unfortunately, closed form expressions for Φ^{-1} are available only assuming *non-generic* differential geometric conditions (e.g., the system needs to be feedback linearizable, differentially flat, or nilpotent). Instead of aiming at "exact" solutions, we focus on computing an *approximate inverse map* using perturbation methods such as power series expansions and averaging theory. Although these tools are only approximate, the resulting algorithms are consistent and complete.

Oscillatory (high frequency, high amplitude) controls for trajectory planning.

We present the approach in three steps and refer to [35] for all the details. As first step, we present a recent coordinate-free averaging result. Let $0 < \epsilon \ll 1$. Assume the control inputs are of the form

$$u_i = \frac{1}{\epsilon} u_i \left(\frac{t}{\epsilon}, t \right),$$

and assume they are T-periodic and zero-mean in the first variable. Define the averaged multinomial iterated integrals of $u_1, \ldots, u_m$ as

$$\overline{\mathcal{U}}_{k_1, \ldots, k_m}(t) = \frac{T^{-1}}{k_1! \ldots k_m!} \int_0^T \left(\int_0^s u_1(\tau, t) d\tau \right)^{k_1} \ldots \left(\int_0^s u_m(\tau, t) d\tau \right)^{k_m} ds \, .$$

Let a, b, c take value in $\{1, \ldots, m\}$. Let $\vec{k}_a$ (resp. $\vec{k}_{ab}$) denote the tuple $(k_1, \ldots, k_m)$ with $k_c = \delta_{ca}$ (resp. $k_c = \delta_{ca} + \delta_{cb}$). Then, over a finite time $q(t) = r(t) + O(\epsilon)$, as $\epsilon \to 0$, where $r(t)$ satisfies

$$\nabla_{\dot r} \dot r = Y_0(r) + k(r)(\dot r) + \sum_{a=1}^m \left(\frac{1}{2} \overline{\mathcal{U}}_{\vec{k}_a}^2(t) - \overline{\mathcal{U}}_{\vec{k}_{aa}}(t) \right) \langle Y_a : Y_a \rangle(r) \qquad (5)$$

$$+ \sum_{a<b} \left(\overline{\mathcal{U}}_{\vec{k}_a}(t) \overline{\mathcal{U}}_{\vec{k}_b}(t) - \overline{\mathcal{U}}_{\vec{k}_{ab}}(t) \right) \langle Y_a : Y_b \rangle(r) \, .$$

As a second step, given $z_a(t)$, $z_{bc}(t)$ arbitrary functions of time, we propose the following inversion procedure

(i) take the functions $\psi_{N(a,b)}(t) = \sqrt{2} \, N(a,b) \cos(N(a,b) \, t)$, where $(a,b) \mapsto N(a,b) \in \{1, \ldots, N\}$ is an enumeration of the pairs of integers (a,b), $a < b$.

(ii) select the following controls in (3),

$$u_a(t, q) = v_a(t, q) + \frac{1}{\epsilon} w_a \left(\frac{t}{\epsilon}, t \right),$$

$$w_a(\tau, t) = - \sum_{c=1}^{a-1} \psi_{N(c,a)}(\tau) + \sum_{c=a+1}^m z_{ac}(t) \psi_{N(a,c)}(\tau) \, ,$$

where $v_a(t, q)$ are still to be chosen.

After computing the averaged iterated integrals of the oscillatory inputs $w_a(t/\epsilon, t)$, equation (5) for the averaged system becomes

$$\nabla_{\dot r} \dot r = Y_0(r) + k(r)(\dot r) + \sum_{a=1}^m v_a(t, r) Y_a(r)$$

$$- \sum_{a=1}^m \overline{\mathcal{U}}_{\vec{k}_{aa}}(t) \langle Y_a : Y_a \rangle(r) + \sum_{a<b} z_{ab}(t) \langle Y_a : Y_b \rangle(r) \, .$$
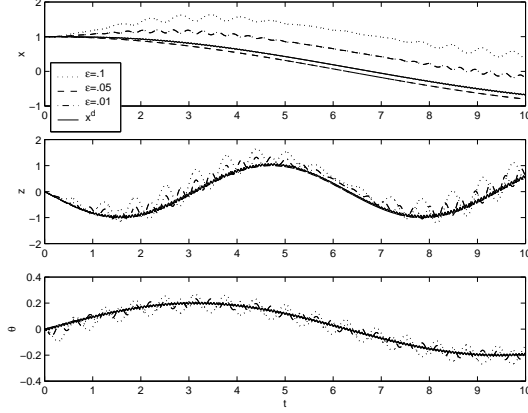
9

Figure 4: Approximate trajectory tracking for an underactuated PVTOL model by means of oscillatory controls. The curve to be tracked is shown solid, and the various oscillating curves correspond to different values of the parameter ϵ

As a third and final step, assume that all the vector fields of the form $\langle Y_b : Y_b \rangle$ belong to span$\{Y_a\}$. Let $\alpha_{ab} : Q \to \mathbb{R}$ be such that $\langle Y_a : Y_a \rangle(q) = \sum_b \alpha_{ab}(q) Y_b(q)$, $q \in Q$. Select

$$ v_a(t,q) = z_a(t) + \frac{1}{2} \sum_{b=1}^m \alpha_{ba}(q) \left(b - 1 + \sum_{c=b+1}^m (z_{bc}(t))^2 \right). $$

Then, we have

$$ \sum_{a=1}^m v_a(t,r) Y_a(r) = \sum_{a=1}^m z_a^d(t) Y_a(r) + \sum_{a=1}^m \overline{\mathcal{U}}_{\tilde{\kappa}_{aa}}(t) \langle Y_a : Y_a \rangle(r), $$

which implies that eq. (5) takes the final form,

$$ \nabla_{\dot{r}} \dot{r} = Y_0(r) + k(r)(\dot{r}) + \sum_{a=1}^m z_a(t) Y_a(r) + \sum_{b<c} z_{bc}(t) \langle Y_b : Y_c \rangle(r), $$

The averaged system now has more available control inputs than the original one. If the input distribution $\mathcal{I} = \text{span}\{Y_a, \langle Y_b : Y_c \rangle\}$ is full rank, then the latter system is fully actuated (i.e., one control input is available for each degree of freedom). If the input distribution $\mathcal{I}$ contains a sufficient number of decoupling vector fields, then the system is kinematically controllable. In both cases, we have reduced the complexity of the motion planning problem.

Remark 3.1 (Small amplitude algorithms based on series expansions) A related approach to motion planning relies on small amplitude periodic forcing; see [6, 34]. The planning problem is solved by approximately inverting the series expansion describing the evolution of the control system (cf. Section 2). This inversion

10

procedure is very similar to the one presented above. Based on it, one can establish two simple primitives of motion to change and maintain velocity, while keeping track of the changes in the configuration. These primitives can then be used as the building blocks to design high-level motion algorithms that solve the point-to-point reconfiguration problem, the static interpolation problem and the local exponential stabilization problem. Fig. 5 shows two examples of the execution of these algorithms.
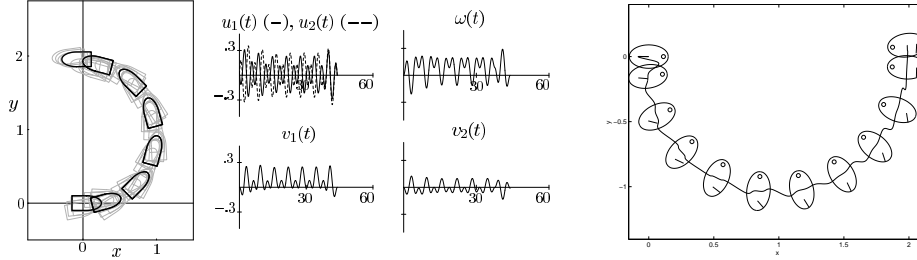

Figure 5: Illustration of the motion planning algorithms via small amplitude periodic forcing for a simple planar body (left) and the blimp model (right). The errors in the final configuration are within the same order of magnitude of the input employed

4   Open problems and possible approaches

Immediate open questions arising from the above-presented results are the following:

Kinematic modeling and control.   The current limitations are as follows: the design problem is now reduced to planning for a kinematic system with the additional constraint of zero-velocity transitions between feasible motions. This additional constraint leads to poor performance when coupled with current randomized planners [20, 24, 28, 29] that switch frequently between the available motions. The zero-velocity switches also create problems for trajectory tracking controllers based on linearization, since the system loses linear controllability at zero-velocity. Finally, there is no notion of time-optimality for these kinematic motions and there is no way of dealing with systems where oscillatory inputs are needed for locomotion (see below for a discussion on this point). Motivated by this analysis, we identify the following open issues:

(i) Develop a catalog of kinematically controllable systems, including planar manipulators with revolute as well as prismatic joints, parallel manipulator, manipulators in three dimensional space and in aerospace and underwater environments (accounting for the different dynamics in such settings). Some preliminary work in this direction can be found in [8]. Analyze and classify the singularities that these vector fields possess as a prerequisite step for planning purposes.

11

(ii) A (left) group action is a map $\psi : G \times Q \to Q$ such that $\psi(e, q) = q$, for all $q \in Q$, where e denotes the identity element in G, and $\psi(g, \psi(h, q)) = \psi(gh, q)$, for all $g, h \in G$, $q \in Q$. Usually $G \subset SE(n)$, and then the action describes a rigid displacement of some components of the robot. An interesting problem would be to identify conditions under which decoupling vector fields can be found which are invariant under such group actions. When this is the case, motion plans can be designed exploiting established "inverse kinematics" methods; see [38, Chapter 3]. This simplification eliminates the need for any numerical procedure if the robot moves in an un-obstructed environment, or further reduces the dimensionality and complexity of the resulting search problem in complex environments.

(iii) To tackle the difficulties inherent with zero-velocity transitions, it would be appropriate to develop randomized planners which require as few switches between decoupling vector fields as possible, and to develop trajectory tracking controllers for these systems able to adequately perform through the singularities.

(iv) Another interesting idea would consist of switching between decoupling vector fields without stopping. In some sense, this is also related to the problem of developing transitions between relative equilibria. *Relative equilibria* are "steady trajectories" that the system admits as feasible solutions. This family of trajectories is of great interest in theory and applications as they provide a rich family of motions with the simplifying property of having constant body-fixed velocity. Relative equilibria for systems in three dimensional Euclidean space include straight lines, circles, and helices. Despite partial results, no method is currently available to design provably stable switching maneuvers from one relative equilibrium to another (or from one decoupling vector field to another without stopping). A necessary preliminary step toward this objective is to analyze the controllability properties of underactuated systems moving along a relative equilibrium or along a decoupling vector field.

Small-amplitude and high-frequency controls. The current limitations are as follows. The implementation of the small amplitude approach requires the computation and manipulation of high order tensors, and the approach has a limited region of convergence. The implementation of the oscillatory control approach presents difficulties in most physical settings because of the required high frequency, high amplitude inputs. Motivated by this analysis, we think that the following are interesting issues to explore:

(i) For the small amplitude controls formulation, open questions include (a) investigate tight estimates for the region of validity of the truncations (simulation studies suggest that there are better bounds than the conservative ones currently available), (b) design base functions optimal with regards to region of convergence and appropriate cost criteria, (c) design inversion algorithms for systems that are not linearly controllable. The latter setting is equivalent to a non-definite quadratic programming problem, i.e., to the problem of finding

sufficient conditions for a vector-valued quadratic form to be surjective (see [5] for a discussion on this subject).

(ii) For the oscillatory controls formulation, standing problems are (a) investigate the use of high-frequency bounded amplitude controls, (b) characterize approximate kinematic controllability and differential flatness via oscillations, (c) investigate physical settings in which oscillatory controls are natural control means, e.g., micro-electromechanical robots, (d) investigate extensions of this coordinate-free perturbation theory to discrete-time nonlinear systems, and to distributed parameter systems and partial differential equations.

(iii) An ambitious program would consist of developing schemes that combine the proposed analytic methods with iterative numerical algorithms. One approach is via homotopy and level set methods [1, 46] as schemes that overcome the limitations induced by the small parameter (small convergence region or high amplitude high frequency). A second direction is to use the planner based on small amplitude controls as a local planner inside a global search algorithm based on randomization; see [22] for some preliminary results on local/global planners.

Acknowledgments

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