EXACT COMPLEXITY OF EXACT-FOUR-COLORABILITY AND OF THE WINNER PROBLEM FOR YOUNG ELECTIONS*

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Abstract We classify two problems: `Exact-Four-Colorability` and the winner problem for Young elections. Regarding the former problem, Wagner raised the question of whether it is DP-complete to determine if the chromatic number of a given graph is exactly four. We prove a general result that in particular solves Wagner's question in the affirmative.

In 1977, Young proposed a voting scheme that extends the Condorcet Principle based on the fewest possible number of voters whose removal yields a Condorcet winner. We prove that both the winner and the ranking problem for Young elections is complete for $\mathrm{P}_{\|}^{\mathrm{NP}}$, the class of problems solvable in polynomial time by parallel access to NP. Analogous results for Lewis Carroll's 1876 voting scheme were recently established by Hemaspaandra et al. In contrast, we prove that the winner and ranking problems in Fishburn's homogeneous variant of Carroll's voting scheme can be solved efficiently by linear programming.

Keywords: Computational complexity; graph colorability; completeness; boolean hierarchy; voting schemes.

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1. Introduction

In this paper, we classify two problems from different fields with respect to their computational complexity: `Exact-Four-Colorability` and the winner problem for Young elections.

Sections 2 and 3 are concerned with `Exact-Four-Colorability`. Let $M_k \subseteq \mathbb{N}$ be a given set that consists of k noncontiguous integers. Exact-M_k-Colorability is the problem of determining whether $\chi(G)$, the chromatic number of a given graph G, equals one of the k elements of the set M_k exactly. In 1987, Wagner [27] proved that Exact-M_k-Colorability is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete, where $M_k = \{6k+1, 6k+3, \ldots, 8k-1\}$ and $\mathrm{BH}_{2k}(\mathrm{NP})$ is the $2k$th level of the boolean hierarchy over NP. In particular, for $k = 1$, it is DP-complete to determine whether $\chi(G) = 7$, where $\mathrm{DP} = \mathrm{BH}_2(\mathrm{NP})$. Wagner raised the question of how small the numbers in a k-element set M_k can be chosen such that Exact-M_k-Colorability still is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete. In particular, for $k = 1$, he asked if it is DP-complete to determine whether $\chi(G) = 4$.

In Section 3, we solve this question of Wagner and determine the precise threshold $t \in \{4, 5, 6, 7\}$ for which the problem Exact-$\{t\}$-Colorability jumps from NP to DP-completeness: It is DP-complete to determine whether $\chi(G) = 4$, yet Exact-$\{3\}$-Colorability is in NP. More generally, for each $k \geq 1$, we show that Exact-M_k-Colorability is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete for $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$.

Sections 4 and 5 are concerned with complexity issues related to voting schemes. More than a decade ago, Bartholdi, Tovey, and Trick initiated the study of electoral systems with respect to their computational properties. In particular, they proved NP hardness lower bounds [2] for determining the winner in the voting schemes proposed by Dodgson (more commonly known by his pen name, Lewis Carroll) and by Kemeny. Since then, a number of related results and improvements of their results have been obtained. Hemaspaandra, Hemaspaandra, and Rothe [15] classified both the winner and the ranking problem for Dodgson elections by proving them complete for $\mathrm{P}_{\|}^{\mathrm{NP}}$, the class of problems solvable in polynomial time by parallel access to an NP oracle. E. Hemaspaandra (as cited in [14]) and Spakowski and Vogel [26] obtained the analogous result for Kemeny elections; a joint paper by E. Hemaspaandra, Spakowski, and Vogel is in preparation. For many further interesting results and the state of the art regarding computational politics, we refer to the survey [14].

In this paper, we study complexity issues related to Young and Dodgson elections. In 1977, Young proposed a voting scheme that extends the Condorcet Principle based on the fewest possible number of voters whose removal makes a given candidate c the Condorcet winner, i.e., c defeats all other candidates by a strict majority of the votes. We prove that both the winner and the ranking problem for Young elections is complete for $\mathrm{P}_{\|}^{\mathrm{NP}}$. To this end, we give a reduction from the problem `Maximum Set Packing Compare`, which we also prove $\mathrm{P}_{\|}^{\mathrm{NP}}$-complete.

In Section 5, we study a homogeneous variant of Dodgson elections that was introduced by Fishburn [9]. In contrast to the above-mentioned result of Hemaspaandra et al. [15], we show that both the winner and the ranking problem for Fishburn's homogeneous Dodgson elections can be solved efficiently by a linear program that is based on an integer linear program of Bartholdi et al. [2].

2. Exact-M_k-Colorability and the Boolean Hierarchy over NP

To classify the complexity of problems known to be NP-hard or coNP-hard, but seemingly not contained in NP $\cup$ coNP, Papadimitriou and Yannakakis [22] introduced DP, the class of differences of two NP problems. They showed that DP contains various interesting types of problems, including *uniqueness problems*, *critical graph problems*, and *exact optimization problems.* For example, Cai and Meyer [5] proved the DP-completeness of `Minimal-3-Uncolorability`, a critical graph problem that asks whether a given graph is not 3-colorable, but deleting any of its vertices makes it 3-colorable. A graph is said to be *k-colorable* if its vertices can be colored using no more than k colors such that no two adjacent vertices receive the same color. The *chromatic number of a graph G*, denoted $\chi(G)$, is defined to be the smallest k such that G is k-colorable. Generalizing DP, Cai et al. [4] defined and studied the boolean hierarchy over NP. Their work initiated many further results on the boolean hierarchy; see e.g., [27, 20, 28, 17] to name just a few. To define the boolean hierarchy, we use the symbols $\wedge$ and $\vee$, respectively, to denote the *complex intersection* and the *complex union* of set classes.

Definition 1 [4] *The* boolean hierarchy over NP *is inductively defined as follows:*

$$\begin{aligned} \mathrm{BH}_1(\mathrm{NP}) &= \mathrm{NP}, \qquad \mathrm{BH}_2(\mathrm{NP}) = \mathrm{NP} \wedge \mathrm{coNP}, \\ \mathrm{BH}_k(\mathrm{NP}) &= \mathrm{BH}_{k-2}(\mathrm{NP}) \vee \mathrm{BH}_2(\mathrm{NP}) \quad \textit{for } k \geq 3, \textit{ and} \\ \mathrm{BH}(\mathrm{NP}) &= \bigcup_{k \geq 1} \mathrm{BH}_k(\mathrm{NP}). \end{aligned}$$

Equivalent definitions in terms of different boolean hierarchy normal forms can be found in the papers [4, 27, 20]; for the boolean hierarchy over arbitrary set rings, we refer to the early work by Hausdorff [13]. Note that DP = $\mathrm{BH}_2(\mathrm{NP})$.

In his seminal paper [27], Wagner provided sufficient conditions to prove problems complete for the levels of the boolean hierarchy. In particular, he established the following lemma for $\mathrm{BH}_{2k}(\mathrm{NP})$.

Lemma 2 [27, Thm. 5.1(3)] *Let A be some* NP*-complete problem, let B be an arbitrary problem, and let $k \geq 1$ be fixed. If there exists a polynomial-time computable function f such that, for all strings $x_1, x_2, \ldots, x_{2k} \in \Sigma^*$ satisfying* *that $x_{j+1} \in A$ implies $x_j \in A$ for each j with $1 \leq j < 2k$, it holds that*

$$\|\{i \mid x_i \in A\}\| \text{ is odd } \iff f(x_1, x_2, \ldots, x_{2k}) \in B, \tag{1}$$

then B is $\mathrm{BH}_{2k}(\mathrm{NP})$*-hard.*

For fixed $k \geq 1$, let $M_k = \{6k+1, 6k+3, \ldots, 8k-1\}$, and define the problem $\texttt{Exact-}M_k\texttt{-Colorability} = \{G \mid \chi(G) \in M_k\}$. In particular, Wagner applied Lemma 2 to prove that, for each $k \geq 1$, $\texttt{Exact-}M_k\texttt{-Colorability}$ is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete. For the special case of $k = 1$, it follows that $\texttt{Exact-}\{7\}\texttt{-Colorability}$ is DP-complete.

Wagner [27, p. 70] raised the question of how small the numbers in a k-element set M_k can be chosen such that $\texttt{Exact-}M_k\texttt{-Colorability}$ still is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete. Consider the special case of $k = 1$. It is easy to see that $\texttt{Exact-}\{3\}\texttt{-Colorability}$ is in NP and, thus, cannot be DP-complete unless the boolean hierarchy collapses; see Proposition 3 below. Consequently, for $k = 1$, Wagner's result leaves a gap in determining the precise threshold $t \in \{4, 5, 6, 7\}$ for which $\texttt{Exact-}\{t\}\texttt{-Colorability}$ jumps from NP to DP-completeness. Closing this gap, we show that it is DP-complete to determine whether $\chi(G) = 4$. More generally, answering Wagner's question for each $k \geq 1$, we show that $\texttt{Exact-}M_k\texttt{-Colorability}$ is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete for $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$.

3. Solving Wagner's Question

Proposition 3 *Fix any $k \geq 1$, and let M_k be any set that contains k noncontiguous positive integers including* 3. *Then,* $\texttt{Exact-}M_k\texttt{-Colorability}$ *is in* $\mathrm{BH}_{2k-1}(\mathrm{NP})$*; in particular, for $k = 1$,* $\texttt{Exact-}\{3\}\texttt{-Colorability}$ *is in* NP.

Hence, $\texttt{Exact-}M_k\texttt{-Colorability}$ is not $\mathrm{BH}_{2k}(\mathrm{NP})$-complete unless the boolean hierarchy, and consequently the polynomial hierarchy, collapses. Proposition 3 easily follows from the fact that it can be tested in polynomial time whether a given graph is 2-colorable; for details of the proof, see [23].

Theorem 4 *For fixed $k \geq 1$, let $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$. Then,* $\texttt{Exact-}M_k\texttt{-Colorability}$ *is* $\mathrm{BH}_{2k}(\mathrm{NP})$*-complete. In particular, for $k = 1$, it follows that* $\texttt{Exact-}\{4\}\texttt{-Colorability}$ *is* DP*-complete.*

Proof. We apply Lemma 2 with A being the NP-complete problem $\texttt{3-SAT}$ and B being $\texttt{Exact-}M_k\texttt{-Colorability}$, where $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$ for fixed k. The standard reduction σ from $\texttt{3-SAT}$ to $\texttt{3-Colorability}$ has the following property [10]:

$$\phi \in \texttt{3-SAT} \implies \chi(\sigma(\phi)) = 3 \quad \text{and} \quad \phi \notin \texttt{3-SAT} \implies \chi(\sigma(\phi)) = 4. \tag{2}$$

Using the PCP theorem, Khanna, Linial, and Safra [19] showed that it is NP-hard to color a 3-colorable graph with only four colors. Guruswami and Khanna [11] gave a novel proof of the same result that does not rely on the

PCP theorem. We use their direct transformation, call it ρ, that consists of two subsequent reductions—first from 3-SAT to the independent set problem, and then from the independent set problem to 3-Colorability—such that $\phi \in$ 3-SAT implies $\chi(\rho(\phi)) = 3$, and $\phi \notin$ 3-SAT implies $\chi(\rho(\phi)) \geq 5$. Guruswami and Khanna [11] note that the graph $H = \rho(\phi)$ they construct always is 6-colorable. In fact, their construction even gives that H always is 5-colorable; hence, we have:

$$\phi \in \text{3-SAT} \implies \chi(\rho(\phi)) = 3 \quad \text{and} \quad \phi \notin \text{3-SAT} \implies \chi(\rho(\phi)) = 5. \tag{3}$$

To see why, look at the reduction in [11]. The graph H consists of tree-like structures whose vertices are replaced by 3×3 grids, which always can be colored with three colors, say 1, 2, and 3. In addition, some leaves of the tree-like structures are connected by leaf-level gadgets of two types, the "same row kind" and the "different row kind." The latter gadgets consist of two vertices connected to some grids, and thus can always be colored with two additional colors. The leaf-level gadgets of the "same row kind" consist of a triangle whose vertices are adjacent to two grid vertices each. Hence, regardless of which 3-coloring is used for the grids, one can always color one triangle vertex, say t_1, with a color $c \in \{1,2,3\}$ such that c is different from the colors of the two grid vertices adjacent to t_1. Using two additional colors for the other two triangle vertices implies $\chi(H) \leq 5$, which proves Equation (3).

The join operation $\oplus$ on graphs is defined as follows: Given two disjoint graphs $A = (V_A, E_A)$ and $B = (V_B, E_B)$, their join $A \oplus B$ is the graph with vertex set $V_{A\oplus B} = V_A \cup V_B$ and edge set $E_{A\oplus B} = E_A \cup E_B \cup \{\{a,b\} \mid a \in V_A \text{ and } b \in V_B\}$. Note that $\oplus$ is an associative operation on graphs and $\chi(A \oplus B) = \chi(A) + \chi(B)$.

Let $\phi_1, \phi_2, \dots, \phi_{2k}$ be $2k$ given boolean formulas satisfying $\phi_{j+1} \in$ 3-SAT $\implies \phi_j \in$ 3-SAT for each j with $1 \leq j < 2k$. Define $2k$ graphs $H_1, H_2, \dots, H_{2k}$ as follows. For each i with $1 \leq i \leq k$, define $H_{2i-1} = \rho(\phi_{2i-1})$ and $H_{2i} = \sigma(\phi_{2i})$. By Equations (2) and (3),

$$\chi(H_j) = \begin{cases} 3 & \text{if } 1 \leq j \leq 2k \text{ and } \phi_j \in \text{3-SAT} \\ 4 & \text{if } j = 2i \text{ for some } i \in \{1,2,\dots,k\} \text{ and } \phi_j \notin \text{3-SAT} \\ 5 & \text{if } j = 2i-1 \text{ for some } i \in \{1,2,\dots,k\} \text{ and } \phi_j \notin \text{3-SAT.} \end{cases} \tag{4}$$

For each i with $1 \leq i \leq k$, define the graph G_i to be the disjoint union of the graphs H_{2i-1} and H_{2i}. Thus, $\chi(G_i) = \max\{\chi(H_{2i-1}), \chi(H_{2i})\}$, for each i with $1 \leq i \leq k$. The construction of our reduction f is completed by defining $f(\phi_1, \phi_2, \dots, \phi_{2k}) = G$, where the graph $G = \bigoplus_{i=1}^{k} G_i$ is the join of the graphs $G_1, G_2, \dots, G_k$. Thus,

$$\chi(G) = \sum_{i=1}^{k} \chi(G_i) = \sum_{i=1}^{k} \max\{\chi(H_{2i-1}), \chi(H_{2i})\}. \tag{5}$$

It follows from our construction that

$$
\begin{aligned}
&\|\{i \mid \phi_i \in \texttt{3-SAT}\}\| \text{ is odd} \\
&\iff (\exists i : 1 \leq i \leq k)\,[\phi_1, \ldots, \phi_{2i-1} \in \texttt{3-SAT} \text{ and } \phi_{2i}, \ldots, \phi_{2k} \notin \texttt{3-SAT}] \\
&\overset{(4),(5)}{\iff} (\exists i : 1 \leq i \leq k) \left[\sum_{j=1}^{k} \chi(G_j) = 3(i-1) + 4 + 5(k-i) = 5k - 2i + 1\right] \\
&\overset{(5)}{\iff} \chi(G) \in M_k = \{3k+1, 3k+3, \ldots, 5k-1\} \\
&\iff f(\phi_1, \phi_2, \ldots, \phi_{2k}) = G \in \texttt{Exact-}M_k\texttt{-Colorability}.
\end{aligned}
$$

Hence, Equation (1) is satisfied. Lemma 2 implies that `Exact-`M_k`-Colorability` is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete. ∎

And now for something completely different [6]: Voting schemes.

4. Hardness of Determining Young Winners

We first give some background from social choice theory. Let C be the set of all candidates (or alternatives). We assume that each voter has strict preferences over the candidates. Formally, the preference order of each voter is strict (i.e., irreflexive and antisymmetric), transitive, and complete (i.e., all candidates are ranked by each voter). An election is given by a *preference profile*, a pair $\langle C, V\rangle$ such that C is a set of candidates and V is the multiset of the voters' preference orders on C. Note that distinct voters may have the same preferences over the candidates. A *voting scheme* (or *social choice function*, SCF for short) is a rule for how to determine the winner(s) of an election; i.e., an SCF maps any given preference profile to society's aggregate *choice set*, the set of candidates who have won the election. For any SCF f and any preference profile $\langle C, V\rangle$, $f(\langle C, V\rangle)$ denotes the set of winning candidates. For example, an election is won according to the *majority rule* by any candidate who is preferred over any other candidate by a strict majority of the voters. Such a candidate is called the Condorcet winner. In 1785, Marie-Jean-Antoine-Nicolas de Caritat, the Marquis de Condorcet, noted in his seminal essay [7] that whenever there are at least three candidates, say A, B, and C, the majority rule may yield cycles: A defeats B and B defeats C, and yet C defeats A. Thus, even though each individual voter has a rational (i.e., transitive or non-cyclic) preference order, society may behave irrationally and Condorcet winners do not always exist. This observation is known as the Condorcet Paradox. The *Condorcet Principle* says that for each preference profile, the winner of the election is to be determined by the majority rule. An SCF is said to be a *Condorcet SCF* if and only if it respects the Condorcet Principle in the sense that the Condorcet winner is elected whenever he or she exists. Note that Condorcet winners are uniquely determined if they exist. Many Condorcet SCFs have been proposed in the social choice literature; for an overview of the most central ones, we refer to the work of Fishburn [9]. They extend the Condorcet Principle in a way that avoids the troubling feature of the majority rule. In this paper, we will focus on only two such Condorcet SCFs, the Dodgson voting scheme [8] and the Young voting scheme [29].

In 1876, Charles L. Dodgson (better known by his pen name, Lewis Carroll) proposed a voting scheme [8] that suggests that we remain most faithful to the Condorcet Principle if the election is won by any candidate who is "closest" to being a Condorcet winner. To define "closeness," each candidate c in a given election $\langle C, V \rangle$ is assigned a score, denoted $\text{DodgsonScore}(C, c, V)$, which is the smallest number of sequential interchanges of adjacent candidates in the voters' preferences that are needed to make c a Condorcet winner. Here, one interchange means that in (any) one of the voters two adjacent candidates are switched. A *Dodgson winner* is any candidate with minimum Dodgson score. Using Dodgson scores, one can also tell who of two given candidates is ranked better according to the Dodgson SCF.

Young's approach to extending the Condorcet Principle is reminiscent of Dodgson's approach in that it is also based on altered profiles. Unlike Dogson, however, Young [29] suggests that we remain most faithful to the Condorcet Principle if the election is won by any candidate who is made a Condorcet winner by *removing the fewest possible number of voters*, instead of doing the fewest possible number of switches in the voters' preferences. For each candidate c in a given preference profile $\langle C, V \rangle$, define $\text{YoungScore}(C, c, V)$ to be the size of a largest subset of V for which c is a Condorcet winner. A *Young winner* is any candidate with a maximum Young score. Homogeneous variants of these voting schemes will be defined in Section 5.

To study computational complexity issues related to Dodgson's voting scheme, Bartholdi, Tovey, and Trick [2] defined the following decision problems.

`Dodgson Winner`

Instance: A preference profile $\langle C, V \rangle$ and a designated candidate $c \in C$.
Question: Is c a Dodgson winner of the election? That is, is it true that for all $d \in C$, $\text{DodgsonScore}(C, c, V) \leq \text{DodgsonScore}(C, d, V)$?

`Dodgson Ranking`

Instance: A preference profile $\langle C, V \rangle$ and two designated candidates $c, d \in C$.
Question: Does c tie-or-defeat d in the election? That is, is it true that

$$\text{DodgsonScore}(C, c, V) \leq \text{DodgsonScore}(C, d, V) \text{ ?}$$

Bartholdi et al. [2] established an NP-hardness lower bound for both these problems. Their result was optimally improved by Hemaspaandra, Hemaspaandra, and Rothe [15] who proved that `Dodgson Winner` and `Dodgson Ranking` are complete for $\text{P}_{\|}^{\text{NP}}$, the class of problems solvable in polynomial time with parallel (i.e., truth-table) access to an NP oracle. As above, we define the corresponding decision problems for Young elections as follows.

`Young Winner`

Instance: A preference profile $\langle C, V \rangle$ and a designated candidate $c \in C$.
Question: Is c a Young winner of the election? That is, is it true that for all $d \in C$, $\text{YoungScore}(C, c, V) \geq \text{YoungScore}(C, d, V)$?

`Young Ranking`

Instance: A preference profile $\langle C, V \rangle$ and two designated candidates $c, d \in C$.
Question: Does c tie-or-defeat d in the election? That is, is it true that

$$\text{YoungScore}(C, c, V) \geq \text{YoungScore}(C, d, V) \ ?$$

The main result in this section is that the problems `Young Winner` and `Young Ranking` are complete for $\mathrm{P}_{\parallel}^{\mathrm{NP}}$. In Theorem 6 below, we give a reduction from the problem `Maximum Set Packing Compare` defined below. For a given familiy $\mathcal{S}$ of sets, let $\kappa(\mathcal{S})$ be the maximum number of pairwise disjoint sets in $\mathcal{S}$.

`Maximum Set Packing Compare`

Instance: Two families $\mathcal{S}_1$ and $\mathcal{S}_2$ of sets such that, for $i \in \{1, 2\}$, each set $S \in \mathcal{S}_i$ is a nonempty subset of a given set B_i.
Question: Does it hold that $\kappa(\mathcal{S}_1) \geq \kappa(\mathcal{S}_2)$?

Theorem 5 `Maximum Set Packing Compare` *is* $\mathrm{P}_{\parallel}^{\mathrm{NP}}$*-complete.*

Theorem 5 is proven (see the full version [24] for details) via a reduction from `Independence Number Compare`, which in turn can be shown $\mathrm{P}_{\parallel}^{\mathrm{NP}}$-complete by the techniques of Wagner [27]; see [25, Thm. 12] for an explicit proof of this result. `Independence Number Compare` has also been used in [16]. To define the problem, let G be an undirected, simple graph. An *independent set of* G is any subset I of the vertex set of G such that no two vertices in I are adjacent. For any graph G, let $\alpha(G)$ be the *independence number of* G, i.e., the size of a maximum independent set of G.

`Independence Number Compare`

Instance: Two graphs G_1 and G_2.
Question: Does it hold that $\alpha(G_1) \geq \alpha(G_2)$?

Now, we prove the main result of this section.

Theorem 6 `Young Ranking` *and* `Young Winner` *are* $\mathrm{P}_{\parallel}^{\mathrm{NP}}$*-complete.*

Proof. It is easy to see that `Young Ranking` and `Young Winner` are in $\mathrm{P}_{\parallel}^{\mathrm{NP}}$. To prove the $\mathrm{P}_{\parallel}^{\mathrm{NP}}$ lower bound, we first give a polynomial-time many-one reduction from `Maximum Set Packing Compare` to `Young Ranking`.

Let $B_1 = \{x_1, x_2, \ldots, x_m\}$ and $B_2 = \{y_1, y_2, \ldots, y_n\}$ be two given sets, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be given families of subsets of B_1 and B_2, respectively. Recall that $\kappa(\mathcal{S}_i)$, for $i \in \{1, 2\}$, is the maximum number of pairwise disjoint sets in $\mathcal{S}_i$; w.l.o.g., we may assume that $\kappa(\mathcal{S}_i) > 2$. We define a preference profile $\langle C, V \rangle$ such that c and d are designated candidates in C, and it holds that:

$$\begin{aligned} \text{YoungScore}(C, c, V) &= 2 \cdot \kappa(\mathcal{S}_1) + 1; && (6) \\ \text{YoungScore}(C, d, V) &= 2 \cdot \kappa(\mathcal{S}_2) + 1. && (7) \end{aligned}$$

Define the set C of candidates as follows: Create the two designated candidates c and d; for each element x_i of B_1, create a candidate x_i; for each element y_i of B_2, create a candidate y_i; finally, create two auxiliary candidates, a and b.

Define the set V of voters as follows:

- **Voters representing $\mathcal{S}_1$:** For each set $E \in \mathcal{S}_1$, create a single voter v_E as follows:
  - Enumerate E as $\{e_1, e_2, \dots, e_{\|E\|}\}$ (renaming the candidates e_i from $\{x_1, x_2, \dots, x_m\}$ for notational convenience), and enumerate its complement $\overline{E} = B_1 - E$ as $\{\overline{e}_1, \overline{e}_2, \dots, \overline{e}_{m-\|E\|}\}$.
  - To make the preference orders easier to parse, we use

    "$\vec{E}$" to represent the text string "$e_1 > e_2 > \cdots > e_{\|E\|}$";

    "$\vec{\overline{E}}$" to represent the text string "$\overline{e}_1 > \overline{e}_2 > \cdots > \overline{e}_{m-\|E\|}$";

    "$\vec{B_1}$" to represent the text string "$x_1 > x_2 > \cdots > x_m$";

    "$\vec{B_2}$" to represent the text string "$y_1 > y_2 > \cdots > y_n$".

  - Create one voter v_E with preference order:

$$\vec{E} > a > c > \vec{\overline{E}} > \vec{B_2} > b > d. \tag{8}$$

- Additionally, create two voters with preference order:

$$c > \vec{B_1} > a > \vec{B_2} > b > d, \tag{9}$$

  and create $\|\mathcal{S}_1\| - 1$ voters with preference order:

$$\vec{B_1} > c > a > \vec{B_2} > b > d. \tag{10}$$

- **Voters representing $\mathcal{S}_2$:** For each set $F \in \mathcal{S}_2$, create a single voter v_F as follows:
  - Enumerate F as $\{f_1, f_2, \dots, f_{\|F\|}\}$ (renaming the candidates f_j from $\{y_1, y_2, \dots, y_n\}$ for notational convenience), and enumerate its complement $\overline{F} = B_1 - F$ as $\{\overline{f}_1, \overline{f}_2, \dots, \overline{f}_{n-\|F\|}\}$.
  - To make the preference orders easier to parse, we use

    "$\vec{F}$" to represent the text string "$f_1 > f_2 > \cdots > f_{\|F\|}$";

    "$\vec{\overline{F}}$" to represent the text string "$\overline{f}_1 > \overline{f}_2 > \cdots > \overline{f}_{n-\|F\|}$".

– Create one voter v_F with preference order:

$$\overrightarrow{F} > b > d > \overrightarrow{\overrightarrow{F}} > \overrightarrow{B_1} > a > c. \tag{11}$$

■ Additionally, create two voters with preference order:

$$d > \overrightarrow{B_2} > b > \overrightarrow{B_1} > a > c, \tag{12}$$

and create $||\mathcal{S}_2|| - 1$ voters with preference order:

$$\overrightarrow{B_2} > d > b > \overrightarrow{B_1} > a > c. \tag{13}$$

We now prove Equation (6): $\mathrm{YoungScore}(C, c, V) = 2 \cdot \kappa(\mathcal{S}_1) + 1$.

Let $E_1, E_2, \ldots, E_{\kappa(\mathcal{S}_1)} \in \mathcal{S}_1$ be $\kappa(\mathcal{S}_1)$ pairwise disjoint subsets of B_1. Consider the following subset $\widehat{V} \subseteq V$ of the voters. $\widehat{V}$ consists of: (a) every voter v_{E_i} corresponding to the set E_i, where $1 \leq i \leq \kappa(\mathcal{S}_1)$; (b) the two voters given in Equation (9); and (c) $\kappa(\mathcal{S}_1) - 1$ voters of the form given in Equation (10).

Then, $||\widehat{V}|| = 2 \cdot \kappa(\mathcal{S}_1) + 1$. Note that a strict majority of the voters in $\widehat{V}$ prefer c over any other candidate, and thus c is a Condorcet winner in $\langle C, \widehat{V} \rangle$. Hence,

$$\mathrm{YoungScore}(C, c, V) \geq 2 \cdot \kappa(\mathcal{S}_1) + 1.$$

Conversely, to prove that $\mathrm{YoungScore}(C, c, V) \leq 2 \cdot \kappa(\mathcal{S}_1) + 1$, we need the following lemma. The proof of Lemma 7 can be found in the full version [24].

Lemma 7 *For any λ with $3 < \lambda \leq ||\mathcal{S}_1|| + 1$, let V_λ be any subset of V such that V_λ contains exactly λ voters of the form (9) or (10) and c is the Condorcet winner in $\langle C, V_\lambda \rangle$. Then, V_λ contains exactly $\lambda - 1$ voters of the form (8) and no voters of the form (11), (12), or (13). Moreover, the $\lambda - 1$ voters of the form (8) in V_λ represent pairwise disjoint sets from $\mathcal{S}_1$.*

To continue the proof of Theorem 6, let $k = \mathrm{YoungScore}(C, c, V)$. Let $\widehat{V} \subseteq V$ be a subset of size k such that c is the Condorcet winner in $\langle C, \widehat{V} \rangle$. Suppose that there are exactly $\lambda \leq ||\mathcal{S}_1|| + 1$ voters of the form (9) or (10) in $\widehat{V}$. Since c, the Condorcet winner of $\langle C, \widehat{V} \rangle$, must in particular outpoll a, we have $\lambda \geq \lceil \frac{k+1}{2} \rceil$. By our assumption that $\kappa(\mathcal{S}_1) > 2$, it follows from $k \geq 2 \cdot \kappa(\mathcal{S}_1) + 1$ that $\lambda > 3$. Lemma 7 then implies that there are exactly $\lambda - 1$ voters of the form (8) in $\widehat{V}$, which represent pairwise disjoint sets from $\mathcal{S}_1$, and $\widehat{V}$ contains no voters of the form (11), (12), or (13). Hence, $k = 2 \cdot \lambda - 1$ is odd, and $\frac{k-1}{2} = \lambda - 1 \leq \kappa(\mathcal{S}_1)$, which proves Equation (6). Equation (7) can be proven analogously. Thus, we have $\kappa(\mathcal{S}_1) \geq \kappa(\mathcal{S}_2)$ if and only if $\mathrm{YoungScore}(C, c, V) \geq \mathrm{YoungScore}(C, d, V)$. Hence, `Young Ranking` is $\mathrm{P}_{||}^{\mathrm{NP}}$-complete. Modifying the above reduction, we can also prove `Young Winner` $\mathrm{P}_{||}^{\mathrm{NP}}$-complete, which completes the proof of Theorem 6. For details of the modified reduction, we refer to the full version [24]. ■

5. Homogeneous Young and Dodgson Voting Schemes

Social choice theorists have studied many "reasonable" properties that any "fair" election procedure arguably should satisfy, including very natural properties such as nondictatorship, monotonicity, the Pareto Principle, and independence of irrelevant alternatives. One of the most notable results in this regard is Arrow's famous Impossibility Theorem [1] stating that the just-mentioned four properties are logically inconsistent, and thus no "fair" voting scheme can exist. In this section, we are concerned with another quite natural property, the homogeneity of voting schemes (see [9, 29]).

Definition 8 *A voting scheme f is said to be* homogeneous *if and only if for each preference profile $\langle C, V\rangle$ and for all positive integers q, it holds that $f(\langle C, V\rangle) = f(\langle C, qV\rangle)$, where qV denotes V replicated q times.*

Homogeneity means that splitting each voter $v \in V$ into q voters, each of whom has the same preference order as v, yields exactly the same choice set of winning candidates. Fishburn [9] showed that neither the Dodgson nor the Young voting schemes are homogeneous. For the Dodgson SCF, he presented a counterexample with seven voters and eight candidates; for the Young SCF, he modified a preference profile constructed by Young with 37 voters and five candidates. Fishburn [9] provided the following limit devise in order to define homogeneous variants of the Dodgson and Young SCFs. For example, the Dodgson scheme can be made homogeneous by defining from the function DodgsonScore for each preference profile $\langle C, V\rangle$ and designated candidate $c \in C$ the function

$$\text{DodgsonScore}^*(C, c, V) = \lim_{q\to\infty} \frac{\text{DodgsonScore}(C, c, qV)}{q}.$$

The resulting SCF is denoted by Dodgson* SCF, and the corresponding winner and ranking problems are denoted by `Dodgson* Winner` and `Dodgson* Ranking`. Analogously, the Young voting scheme defined above can be made homogeneous by defining YoungScore*. Remarkably, Young [29] showed that the corresponding problem `Young* Winner` can be solved by a linear program. Hence, the problem `Young* Winner` is efficiently solvable, since the problem `Linear Programming` can be decided in polynomial time [12], see also [18]. We establish an analogous result for the problems `Dodgson* Winner` and `Dodgson* Ranking`. The proof of Theorem 9 can be found in the full version [24].

Theorem 9 `Dodgson* Winner` *and* `Dodgson* Ranking` *can be solved in polynomial time.*

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