Clones in Graphs

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Abstract Finding structural similarities in graph data, like social networks, is a far-ranging task in data mining and knowledge discovery. A (conceptually) simple reduction would be to compute the automorphism group of a graph. However, this approach is ineffective in data mining since real world data does not exhibit enough structural regularity. Here we step in with a novel approach based on mappings that preserve the maximal cliques. For this we exploit the well known correspondence between bipartite graphs and the data structure formal context (G, M, I) from Formal Concept Analysis. From there we utilize the notion of clone items. The investigation of these is still an open problem to which we add new insights with this work. Furthermore, we produce a substantial experimental investigation of real world data. We conclude with demonstrating the generalization of clone items to permutations.

Keywords: Social Network Analysis, Formal Concept Analysis, Clones

1 Introduction

The identification of (structural) similar entities in graph data sets is a particularly relevant task in data analysis: it provides insights into entities in the data (e.g., in members of social networks); it allows grouping entities and even reducing data sets by removing redundant (structurally equivalent) elements (factorization). For bipartite graph data, a notion of structural similarity that suggests itself is that of clone items, known from the realm of Formal Concept Analysis (FCA). The latter is a mathematical toolset for qualitative data analysis, relying on algebraic notions s.a. lattices and closure systems. Here, clone items are entities from the same partition that are completely interchangeable within the family of that partition's closed subsets.

In this paper, we follow up on a long-standing open problem of FCA, collected at ICFCA 2006, regarding the meaning of clone items in real world graph data.

Authors are given in alphabetical order. No priority in authorship is implied.

⁴ http://www.upriss.org.uk/fca/problems06.pdf

The notion of clones was initially proposed⁵ in "Clone items: a pre-processing information for knowledge discovery" by R. Medina and L. Nourine. Subsequently, a plethora of desirable properties of clone items has been shown, such as, "hidden combinatorics" [7] that allow factorizations of data structures containing clones, computational properties investigations, like [14], or the use of clones in association rule mining [13]. Finally, the question of semantics was addressed by [11], who investigated clones in three well-known data sets (*Mushroom, Adults*, and *Anonymous* from the UCI Machine Learning Repository [9]). Following the observation that two data sets were free of clones whereas the mushroom data set had only few, [11] introduced nearly clones relying rather on statistical than on structural properties. However, despite these previous efforts, the question – are clone items frequent in natural graph data sets – in particular in social network data – has not yet been answered in general.

The contributions of this paper are threefold: First, we provide a prove for the characterization of clone items on the level of formal contexts that allows us to easily compute clone items in large data sets. Second, we investigate a diverse variety of public realworld data sets coming from different domains and exhibiting different properties. We show that clones are not common in these data sets and conclude that in their present form, clones are not as useful as one would have hoped, regarding the efforts made in previous literature. Third, to resolve this dilemma, we point out a more general notion of clones. For this we fall back to permutations on the set of attributes in a formal context, providing a natural extension of the clone property. These higher order clones are able to identify more complicated "clone structures" and should be the next step in the investigation of relational data structures.

This work is structured as follows. In Section 2, we recall basic notations of FCA and show the correspondence to graphs. Then, in Section 3, we provide a characterization of clone items on the level of formal contexts. Following this, in Section 3.1 we demonstrate how the notion of clones can be applied in the realm of graphs. Subsequent to experiments on various data sets, in Section 4, we extend the notion of clone items to higher order clones. Eventually, we conclude our work with Section 5.

2 Preliminaries

We give a short recollection of the ideas from formal concept analysis as introduced in [5, 18] that are relevant in this work. We use the common presentation of formal contexts by $\mathbb{K} = (G, M, I)$, where G and M are sets and $I \subseteq G \times M$. The elements of G are called objects, those of M are called attributes, and $(g,m) \in I$ signifies that object g has the attribute m. The correspondence to a bipartite graph (network) is at hand. Let $H = (U \cup W, E)$ be such an undirected bipartite graph with $U \cap W = \emptyset$ where U is a set of entities (often users), W some set of common properties, and $E \subseteq \{\{u,w\} \mid u \in U, w \in W\}$ the set of edges between U and W. There are two natural ways of identifying H as a formal context. In the following, we choose $\mathbb{K}(H) = (U,W,I)$ as the to H associated formal context, G where for G where for G where for G where for G where G and G where G where G is a set of G where G is a context of G is a context of G is a context of G and G is a context of G.

⁵ This work is noted to be submitted (e. g, in [7]), but has never been published.

⁶ The second way yields the dual context $\mathbb{K}(H) = (W,U,I)$.

 $w \in W$, we have $(u,w) \in I : \Leftrightarrow \exists e \in E : u \in e \land w \in e$. For the case of a non-bipartite Graph G = (V,E) we simply construct the formal context by $\mathbb{K} = (V,V,I)$ with $(u,v) \in I \Leftrightarrow \{u,v\} \in E$ for all $u,v \in V$. In the following we use the terms network, (bipartite) graph, and formal context interchangeably in the sense above.

We will utilize the common derivation operators $\cdot': \mathcal{P}(G) \to \mathcal{P}(M), A \mapsto B \coloneqq \{m \in M \mid \forall g \in A : (g,m) \in I\}$ and $\cdot': \mathcal{P}(M) \to \mathcal{P}(G), B \mapsto A \coloneqq \{g \in G \mid \forall m \in B : (g,m) \in I\}$. Having those operations we call a formal context $\mathbb{K} = (G,M,I)$ object clarified iff $\forall g,h \in G,g \neq h:g' \neq h'$, attribute clarified iff $\forall m,n \in M,m \neq n:m' \neq n'$ and clarified iff it is both. In this definition we used g' as shorthand for $\{g\}'$. Clarification will later on correspond to a particular trivial kind of clones. Similarly we call a clarified context \mathbb{K} object reduced if for all $g \in G$ there is no $S \subseteq G \setminus \{g\}$ such that g' = S'. We call \mathbb{K} attribute reduced iff for all $m \in M$ there is no $S \subseteq M \setminus \{m\}$ such that m' = S'. And, we call this \mathbb{K} reduced iff \mathbb{K} is attribute and object reduced.

A pair (A,B) where $A\subseteq G, B\subseteq M$ with A'=B and B'=A is called a formal concept. Here, A is called the concept extent and B is called the concept intent. The set of all these formal concepts, i.e., $\mathfrak{B}(\mathbb{K}) \coloneqq \{(A,B) \mid A\subseteq G, B\subseteq M, A'=B, B'=A\}$ gives rise to an order structure (\mathfrak{B},\leq) using $(A,B)\leq (C,D)$: $\Leftrightarrow A\subseteq C$, called concept lattice. For clone items we are particularly interested in the two entailed closure systems, i.e, in the object closure system $\mathfrak{G}(\mathbb{K}) \coloneqq \{A \in G \mid (A,B) \in \mathfrak{B}(\mathbb{K})\}$ and the attribute closure system $\mathfrak{M}(\mathbb{K}) \coloneqq \{B \in M \mid (A,B) \in \mathfrak{B}(\mathbb{K})\}$. We may denote those by \mathfrak{G} and \mathfrak{M} whenever the according context is implicitly given.

Clones Besides the original definition of what clone items are there will be some graduations useful to graphs. We start with the common definition. Given a formal context $\mathbb{K} = (G, M, I)$ and two items $a, b \in M$, we say a is clone to b in \mathfrak{M} if $\forall X \in \mathfrak{M} \colon \varphi_{a,b}(X) \in \mathfrak{M}$, with:

$$\varphi_{a,b}(X) \coloneqq \begin{cases} X \backslash \{a\} \cup \{b\} & \text{if } a \in X \land b \not\in X \\ X \backslash \{b\} \cup \{a\} & \text{if } a \not\in X \land b \in X \\ X & \text{else} \end{cases}$$

We may denote this property by $a \sim_{\mathbb{K}} b$ and whenever the context is distinctive $a \sim b$. It is obvious that \sim is a reflexive and symmetric relation on $M \times M$. Actually, it is also transitive, which can be shown easily, hence \sim is an equivalence relation. Since every $a \in M$ is a clone to itself we say an a is a proper clone iff there is a $b \in M \setminus \{a\}$ such that $a \sim b$. In a not-clarified formal context there might be some $m, n \in M, m \neq n$ such that m' = n'. Those elements are proper clones. However, this is obvious and not revealing any hidden structure besides the fact that two identical copies are present. Therefore we call a proper clone $a \in M$ trivial iff there is a $b \in M \setminus \{a\}$ with a' = b'.

A this point one may ask if it is hard to construct a formal context having a significant number of non-trivial clones. This is very easy as the following example discloses.

Example 2.1. The nominal scales, i.e., $(\{1,...,n\},\{1,...,n\},=)$ and the contranominal-scale $(\{1,...,n\},\{1,...,n\},\neq)$ provide formal contexts where every attribute

element is a non-trivial clone. Furthermore, the union of two formal contexts, i.e., $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K} := (G_2, M_2, I_2)$ becomes $\mathbb{K}_1 \cup \mathbb{K}_2 := (G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2)$, preserves the clones from \mathbb{K}_1 and \mathbb{K}_2 .

All the above can be defined similarly for elements of G using the dual-context, i.e., the context where objects and attributes are interchanged. We therefore omit the explicit definitions and continue assuming the necessary definitions are made. However, we may provide some wording to differentiate between clones in $\mathfrak M$ and clones in $\mathfrak G$ for some formal context (G,M,I). When necessary we call the former attribute clone and the latter object clone.

3 Theoretical observations

In this section, we derive some crucial properties of clones as well as a characterization of the clone property on the level of the context table. These theoretical results allow a fast computation of clones and help understanding the nature of clones in data. The first shows that for attributes with $a \sim b$ the object sets a' and b' are incomparable.

Lemma 3.1 (Clones are incomparable). Let $\mathbb{K} = (G, M, I)$ be a formal context and $a, b \in M$. If $a \sim b$, then from $a' \subseteq b'$ follows a' = b'.

Proof. Using $a' \subseteq b'$ we show $b' \subseteq a'$. We examine the mapping

$$\varphi_{ab}(b'') = \begin{cases} b'' & \text{if } a \in b'' \\ b'' \setminus \{b\} \cup \{a\} & \text{if } a \notin b''. \end{cases}$$

We show that the second case is invalid. From $a \sim b$ and $\varphi_{ab}(b'')$ being a closure we deduce $a'' \subseteq b'' \setminus \{b\} \cup \{a\}$. Since $a' \subseteq b'$, we have $b'' \subseteq a''$ and together we yield $b \in b'' \subseteq a'' \subseteq b'' \setminus \{b\} \cup \{a\}$ contradicting the case. Hence, only the first case can exist, meaning $a \in b''$, thus obviously $b' \subseteq a'$.

The next results indicates, that reducible elements of a formal context can be ignored in the search for clones.

Lemma 3.2 (Clone irreducability). Let $\mathbb{K} = (G, M, I)$ be a clarified formal context and attributes $a, b \in M : a \neq b$ with $a \sim b$. Then a is irreducible in \mathbb{K} .

Proof. Assume a is reducible, i. e., there exists a set of attributes $N \subseteq M$ with $a \notin N$ and $\bigcap_{n \in N} n' = a'$. As \mathbb{K} is clarified, we have $a' \neq b'$, thus from Lemma 3.1 follows $b \notin a''$. Therefore $\varphi_{a,b}(a'') = a'' \setminus \{a\} \cup \{b\}$. From the reducibility assumption follows

$$\forall n\!\in\!N\!:\!n'\!\supseteq\!a'\!\Longrightarrow\!n\!\in\!a''\!\stackrel{n\neq a}{\Longrightarrow}\!n\!\in\!a''\!\setminus\!\{a\}\!\cup\!\{b\}\!=\!\varphi_{a,b}(a'').$$

Thus, $a' = \bigcap_{n \in N} n' \supseteq \varphi_{a,b}(a'')'$, which means $a'' \subseteq \varphi_{a,b}(a'') = a'' \setminus \{a\} \cup \{b\}$. Clearly, this means a = b contradicting the lemma's assumption.

While clarifying a context removes the non-trivial clones, additionally reducing that context does not change the clone relationship any further. Therefore, for finding non-trivial clones it suffices considering reduced contexts. Next, we describe for such contexts how clones can be identified directly from the context's table. [7] already found it is sufficient to check join-irreducible intents to check the clone property. The respective result there (Proposition 1) is formulated for the dual version of formal contexts, i. e., where G and M are interchanged. Also, for the proof the authors of [7] refer to a manuscript that had been submitted (at the time) but appears to have never been published. For the sake of completeness, we present a variation of their result in the common notion of a formal context and present a proof. Here, we already use the fact that in a reduced context, the join irreducible concepts are exactly the object concepts.

Theorem 3.1. Let $\mathbb{K} = (G, M, I)$ be a reduced formal context and $a, b \in M$ with $a \neq b$. The following are equivalent:

- 1. $a \sim b$
- 2. For each object $g \in G$, there is an object $h \in G$ such that $\varphi_{a,b}(g') = h'$.

Proof. First we show, $1 \Longrightarrow 2$. For $a,b \in g'$ or $a,b \notin g'$, the claim is obvious (using $b \coloneqq g$). Without loss of generality, we can assume $a \in g'$ and $b \notin g'$, thus $\varphi_{a,b}(g') = g' \setminus \{a\} \cup \{b\}$.

As $\varphi_{a,b}(g')$ is an intent, there exists a set of objects $H \subseteq G$ with $H' = \varphi_{a,b}(g') = g' \setminus \{a\} \cup \{b\}$. We can partition H into $H_a := \{h \in H \mid a \in h'\}$ and $H_{\bar{a}} := \{h \in H \mid a \notin h'\}$. As clearly $a \notin \varphi_{a,b}(g')$, $H_{\bar{a}}$ cannot be empty. We yield:

$$g'\backslash\{a\}\cup\{b\} = \bigcap_{h\in H_a} h'\cap \bigcap_{h\in H_{\bar{a}}} h'$$

$$H_{\bar{a}}\neq\emptyset, b\notin g' \Longrightarrow \qquad g'\backslash\{a\} = \bigcap_{h\in H_a} h'\cap \bigcap_{h\in H_{\bar{a}}} (h'\backslash\{b\})$$

$$a\in g' \Longrightarrow \qquad g' = \bigcap_{h\in H_a} h'\cap \bigcap_{h\in H_{\bar{a}}} (h'\backslash\{b\}\cup\{a\})$$

$$b\in\varphi_{a,b}(g') = H' \Longrightarrow \qquad g' = \bigcap_{h\in H_a} h'\cap \bigcap_{h\in H_{\bar{a}}} \varphi_{a,b}(h')$$

As g is irreducible, we either have an object $h \in H_a$ with g' = h' or an object $h \in H_{\bar{a}}$ with $g' = \varphi_{a,b}(h')$. Clearly, the former cannot be true, as $b \in h'$ for $h \in H$ and $b \notin g'$. From the latter follows $\varphi_{a,b}(g') = h'$.

Next, we show $2. \Longrightarrow 1$: Let $N \subseteq M$ be an intent of \mathbb{K} , i. e., there is a set of objects $H \subseteq G$ such that N = H'. We show that $\varphi_{a,b}(N)$ is an intent. This is trivial for the cases $a,b \in N$ and $a,b \notin N$. Without loss of generality, we assume $a \in N$ and $b \notin N$. Then

$$\varphi_{a,b}(N) = \varphi_{a,b}(H') = H' \setminus \{a\} \cup \{b\} = \bigcap_{h \in H_b} (h' \setminus \{a\} \cup \{b\}) \cap \bigcap_{h \in H_{\bar{b}}} (h' \setminus \{a\} \cup \{b\})$$

with H_b and $H_{\bar{b}}$ defined as H_a and $H_{\bar{a}}$. For $h \in H_b$ it holds $h' \setminus \{a\} \cup \{b\} = h' \setminus \{a\}$. As $H_{\bar{b}} \neq \emptyset$ (a.p., $b \notin N = H'$) we yield $\varphi_{a,b}(N) = \bigcap_{h \in H_b} h' \cap \bigcap_{h \in H_{\bar{b}}} (h' \setminus \{a\} \cup \{b\})$.

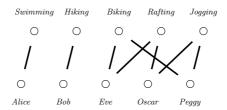


Figure 1. Example of a social network graph exhibting various clones itmes. Edges connect a person with his or her activity. Equivalence classes for attribute clones are {Swimming,Hiking},{Biking, Rafting, Jogging}.

Since $a \in H'$, for $h \in H_{\bar{b}}$: $h' \setminus \{a\} \cup \{b\} = \varphi_{a,b}(h')$, which by 2. is g' for some $g \in G$. Thus $\varphi_{a,b}(N)$ is the intersection of intents and therefore itself an intent.

The theorem characterizes clones on the context level: Two attributes a and b are clones if for each object $g \in G$ whose row contains only one of the two attributes, there is another object $h \in G$ such that its row contains only the other of the two attributes, while the remaining parts of the rows are identical, i. e., $g' \setminus \{a\} = h' \setminus \{b\}$.

3.1 Clones in Graph Data

Clones in Social Networks In the following, we identify any given graph with the formal context counterpart $\mathbb{K} = (U, W, I)$, as in Section 2. Transferring the definitions from Section 2, we obtain what clones in graphs, in particular in social networks, are. For the special case of social networks we call object clones user clones and attribute clones are either some property clone, in the bipartite case, or also user clones, in the single-mode case.

Example 3.1 (Social Network). In Figure 1 we show a small artificial example of a possible social network. Represented as context as described in Section 2 we get with $M = \{ \mathbf{S} \text{wimming}, \mathbf{H} \text{iking}, \mathbf{B} \text{iking}, \mathbf{R} \text{afting}, \mathbf{J} \text{ogging} \}$ the closure system $\mathfrak{M}(\mathbb{K}) = \{ \{ \mathbf{S} \}, \{ \mathbf{H} \}, \{ \mathbf{B} \}, \{ \mathbf{R} \}, \{ \mathbf{B}, \mathbf{R} \} \}, \{ \mathbf{B}, \mathbf{J} \}, \{ \mathbf{R}, \mathbf{J} \} \}$. The associated clone classes are denoted in Figure 1.

Data set description Almost all of the following data sets can be obtained from the UCI Machine Learning Repository [9]. We consider nine social network graphs and two non social network data sets: **zoo** [9]: 101 animals and seventeen attributes (fifteen Boolean and two numerical). All attributes were nominal scaled, resulting in a set with 43 attributes; **cancer** [12]: 699 instances of breast cancer diagnoses with ten numerical attributes, which were nominal scaled; **facebooklike** [15]: 337 forum users with 522 topics they communicated on; **southern** [17]: classical small world social network consisting of fourteen woman attending eighteen different events; **club** [3]: 25 corporate executive officers and fifteen social clubs in which

Table 1. Properties of the considered (social) networks and data sets and results for clone experiment. With G-t we denote trivial clones whereas clones denote non-trivial clones.

Name	U $ M $	density	$\# \ G\text{-clones}$	# M-clones	$\# \ G\text{-t-clones}$	# M-t-Clones
zoo	101 43	0.390	0	0	42	2
cancer	699 92	0.110	0	0	236	0
facebooklike	377 522	0.014	7	0	24	83
southern	18 14	0.352	0	0	1	1
aplnm	79 188	0.061	0	0	1	21
club	25 15	0.250	0	0	0	0
movies	62 39	0.079	0	0	1	0
jazz	198 198	0.068	7	7	0	0
dolphins	62 62	0.082	0	0	2	2
hightech	33 33	0.148	0	0	1	1
wiki	764 605	0.006	234	234	73	30

they are involved in; **movies** [4]: 39 composers of film music and their relations to 62 producers; **aplnm** [1]: 79 participants of the *Lange Nacht der Musik* in 2013 and the 188 events they participated in; **jazz** [6]: 198 jazz musicians and their collaborations; **dolphin** [10]: 62 bottlenose dolphins with contacts amongst each other; **hightech** [16]: Some (one-mode) social network with 33 users from within the parameters of a social network but with no further insights provided; **wiki** [8, 16]: 764 voters on Wikipedia with 605 users to be voted on.

For comparison, we also investigate randomized versions of all those data sets, generated using a coin draw process. This may imply that the resulting formal contexts are prone to the stegosaurus phenomenon. However, no unbiased method for generating formal context for a given number of objects, attributes, and density is known [2].

Computation Computing the attribute (object) clones for a given formal context (G,M,I) would imply to know the associated attribute (object) closure system. However, computing those is computational infeasible for contexts of a particular size or greater. To cope with this barrier we utilize Lemma 3.2 and Theorem 3.1. Hence, instead of checking all elements of a closure system we only need to check the irreducibles. Therefore we checked brute force all combinations of attributes (objects) for every given data set by checking the according irreducibles.

In particular we computed for every data set the number of trivial and non-trivial object clones, and attribute clones. The results are shown in Table 1. In addition we also computed the number of trivial and non-trivial clones for the object/attribute-projections for every formal context. However, besides creating more trivial clones no further insights could be grasped from this. Also, the experiment on randomly generated formal contexts had not different outcome. Therefore we omitted presenting the particular results for the latter two.

Discussion The most obvious result for all data sets alike is that non-trivial clones are very infrequent. Omitting the wiki data set only two data sets have clones at all, in particular a very small number of object clones compared to the size of the network. We investigated the exception by the wiki data set further and discovered a large nominal scale as subcontext responsible for the vast amount of clones. Since the wiki data set is the result of a collection of voting processes this would represent single votes. For trivial clones we have diverse observations. Some networks like facebooklike have a significant amount of trivial clones. Others of comparable size, however, do not, like jazz. Since those clones do not reveal any hidden structure but the fact that copies of users or properties are present in the network, we consider these clones uninteresting.

For the object and attribute projections we obtain almost the same results. Almost no non-trivial clones are present. Though, the number of trivial clones has increased in almost all the networks. This could be another indication that simple one-mode projections are insufficient for analyzing bipartite networks.

All in all, the notion of non-trivial clones seems insufficient for the investigation of graphs. The explanation for this is that the structural requirements for two attributes being clone are too strong, cf. theoretical results in Section 3. However, it strikes the question if there is a generalization which is softening those requirements while preserving enough structure.

4 Generalized Clones

The results from the previous section motivate finding a more general clone notion for formal contexts. In [7] the authors provided an interesting generalization of clones in a formal context. They proposed P-Clones, i. e., clones with respect to the family of pseudo intents, and A-Clones, i. e., clones in a particular kind of atomized context. Both approaches are based on using some kind of modified family of sets. Another course of action was taken in [11], in which the author used a measure of "cloneniness" based on the number of incorrect mapped sets. We take a different approach, using the original set of closures – the intents – based on the following observation.

Remark 4.1 (Clone permutation). Every pair (a,b) of elements $a,b \in M$ with $a \sim b$ for a given formal context (G,M,I) gives rise to a permutation $\sigma: M \to M, m \mapsto \sigma(m)$, with $\sigma(a) = b$, $\sigma(b) = a$, and $\sigma(m) = m$ for $m \in M \setminus \{a,b\}$. We denote such permutations as clone permutations.

Since for every $a \in M$ we have $a \sim a$, the set of clone permutations S for a given formal context (G,M,I) contains the identity. For any two elements $a,b \in M$ with $a \sim b$ we can represent the associated clone permutation σ by $\sigma := (ab)$ using the reduced cycle notation. From this we note that the set of all pairs of proper clones corresponds to a particular subset of permutations on M where every permutation σ contains exactly one two-cycle. This gives rise to two possible generalizations. Both associated computational problems require sophisticated algorithms to be developed.

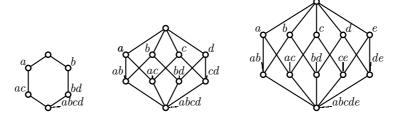


Figure 2. Example for clone-free closure system on four attributes (left, middle) and on five attributes (right).

Multiple two-cycles We motivate this approach using the lattice for a closure system on $M = \{a,b,c,d\}$ represented in Figure 2 (left). In this closure system there are no proper clones. However, we can find a permutation σ that preserves the closure system. For example, the permutation $\sigma = (ab)(cd)$, which is a permutation of two disjoint cycles of length two. This permutation is not representable by exactly one cycle of length two. Hence, we propose permutations representable as products of cycles of length two as one generalization of clones. Yet, this immediately gives rise to the idea of higher order permutations.

Higher Order Again, we want to motivate this generalization by providing an example. In Figure 2 (middle), we show the lattice for a closure system $M = \{a,b,c,d\}$. This closure system is free of (proper) clones. However, we find a permutation $\sigma = (ab)(cd)$ in the above described manner. In addition we find a permutation of order four, i. e., $\sigma^4 = id$, preserving the closure system, e. g., $\sigma = (acbd)$. In the same figure on the right we observe a permutation of order five, i.e., $\sigma = (acedb)$, answering the natural question for a permutation with odd order.

5 Conclusion

While starting the investigation the authors of this work were confident to discover clones in graph data sets, at least for graphs of a particular minimal size. In order to cope with the computational complexity of closure systems we utilized results from [7] and expressed them in terms of statements about formal contexts. However, our investigation did reveal the absence of clones in real world graph like data. The only significant observation was the emergence of trivial clones while projecting bipartite social networks to one set of nodes.

This setback, though, led us to discover two more general notions of clones, which can cope with more structural requirements. Investigating those more thoroughly should be the next step in clone related research, building on the theoretical results in Section 3. To this end, we finish our work with the following three open questions. **Question 1:** To which graph theoretical notion could the idea of clone permutation correspond to? **Question 2:** Does the set of all valid clone permutations on a closure set always form a group and if no, why not? **Question 3:** If

yes, can this group provide new insights into the structure of closure systems or of social networks?

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