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BEM-based Finite Element Approaches on Polytopal Meshes

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To Anna

Preface

The BEM-based finite element method has been developed within the last decade, and it is one of the first methods designed for the approximation of boundary value problems on polygonal and polyhedral meshes. This is possible due to the use of implicitly defined ansatz functions which are treated locally by means of boundary integral operators and are realized with the help of boundary element methods (BEM) in the computations.

When I started my doctoral studies in 2009, it was an appealing but also a somehow abstruse idea to generalize the well-known finite element method (FEM) to polygonal and polyhedral meshes. To that time, I was not aware of any other attempts in this direction. A few years later, the virtual element method came up, and I learned about other approaches that just started in parallel to work on general meshes. The discretization of differential equations on polygonal and polyhedral meshes became a hot research area, and the amount of publications and organized conferences is continuously increasing. The beauty of these ideas caught my attention and shaped my research interests. My resulting theoretical and computational contributions of the last 10 years have found their way into this monograph which is based on my habilitation thesis.

The monograph presents an introduction, a mathematical analysis and applications of the BEM-based finite element approach. It is intended to researchers in the field of numerics for partial differential equations in the wide community of mathematicians and mathematically aware engineers using finite element methods but also to advanced graduate students who are interested to deepen their knowledge in the field.

After the discussion of polytopal meshes, the BEM-based FEM is introduced for high-order approximation spaces over general elements, and its error analysis is given on uniform meshes for smooth solutions (Chap. 2). The method is studied for uniform, adaptive and anisotropic discretizations, where several error estimation techniques and interpolation operators are derived and the notion of anisotropic polytopal elements is introduced (Chap. 3). Beside these, the numerical treatment of boundary integral equations is discussed (Chap. 4). Local error estimation techniques give rise to adaptive mesh refinement as in classical finite element strategies.

Here, the flexibility of polytopal meshes yields benefits in the refinement process and in the application of post-processing operators as in goal-oriented techniques, for instance (Chap. 5). Finally, some recent developments on mixed finite element formulations and problem-adapted approximation spaces are highlighted (Chap. 6). Throughout the monograph, all theoretical results are motivated and validated by numerous numerical experiments and tests, which demonstrate the applicability and the flexibility of the BEM-based FEM.

This monograph and the involved research could not have been realized without the encouragement and support of various persons. Among all these friends in the scientific community, I thank, in particular, Sergej Rjasanow and his research group for their open-mindedness and helpfulness which created a great and stimulating working atmosphere. My deepest gratitude, however, is reserved for my wife, Anna Benz-Weißer, whose logical mind, sound and thoughtful advice, sympathetic ear and unselfish support are of indispensable value for me.

Saarbrücken, Germany Steffen Weißer
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Contents

1 Introduction 1
1.1 Overview 1
1.2 Outline 5
1.3 Mathematical Preliminaries 6
1.3.1 Function Spaces and Trace Operators 7
1.3.2 Galerkin Formulations 13

2 Finite Element Method on Polytopal Meshes 17
2.1 Preliminaries 17
2.2 Polygonal and Polyhedral Meshes 18
2.2.1 Mesh Regularity and Properties in 2D 20
2.2.2 Mesh Regularity and Properties in 3D 24
2.2.3 Mesh Refinement 31
2.3 Trefftz-Like Basis Functions 33
2.3.1 Node and Edge Basis Functions 34
2.3.2 Element Basis Functions 38
2.3.3 Final Approximation Space 40
2.3.4 Simple Generalization to 3D 41
2.4 Interpolation Operators 43
2.5 Galerkin Formulation and Convergence Estimates 53
2.6 Numerical Examples 58

3 Interpolation of Non-smooth Functions and Anisotropic Polytopal Meshes 65
3.1 Preliminaries 65
3.2 Trace Inequality and Best Approximation 68
3.3 Quasi-Interpolation of Non-smooth Functions 72
3.3.1 Clément-Type Interpolation 72
3.3.2 Scott–Zhang-Type Interpolation 75
3.4 Anisotropic Polytopal Meshes 77
3.4.1 Characterisation of Anisotropy and Regularity 77
3.4.2 Approximation Space 86

3.4.3 Anisotropic Trace Inequality and Best Approximation 87
3.4.4 Quasi-Interpolation of Anisotropic Non-smooth Functions 90
3.4.5 Interpolation of Anisotropic Smooth Functions 95
3.4.6 Numerical Assessment of Anisotropic Meshes 98

4 Boundary Integral Equations and Their Approximations 107
4.1 Preliminaries 107
4.2 Boundary Integral Formulations 108
4.2.1 Direct Approach for Dirichlet Problem 111
4.2.2 Indirect Approach for Dirichlet Problem 112
4.2.3 Direct Approach for Neumann Problem 113
4.3 Boundary Element Method 115
4.3.1 Dirichlet Problem 116
4.3.2 Neumann Problem 121
4.4 Nyström Approach 124
4.4.1 Domains with Smooth Boundary 124
4.4.2 Domains with Corners 126
4.5 Application in BEM-Based FEM 130
4.5.1 Incorporation of Local Solvers and Quadrature on Polytopes 130
4.5.2 Numerical Examples and Comparison 135

5 Adaptive BEM-Based Finite Element Method 141
5.1 Preliminaries 141
5.2 Residual Based Error Estimator 145
5.2.1 Reliability 148
5.2.2 Efficiency 149
5.2.3 Numerical Experiments 152
5.3 Goal-Oriented Error Estimation 157
5.3.1 DWR Method for Linear Goal Functionals 158
5.3.2 Preparation for Post-processing: Special Meshes 161
5.3.3 Approximation of the Primal and Dual Solution 161
5.3.4 The Localized Error Estimators 165
5.3.5 Numerical Tests 168

6 Developments of Mixed and Problem-Adapted BEM-Based FEM 177
6.1 Mixed Formulations Treated by Means of BEM-Based FEM 177
6.1.1 Mixed Formulation 178
6.1.2 $\mathbf{H}$(div)-Conforming Approximation Space 180
6.1.3 Approximation of Mixed Formulation 185
6.1.4 Realization and Numerical Examples 188
6.2 3D Generalization with Application to Convection-Diffusion-Reaction Equation 195
6.2.1 Generalization for Diffusion Problem 195
6.2.2 Application to Convection-Diffusion-Reaction Problem 197

6.2.3 Realization of the Basis Functions 200
6.2.4 Fully Discrete Galerkin Formulation 205
6.2.5 Numerical Experiments: Diffusion Problem 207
6.2.6 Numerical Experiments: Convection-Diffusion Problem 211

References 223

Index 233

Acronyms

List of abbreviations which are sorted according to their context

Boundary Value Problem

Γ	Boundary of the domain Ω
Γ_D	Dirichlet boundary with $\Gamma_D \subset \Gamma$, $\lvert\Gamma_D\rvert > 0$
Γ_N	Neumann boundary with $\Gamma_N \subset \Gamma$
Ω	Domain of the boundary value problem with $\Omega \subset \mathbb{R}^d$, which is open and bounded
a	Scalar valued diffusion coefficient with $0 < a_{\min} \leq a \leq a_{\max}$
d	Space dimension with $d = 2, 3$
f	Source function
g_D	Dirichlet data
g_N	Neumann data
$\mathbf{n}$	Outward unit normal vector on Γ
$\mathbf{x}, \mathbf{y}$	Points in $\mathbb{R}^d$ with $\mathbf{x} = (x_1, \ldots, x_d)^\top$

Function Spaces

Let $\omega \subset \overline{\Omega}$ be either a domain or a lower-dimensional manifold.

$\mathscr{P}^k(\omega)$	Space of polynomials with degree smaller or equal $k \in \mathbb{N}_0$ with the convention $\mathscr{P}^{-1}(\omega) = \{0\}$
$\mathscr{P}^k_{\mathrm{pw}}(\mathscr{T}_h(K))$	Space of continuous functions on K which are polynomials of degree smaller or equal k on each triangle ($d = 2$) or tetrahedron ($d = 3$) of the auxiliary discretization $\mathscr{T}_h(K)$ of the polytopal element K

$\mathscr{P}^k_{\mathrm{pw}}(\partial K)$	Space of continuous functions on ∂K which are polynomials of degree smaller or equal k on each edge ($d = 2$) or triangular face ($d = 3$) of the polytopal element K; $\mathscr{P}^k_{\mathrm{pw}}(\mathscr{B}_h)$ for a general boundary mesh $\mathscr{B}_h$
$\mathscr{P}^k_{\mathrm{pw,d}}(\partial K)$	Space of $L_2(\partial K)$ functions which are polynomials of degree smaller or equal k on each edge ($d = 2$) or triangular face ($d = 3$) of the polytopal element K; $\mathscr{P}^k_{\mathrm{pw,d}}(\mathscr{B}_h)$ for a general boundary mesh $\mathscr{B}_h$
$C^0(\omega)$	Space of continuous functions on ω
$C^k(\omega)$	Space of $k \in \mathbb{N}$ times continuously differentiable functions on ω
$C^{k,\kappa}(\omega)$	Space of Hölder continuous functions with $k \in \mathbb{N}_0$ and $\kappa \in (0, 1]$ on ω and in particular the space of Lipschitz continuous functions $C^{0,1}(\omega)$
$H^s(\omega)$	Sobolev space of order $s \in \mathbb{R}$ over ω (see Sect. 1.3)
$H^1_0(\omega)$	Space of $H^1(\omega)$ functions with vanishing trace on $\partial\omega$
$H^1_*(K)$	Space of $H^1(K)$ functions with vanishing mean values on ∂K
$H^{1/2}_*(\partial K)$	Space of $H^{1/2}(\partial K)$ functions with vanishing mean values
$H^1_\Delta(\mathscr{K}_h)$	Space of $H^1(\Omega)$ functions which are weakly harmonic on each $K \in \mathscr{K}_h$
$H^1_D(\Omega)$	Space of $H^1(\Omega)$ functions with vanishing trace on the Dirichlet boundary Γ_D
$\mathbf{H}(\mathrm{div}, \Omega)$	Space of vector valued L_2-functions with divergence in $L_2(\Omega)$
$\mathbf{H}_N(\mathrm{div}, \Omega)$	Space of $\mathbf{H}(\mathrm{div}, \Omega)$ functions with vanishing normal trace on Γ_N
$L_2(\omega)$	Lebesgue space of square integrable functions on ω
$L_\infty(\omega)$	Lebesgue space of measurable functions which are bounded almost everywhere on ω
$W^2_\infty(\omega)$	Sobolev space of functions which are, together with their first and second derivatives, measurable and bounded almost everywhere on ω

Polytopal Mesh

$c_{\mathscr{K}}, c_{\mathscr{F}}$	Regularity parameter for stable polytopal meshes (see Definitions 2.2, 2.12 and 2.17)
$\mathscr{E}_h$	Set of all edges
$\mathscr{E}_{h,\Omega}$	Set of all edges in the interior of Ω
$\mathscr{E}_{h,D}$	Set of all edges on the Dirichlet boundary Γ_D
$\mathscr{E}_{h,N}$	Set of all edges on the interior of the Neumann boundary Γ_N
$\mathscr{F}_h$	Set of all faces for 3D, might also refer to the set of edges
$\mathscr{F}_{h,\Omega}$	Set of all faces in the interior of Ω
$\mathscr{F}_{h,D}$	Set of all faces on the Dirichlet boundary Γ_D
$\mathscr{F}_{h,N}$	Set of all faces on the interior of the Neumann boundary Γ_N
$\mathscr{E}(K)$	Set of edges which belong to the element K

$\mathscr{E}(F)$	Set of edges which belong to the face F
$\mathscr{F}(K)$	Set of faces which belong to the element K
Γ_S	Skeleton of the discretization
$\mathscr{K}_h$	Set of all polygonal or polyhedral elements in the mesh
$\mathscr{N}_h$	Set of all nodes
$\mathscr{N}_{h,\Omega}$	Set of all nodes in the interior of Ω
$\mathscr{N}_{h,D}$	Set of all nodes on the Dirichlet boundary Γ_D
$\mathscr{N}_{h,N}$	Set of all nodes on the interior of the Neumann boundary Γ_N
$\mathscr{N}(K)$	Set of nodes which belong to the element K
$\mathscr{N}(E)$	Set of nodes which belong to the edge E
$\mathscr{N}(F)$	Set of nodes which belong to the face F
ρ_K, ρ_F	Radius of inscribed ball/circle of an element or face see (Definitions 2.1, 2.10 and 2.11)
$\sigma_{\mathscr{K}}, \sigma_{\mathscr{F}}$	Uniform bound of aspect ratio in regular meshes (see Definitions 2.1, 2.10 and 2.11)
E	Edge in the mesh with $E = \overline{\mathbf{z}_b\mathbf{z}_e}$
F	Polygonal face of a polyhedral element, might also be an edge depending on the context
h_E	Length of an edge
h_F	Diameter of a face
h_K	Diameter of an element
K	Polygonal or polyhedral element
$\mathbf{z}$	Node in the mesh
$\mathbf{z}_K, \mathbf{z}_F$	Centre of inscribed ball/circle of an element or face (see Definitions 2.1, 2.10 and 2.11)

Approximation Spaces, Interpolation and Projection

$\psi_{\mathbf{z}}$	Nodal basis function
ψ_E	Edge basis function
ψ_F	Face basis function
ψ_K	Element basis function
Ψ_h^k	Basis of V_h^k with $\Psi_h^k = \Psi_{h,H}^K \cup \Psi_{h,B}^k$
V_h^k	Approximation space of order k over polytopal mesh with the decomposition $V_h^k = V_{h,H}^k \oplus V_{h,B}^k$
$V_{h,D}^k$	Subspace of V_h^k with functions vanishing on the Dirichlet boundary Γ_D
$V_{h,H,D}^k$	Subspace of $V_{h,H}^k$ with functions vanishing on the Dirichlet boundary Γ_D
$\mathfrak{I}_h^k$	Interpolation operator from $H^2(\Omega)$ into V_h^k with $\mathfrak{I}_h^k = \mathfrak{I}_{h,H}^k + \mathfrak{I}_{h,B}^k$
$\mathfrak{I}_C$	Clément interpolation operator
$\mathfrak{I}_{SZ}$	Scott–Zhang interpolation operator
Π_ω	L_2-projection over ω into the space of constants
$\omega_{\mathbf{z}}$	Neighbourhood of all elements adjacent to the node $\mathbf{z}$

ω_E	Neighbourhood of all elements adjacent to the edge E
ω_F	Neighbourhood of all elements adjacent to the face F
ω_K	Neighbourhood of all elements adjacent to the element K
ω_E^*	Neighbourhood of all elements whose closure contain the edge E
ω_F^*	Neighbourhood of all elements whose closure contain the face F
ω_K^*	Neighbourhood of all elements that share an edge ($d = 2$) or face ($d = 3$) with K
$C_P(\omega)$	Poincaré constant for the domain ω

Boundary Integral Equations

Φ_D	Basis of $\mathscr{P}^k_{\mathrm{pw}}(\mathscr{B}_h)$
$\Phi_{D,F}$	Set of basis functions from Φ_D with support in $F \in \mathscr{F}_h$
Φ_N	Basis of $\mathscr{P}^{k-1}_{\mathrm{pw,d}}(\mathscr{B}_h)$
$\mathscr{P}^k_{\mathrm{pw}}(\mathscr{B}_h)$	Space of continuous functions which are piecewise polynomials of degree smaller or equal k over the boundary mesh $\mathscr{B}_h$ with basis Φ_D
$\mathscr{P}^{k-1}_{\mathrm{pw,d}}(\mathscr{B}_h)$	Space of $L_2(\partial K)$ functions which are piecewise polynomials of degree smaller or equal $k-1$ over the boundary mesh $\mathscr{B}_h$ with basis Φ_N
$\varsigma(\mathbf{x})$	Scalar term on ∂K (see (4.5) and Remark 4.1)
γ_0^K	Trace operator, which gives the trace of a function defined on K on the boundary ∂K
γ_1^K	Conormal derivative operator, which gives the outer normal derivative on ∂K for sufficiently smooth functions defined on K (might be scaled by diffusion matrix A)
$\widetilde{\gamma_1^K}$	Modified conormal derivative operator on ∂K for convection-diffusion-reaction equation (see (6.34))
$\mathbf{D}_K$	Hypersingular integral operator
$\mathbf{D}_{K,h}$	Discretization of the hypersingular integral operator
$\widetilde{\mathbf{D}}_{K,h}$	Regularized discretization of the hypersingular integral operator
$\mathbf{K}_K$	Double-layer potential operator
$\mathbf{K}'_K$	Adjoint double-layer potential operator
$\mathbf{K}_{K,h}$	Discretization of the double-layer potential operator
$\mathbf{M}_{K,h}$	Massmatrix for Φ_N and Φ_D
$\mathbf{M}^{DD}_{K,h}$	Massmatrix for Φ_D and Φ_D
$\mathbf{M}^{NN}_{K,h}$	Massmatrix for Φ_N and Φ_N
$\mathbf{n}_K(\mathbf{x})$	Outward unit normal vector on ∂K at the point $\mathbf{x} \in \partial K$
$\mathbf{P}_K$	Poincaré–Steklov operator (Neumann-to-Dirichlet map)
$\mathbf{S}_K$	Steklov–Poincaré operator (Dirichlet-to-Neumann map)
$\mathbf{S}_{K,h}$	Symmetric discretization of the Steklov–Poincaré operator
$\mathbf{S}^{\mathrm{unsym}}_{K,h}$	Unsymmetric discretization of the Steklov–Poincaré operator
$\widetilde{\mathbf{S}}_K$	Approximation of the Steklov–Poincaré operator

$\mathbf{V}_{K,h}$	Discretization of the single-layer potential operator
$\mathbf{V}_K$	Single-layer potential operator

Adaptivity and Error Estimates

$\|\cdot\|_b$	Energy norm induced by the bilinear form of the variational formulation
$\|\cdot\|_{b,\omega}$	Energy norm with restricted integration domain ω
$[\![u_h]\!]_F$	Jump of the conormal derivative of u_h over F, where F is either an edge ($d = 2$) or a face ($d = 3$)
R_F	Edge/face residual
R_K	Element residual

Miscellaneous

δ_{ij}	Kronecker symbol
$\|\cdot\|_F$	Frobenius norm of a matrix
$\sigma_{\mathscr{T}}$	Uniform bound of aspect ratio in auxiliary discretization $\mathscr{T}_h(\mathscr{K}_h)$
$\mathscr{T}_h(K)$	Auxiliary discretization of K with triangles or tetrahedrons (see Sects. 2.2.1 and 2.2.2); we also use $\mathscr{T}_h(\cdot) = \mathscr{T}_0(\cdot)$
$\mathscr{T}_h(\mathscr{K}_h)$	Admissible auxiliary discretization of Ω with triangles or tetrahedrons constructed by $\mathscr{T}_h(K)$, $K \in \mathscr{K}_h$
$\mathscr{T}_l(F)$	Auxiliary triangulation of a face with mesh level l
$\mathscr{T}_l(\partial K)$	Admissible auxiliary triangulation of the surface of an element constructed by $\mathscr{T}_l(F)$, $F \in \mathscr{F}(K)$
$\mathscr{T}_l(K)$	Auxiliary discretization into tetrahedra of element constructed by $\mathscr{T}_l(\partial K)$
$\mathscr{T}_l(\mathscr{K}_h)$	Admissible auxiliary discretization of Ω into tetrahedra constructed by $\mathscr{T}_l(K)$, $K \in \mathscr{K}_h$
$\mathfrak{F}_K$	Affine mapping which serves for the scaling and depends on the context
$M_{\mathrm{Cov}}(K)$	Covariance matrix of element K
$\overline{\mathbf{x}}_K$	Barycentre of element K