4-CONNECTED TRIANGULATIONS ON FEW LINES[*†]

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Abstract. We show that every 4-connected plane triangulation with n vertices can be drawn such that edges are represented by straight segments and the vertices lie on a set of at most $\sqrt{2n}$ lines each of them horizontal or vertical. The same holds for plane graphs on n vertices without separating triangle.

The proof is based on a corresponding result for diagrams of planar lattices which makes use of orthogonal chain and antichain families.

1 Introduction

Given a planar graph G we denote by $\pi(G)$ the minimum number ℓ such that G has a plane straight-line drawing in which the vertices can be covered by a collection of ℓ lines. Clearly $\pi(G) = 1$ if and only if G is a forest of paths. The set of graphs with $\pi(G) = 2$, however, is already surprisingly rich, it contains trees, outerplanar graphs and subgraphs of grids [1, 9].

The parameter $\pi(G)$ has received some attention in recent years, here is a list of known results:

- It is NP-complete to decide whether $\pi(G) = 2$ (Biedl et al. [2]).
- For a stacked triangulation G, a.k.a. planar 3-tree or Apollonian network, let d_G be the stacking depth (e.g. K_4 has stacking depth 1). On this class lower and upper bounds on $\pi(G)$ are $d_G + 1$ and $d_G + 2$ respectively, see Biedl et al. [2] and for the lower bound also Eppstein [8, Thm. 16.13].
- Eppstein [9] constructed a planar, cubic, 3-connected, bipartite graph G_ℓ on $O(\ell^3)$ vertices with $\pi(G_\ell) \geq \ell$.

Related parameters have been studied by Chaplick et al. [4, 5].

The main result of this paper is the following theorem.

Theorem 1 *If G is a 4-connected plane triangulation on n vertices, then $\pi(G) \leq \sqrt{2n}$.*

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The result is not far from optimal since, using a small number of additional vertices and many additional edges, Eppstein's graph G_ℓ can be transformed into a 4-connected plane triangulation, i.e., in the class we have graphs with $\pi(G) \in \Omega(n^{1/3})$. Figure 1 shows a section of such an extension of G_ℓ.

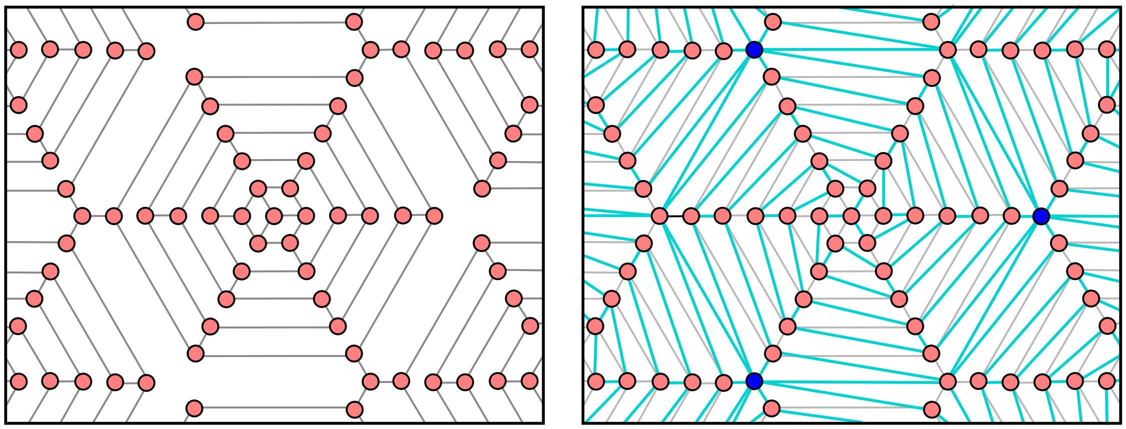

Figure 1: A section of Eppstein's graph G_ℓ (left) and its extension to a 4-connected triangulation (right), extra vertices are shown in blue and additional edges in cyan.

A plane graph on at least 5 vertices and without separating triangles can be extended to a 4-connected triangulation by adding edges and at most one extra vertex, see [3]. Graphs requiring the extra vertex are wheels with an outer cycle of length at least 4 and subgraphs of wheels obtained by removing some cycle edges. These exceptional graphs can be drawn on 3 lines. Plane graphs with at most 4 vertices can be drawn on 2 lines. Therfore, we get:

Corollary 1 *If G is a plane graph on n vertices without separating triangles, then* $\pi(G) \leq \sqrt{2n}$.

The proof of Theorem 1 makes use of transversal structures, these are special colorings of the edges of a 4-connected inner triangulation of a 4-gon with colors red and blue.

In Section 2.1 we survey transversal structures. The red subgraph of a transversal structure can be interpreted as the diagram of a planar lattice. Background on posets and lattices is given in Section 2.2. Dimension of posets and the connection with planarity are covered in Section 2.3. In Section 2.4 we survey orthogonal partitions of posets. The theory implies that every poset on n elements can be covered by at most $\sqrt{2n} - 1$ subsets such that each of the subsets is a chain or an antichain.

In Section 3 we prove that the diagram of a planar lattice on n elements has a straight-line drawing with vertices placed on a set of $\sqrt{2n} - 1$ lines. All the lines used for the construction are either horizontal or vertical.

Finally in Section 4 we prove the main result: transversal structures can be drawn on at most $\sqrt{2n} - 1$ lines. In fact, the red subgraph of the transversal structure has such a drawing by the result of the previous section. It is rather easy to add the blue edges to this drawing and thus prove Theorem 1.

2 Preliminaries

2.1 Transversal structures

Let G be an internally 4-connected inner triangulation of a 4-gon, in other words G is a plane graph with quadrangular outer face, triangular inner faces, and no separating triangle. Let s, a, t, b be the outer vertices of G in clockwise order. A *transversal structure* for G is an orientation and 2-coloring of the inner edges of G such that

(1) All edges incident to s, a, t and b are red outgoing, blue outgoing, red incoming, and blue incoming, respectively.

(2) The edges incident to an inner vertex v come in clockwise order in four non-empty blocks consisting solely of red outgoing, blue outgoing, red incoming, blue incoming edges, respectively.

Figure 2 illustrates the properties and shows an example. Transversal structures have several applications in graph drawing [18], [13], [14]. In particular it has been shown that every internally 4-connected inner triangulation of a 4-gon admits a transversal structure. Fusy [14] used transversal structures to prove the existence of straight-line drawings with vertices being placed on integer points (x, y) with $0 \le x \le W$, $0 \le y \le H$, and $H + W \le n - 1$.

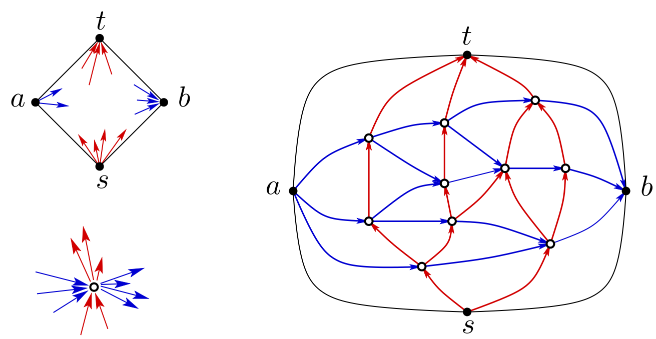

Figure 2: The two local conditions and an example of a transversal structure.

An orientation of a graph G is said to be acyclic if it has no directed cycle. Given an acyclic orientation of G, a vertex having no incoming edge is called a *source*, and a vertex having no outgoing edge is called a *sink*. A *bipolar orientation* is an acyclic orientation with a unique source s and a unique sink t, cf. [6]. Bipolar orientations of plane graphs are also required to have s and t incident to the outer face. A bipolar orientation of a plane graph has the property that at each vertex v the outgoing edges form a contiguous block and the incoming edges form a contiguous block. Moreover, each face f of G has two special vertices s_f and t_f such that the boundary of f consists of two non-empty oriented paths from s_f to t_f.

Let $G = (V, E)$ be an internally 4-connected inner triangulation of a 4-gon with outer vertices s, a, t, b in clockwise order, and let E_R and E_B respectively be the red and blue oriented edges of a transversal structure on G. We define $E_R^+ = E_R \cup \{(s, a), (s, b), (a, t), (b, t)\}$

and $E_B^+ = E_B \cup \{(a,s),(a,t),(s,b),(t,b)\}$, i.e., we think of the outer edges as having both, a red direction and a blue direction. The following has been shown by Kant and He [18] and Fusy [13].

Proposition 1 *The red graph $G_R = (V, E_R^+)$ and the blue graph $G_B = (V, E_B^+)$ both come with a bipolar orientation inherited from the transversal structure. G_R has source s and sink t, and G_B has source a and sink b.*

The following two properties are easy consequences of the previous discussion.

(R) The red and the blue graph are both transitively reduced, i.e., if (v, v') is an edge, then there is no directed path $v, u_1, \ldots, u_k, v'$ with $k \geq 1$.

(F) For every blue edge $e \in E_B$ there is a face f in the red graph such that e has one endpoint on each of the two oriented s_f to t_f paths on the boundary of f.

2.2 Posets

We assume basic familiarity with concepts and terminology for posets, referring the reader to the Trotter's monograph [21] and survey [22] for additional background material. In this paper we consider a poset $P = (X, <)$ as being equipped with a *strict* partial order.

A *cover relation* of P is a pair (x, y) with $x < y$ such that there is no z with $x < z < y$, we write $x \prec y$ to denote a cover relation of the two elements. A *diagram* (a.k.a. Hasse diagram) of a poset is an upward drawing of its transitive reduction. That is, X is represented by a set of points in the plane and a cover relation $x \prec y$ is represented by a y-monotone curve going upwards from x to y. In general these curves (edges) may cross each other but must not touch any vertices other than their endpoints. A diagram uniquely describes a poset, therefore, we usually show diagrams in our figures. A poset is said to be *planar* if it has a planar diagram.

It is well known that in discussions of graph planarity, we can restrict our attention to straight-line drawings. In fact Schnyder [20] proved that a plane graph on n vertices admits a plane straight-line drawing with vertices on an $(n-2) \times (n-2)$ grid. Discussions of planarity for posets can also be restricted to straight-line drawings; however, this may come at some cost in visual clarity. Di Battista et al. [7] have shown that an exponentially large grid may be required for upward planar drawings of directed acyclic planar graphs with straight lines. In the next subsection we will see that for certain planar posets the situation is more favorable.

2.3 Dimension of planar posets

Let $P = (X, <_P)$ be a poset. A *linear extension* of P is a total order $L = (X, <_L)$ extending P, i.e., if $x <_P y$, then $x <_L y$. A *realizer* of P is a collection $L_1, L_2, \ldots, L_t$ of linear extensions of P such that $P = L_1 \cap L_2 \cap \cdots \cap L_t$. The *dimension* of $P = (X, <)$, denoted $\dim(P)$, is the least positive integer t such that P has a realizer of size t. Obviously, a poset P has dimension 1 if and only if it is a chain (total order). Also, there is an elementary characterization of posets of dimension at most 2 that we shall use.

Proposition 2 *A poset $P = (X, <)$ has dimension as most 2 if and only if its incomparability graph is also a comparability graph.*

There are a number of results concerning the dimension of posets with planar order diagrams. Recall that an element is called a *zero* of a poset P when it is the unique minimal element. Dually, a *one* is a unique maximal element. A finite poset which is also a lattice, i.e., which has well defined meet and join operations, always has both a zero and a one. Figure 3 shows some posets and lattices.

The following result may be considered part of the folklore of the subject.

Theorem 2 *Let P be a finite lattice. Then P is planar if and only if it has dimension at most 2.*

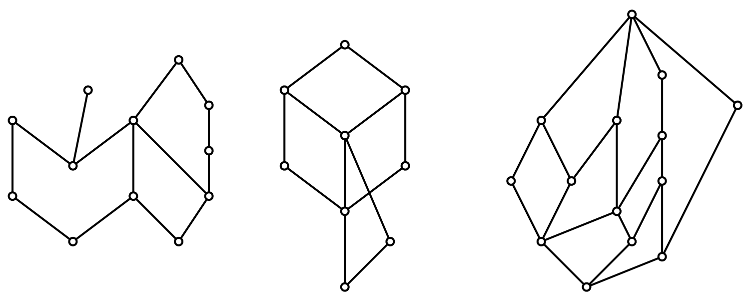

Figure 3: Three posets with their diagrams: (left) a planar poset of dimension 3, (middle) a non-planar lattice, and (right) a planar lattice.

The complete proof of the theorem can be found in Trotter's book [21]. For the proof of the reverse direction let P be a lattice of dimension at most 2. Let L_1 and L_2 be linear orders on X so that $P = L_1 \cap L_2$. For each $x \in X$, and each $i = 1, 2$, let x_i denote the height of x in L_i. Then a planar diagram of P is obtained by locating each $x \in X$ at the point in the plane with integer coordinates (x_1, x_2) and joining points x and y with a straight line segment when one of x and y covers the other in P. Figure 4 shows an example. A pair of crossing edges in this drawing would violate the lattice property, indeed if $x \prec y$ and $x' \prec y'$ are two covers whose edges cross, then $x \leq y'$ and $x' \leq y$ whence there is no unique least upper bound for x and x'.

A planar digraph D with a unique sink and source, both of them on the outer face, and no transitive edges is the digraph of a planar lattice. Hence, the above discussion directly implies the following classical result.

Proposition 3 *A planar digraph D on n vertices with a unique sink and source on the outer face and no transitive edges has an upward drawing on an $(n - 1) \times (n - 1)$ grid.*

To the best of our knowledge the area problem for diagrams of general planar posets is open.

In this paper we will, henceforth, use the terms 2-dimensional poset and planar lattice respectively to refer to a poset $P = (X, <)$ together with a fixed ordered realizer $[L_1, L_2]$. As shown in the proof of Theorem 2, fixing the realizer of a 2-dimensional lattice can be

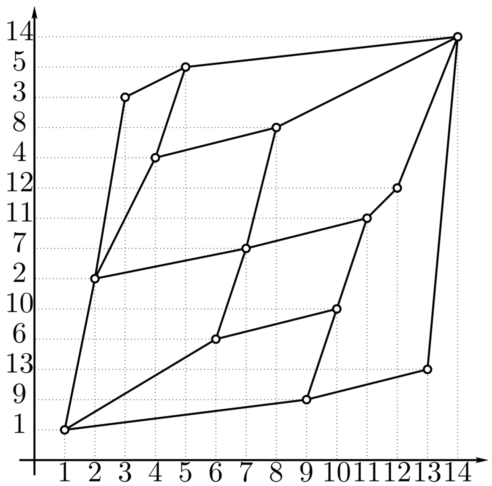

Figure 4: The planar lattice from Fig. 3 with a realizer L_1, L_2.

interpreted as fixing a plane drawing of the diagram. By fixing the realizer of P we also have a well-defined *primary conjugate*, this is the poset Q on X with realizer $[L_1, \overline{L_2}]$, where $\overline{L_2}$ is the reverse of L_2. Define the *left of relation* on X such that x is left of y if and only if $x = y$ or $x \,||\, y$, i.e., x and y are incomparable in P, and $x < y$ in Q. In the drawing of P obtained from $[L_1, L_2]$ element x is left of y if x is in the upper-left quadrant of y.

2.4 Orthogonal partitions of posets

Let $P = (X, <)$ be a finite poset. Dilworth's theorem states that *the maximum size of an antichain equals the minimum number of chains partitioning the elements of P.*

Greene and Kleitman [17] found a nice generalization of Dilworth's result. A *k-antichain* is defined as a family of k pairwise disjoint antichains.

Theorem 3 *For any partially ordered set P and any positive integer k*

$$\max \sum_{A \in \mathcal{A}} |A| = \min \sum_{C \in \mathcal{C}} \min(|C|, k)$$

where the maximum is taken over all k-antichains $\mathcal{A}$ and the minimum over all chain partitions $\mathcal{C}$ of P.

Greene [16] stated the dual of this theorem. Let a *ℓ-chain* be a family of ℓ pairwise disjoint chains.

Theorem 4 *For any partially ordered set P and any positive integer ℓ*

$$\max \sum_{C \in \mathcal{C}} |C| = \min \sum_{A \in \mathcal{A}} \min(|A|, \ell)$$

where the maximum is taken over all ℓ-chains $\mathcal{C}$ and the minimum over all antichain partitions $\mathcal{A}$ of P.

A further theorem of Greene [16] can be interpreted as a generalization of the Robinson-Schensted correspondence and its interpretation given by Greene [15].

To a partially ordered set P with n elements there is an associated integer partition λ of n, such that for the Ferrer's diagram $G(P)$ corresponding to λ we get:

Theorem 5 *The number of squares in the ℓ longest columns of $G(P)$ equals the maximal number of elements covered by an ℓ-chain of P and the number of squares in the k longest rows of $G(P)$ equals the maximal number of elements covered by a k-antichain of P.*

Figure 5 shows an example, in this case the Ferrer's diagram $G(P)$ corresponds to the partition $6{+}3{+}3{+}1{+}1 \models 14$. Several proofs of Greene's results are known [10], [12], [19]. For a not so recent, but at its time comprehensive survey we recommend West [23].

The approach taken by András Frank [12] is particularly elegant. Following Frank we call a chain family $\mathcal{C}$ and an antichain family $\mathcal{A}$ of a poset $P = (X, <)$ an *orthogonal pair* iff

1. $X = \Big(\bigcup_{A\in\mathcal{A}} A\Big) \cup \Big(\bigcup_{C\in\mathcal{C}} C\Big)$, and
2. $|A \cap C| = 1 \quad$ for all $A \in \mathcal{A},\ C \in \mathcal{C}$.

If $\mathcal{C}$ is orthogonal to a k-antichain $\mathcal{A}$ and $\mathcal{C}^+$ is obtained from $\mathcal{C}$ by adding the rest of P as singletons, then

$$\sum_{A\in\mathcal{A}} |A| = \sum_{C\in\mathcal{C}^+} \sum_{A\in\mathcal{A}} |A \cap C| = \sum_{C\in\mathcal{C}^+} \min(|C|, k).$$

Thus $\mathcal{C}^+$ is a k optimal chain partition in the sense of Theorem 3. Similarly an ℓ optimal antichain partition in the sense of Theorem 4 can be obtained from an orthogonal pair $\mathcal{A}, \mathcal{C}$ where $\mathcal{C}$ is an ℓ-chain.

Using the minimum cost flow algorithm of Ford and Fulkerson [11], Frank proved the existence of a sequence of orthogonal chain and antichain families. This sequence is rich enough to allow the derivation of the whole theory. The sequence consists of an orthogonal pair for every point from the boundary of $G(P)$. With the point (ℓ, k) from the boundary of $G(P)$ we get an orthogonal pair $\mathcal{A}, \mathcal{C}$ such that $\mathcal{A}$ is a k-antichain and $\mathcal{C}$ an ℓ-chain, see Figure 5. Since $G(P)$ is the Ferrer's diagram of a partition of n we can find a point (ℓ, k) on the boundary of $G(P)$ with $\ell + k \leq \sqrt{2n} - 1$ (This is because every Ferrer's shape of a partition of m which contains no point (x, y) with $x + y \leq s$ on the boundary contains the shape of the partition $(1, 2, \ldots, s + 1)$. From $m \geq \binom{s+2}{2}$ we get $s + 1 < \sqrt{2m}$).

We will use the following corollary of the theory:

Corollary 2 *Let $P = (X, <)$ be a partial order on n elements, then there is an orthogonal pair $\mathcal{A}, \mathcal{C}$ where $\mathcal{A}$ is a k-antichain and $\mathcal{C}$ an ℓ-chain and $k + \ell \leq \sqrt{2n} - 1$.*

For our application we will need some additional structure on the antichains and chains of an orthogonal pair $\mathcal{A}, \mathcal{C}$.

The *canonical antichain partition* of a poset $P = (X, <)$ is constructed by recursively removing all minimal elements from P and make them one of the antichains of the partition.

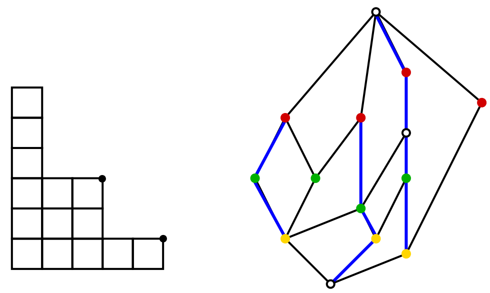

Figure 5: The Ferrer's shape of the lattice L from Fig. 4 together with two orthogonal pairs of L corresponding to the boundary points $(3,3)$ and $(5,1)$ respectively of $G(L)$; chains of $\mathcal{C}$ are blue, antichains of $\mathcal{A}$ are red, green, and yellow.

More explicitely $A_1 = \mathrm{Min}(X)$ and $A_j = \mathrm{Min}\big(X \setminus \bigcup\{A_i : 1 \le i < j\}\big)$ for $j > 1$. Note that by definition for each element $y \in A_j$ with $j > 1$ there is some $x \in A_{j-1}$ with $x < y$. Due to this property there is a chain of h elements in P if the canonical antichain partition consists of h non-empty antichains. This in essence is the dual of Dilworth's theorem, i.e., the statement: *the maximal size of a chain equals the minimal number of antichains partitioning the elements of P.*

Lemma 1 *Let $\mathcal{A}, \mathcal{C}$ be an orthogonal pair of $P = (X, <)$ and let $P_{\mathcal{A}}$ be the order induced by P on the set $X_{\mathcal{A}} = \bigcup\{A : A \in \mathcal{A}\}$. If $\mathcal{A}'$ is the canonical antichain partition of $P_{\mathcal{A}}$, then $|\mathcal{A}| = |\mathcal{A}'|$ and $\mathcal{A}', \mathcal{C}$ is again an orthogonal pair of P.*

Proof. Let $\mathcal{A}$ be the family $A_1, \ldots, A_k$. Starting with this family we will change the antichains in the family while maintaining the invariant that the family of antichains together with $\mathcal{C}$ forms an orthogonal pair. At the end of the process the family of antichains will be the canonical antichain partition of $P_{\mathcal{A}}$.

The first phase of changes is the uncrossing phase. We iteratively choose two antichains A_i, A_j with $i < j$ from the present family and let $B_i = \{y \in A_i :$ there is an $x \in A_j$ with $x < y\}$ and $B_j = \{x \in A_j :$ there is a $y \in A_i$ with $x < y\}$. Define $A'_i = A_i - B_i + B_j$ and $A'_j = A_j - B_j + B_i$. It is easy to see that A'_i and A'_j are antichains and that the family obtained by replacing A_i, A_j by A'_i, A'_j is orthogonal to $\mathcal{C}$. This results in a family of k antichains such that if $i < j$ and $x \in A_i$ and $y \in A_j$ are comparable, then $x < y$.

The second phase is the push-down phase. In this phase we go through $i \in [k-1]$ and let $B = \{y \in A_{i+1} :$ there is no $x \in A_i$ with $x < y\}$. We then define $A'_{i+1} = A_{i+1} - B$ and $A'_i = A_i + B$. It is again easy to see that A'_i and A'_{i+1} are antichains and that the family obtained by replacing A_i, A_{i+1} by A'_i, A'_{i+1} is orthogonal to $\mathcal{C}$. This results in a family of k antichains such that if $y \in A_{i+1}$, then there is an $x \in A_i$ with $x < y$. This implies that $A_j = \mathrm{Min}(X_{\mathcal{A}} \setminus \bigcup\{A_i : 1 \le i < j\})$, hence the family is the canonical antichain partition. □

Let $P = (X, <)$ be a 2-dimensional poset with realizer $[L_1, L_2]$ and recall that the primary conjugate has realizer $[L_1, \overline{L_2}]$. The order Q corresponds to a transitive relation on the complement of the comparability graph of P, in particular chains of P and antichains of Q are in bijection.

The canonical antichain partition of Q yields the *canonical chain partition* of P. The canonical chain partition $C_1, \ldots, C_w$ of P can be characterized by the property that for each $1 \leq i < j \leq w$ and each element $y \in C_j$ there is some $x \in C_i$ with $x \,||\, y$ and in L_1 element x comes before y. In particular C_1 is a maximal chain of P.

Let $\mathcal{A}, \mathcal{C}$ be an orthogonal pair of the 2-dimensional $P = (X, <)$. Applying the proof of Lemma 1 to the orthogonal pair $\mathcal{C}, \mathcal{A}$ of Q we obtain:

Lemma 2 *Let $\mathcal{A}, \mathcal{C}$ be an orthogonal pair of $P = (X, <)$ and let $P_{\mathcal{C}}$ be the order induced by P on the set $X_{\mathcal{C}} = \bigcup\{C : C \in \mathcal{C}\}$. If $\mathcal{C}'$ is the canonical chain partition of $P_{\mathcal{C}}$, then $|\mathcal{C}| = |\mathcal{C}'|$ and $\mathcal{C}', \mathcal{A}$ is again an orthogonal pair of P.*

In a context where edges of the diagram are of interest, it is convenient to work with maximal chains. The canonical chain partition $C_1, \ldots, C_w$ of a 2-dimensional P induces a *canonical chain cover* of P which consists of maximal chains. With chain C_i associate a chain C_i^+ which is obtained by successively adding to C_i all compatible elements of $C_{i-1}, C_{i-2}, \ldots$ in this order. Alternatively C_i^+ can be described by looking at the conjugate Q^d of P with realizer $[\overline{L_1}, L_2]$ (this is the dual of the primary conjugate Q), and defining C_i^+ as the first chain in the canonical chain partition of the order induced by $\bigcup\{C_j : 1 \leq j \leq i\}$), the chain C_i^+ corresponds to the antichain of minimal elements of the order induced by Q^d on $\bigcup\{C_j : 1 \leq j \leq i\}$). The maximality of C_i^+ follows from the characterization of the canonical chain partition given above.

3 Drawing Planar Lattices on Few Lines

In this section we prove that planar lattices with n elements have a straight-line diagram with all vertices on a set of $\sqrt{2n} - 1$ horizontal and vertical lines. The following proposition covers the case where the lattice has an antichain partition of small size. We assume that a planar lattice is given with a realizer $[L_1, L_2]$ and, hence, with a fixed plane drawing of its diagram.

Proposition 4 *For every planar lattice $L = (X, <)$ with a plane diagram D_L and any order preserving map $h : X \to \mathbb{R}$ there is a plane straight-line drawing Γ of D_L such that each element $x \in X$ is represented by a point with y-coordinate $h(x)$. Additionally all elements of the left boundary chain of D_L are aligned vertically in the drawing.*

Proof. Let $C_1, \ldots, C_w$ be the canonical chain partition and $C_1^+, \ldots, C_w^+$ be the corresponding canonical chain cover. Define S_i as the suborder of L induced by $\bigcup\{C_j : 1 \leq j \leq i\}$ and note that S_i is a sublattice of L with left boundary chain $C_1 = C_1^+$ and right boundary chain C_i^+.

Embed the elements of C_1 on a vertical line $\mathbf{g}_1$ (e.g. the line $y = 0$) with points as prescribed by h. This is a drawing Γ_1 of S_1. Suppose that a drawing Γ_i of the diagram S_i is constructed. The right boundary path γ_i of Γ_i is a polygonal y-monotone path. Embed the elements of C_{i+1} on a vertical line $\mathbf{g}_{i+1}$ with points as prescribed by h. We need a position for $\mathbf{g}_{i+1}$ to the right of γ_i such that all the diagram edges connecting C_{i+1} to C_i^+ can be inserted to obtain a crossing free drawing Γ_{i+1} of the diagram of S_{i+1}.

Let E_i be the set of diagram edges connecting C_{i+1} to C_i^+. For each $e \in E_i$ there are points $p \in \gamma_i$ and $q \in \mathbf{g}_{i+1}$ representing the endpoints. Let K_p be an open cone with apex p which intersects γ_i only at p and contains a horizontal ray to the right. Let b_e be the minimal horizontal distance of γ_i and $\mathbf{g}_{i+1}$ such that $q \in K_p$. Let $\beta = \max(b_e : e \in E_i)$. If we place γ_i and $\mathbf{g}_{i+1}$ at horizontal distance β, then the edges of E_i can be drawn such that they do not interfere (introduce crossings) with γ_i. We claim that there is no crossing of edges of E_i. Let $e = (p, q)$ and $e' = (p', q')$ be two drawn edges from E_i. Since they are edges of a planar diagram, they are cover edges and have endpoints on two chains. We can assume that $h(p) < h(p')$ which implies $h(q) < h(q')$. If the two edges cross, their crossing point is in $K_p \cap K_{p'}$. Let c be the leftmost point of $\overline{K_p} \cap \overline{K_{p'}}$ (here we take the closed cones). Note that $h(p) < h(c) < h(p')$. This implies that at the x-coordinate of c edge e' is above e. At the crossing they change their vertical order, whence, to the right of the crossing e is above e'. This implies that $h(q) > h(q')$, a contradiction. Hence we have a planar drawing Γ_{i+1} of the diagram of S_{i+1}. With induction we obtain the drawing $\Gamma = \Gamma_w$ of D_L. $\square$

A particularly interesting order preserving map $h : X \to \mathbb{R}$ is the height function. The height $h(x)$ of an element x of P can be defined via the canonical antichain partition $(A_1, \ldots, A_k)$ of L, namely $h(x) = i$ if and only if $x \in A_i$. Proposition 4 shows that $\pi(L)$ is upper bounded by the height of L, i.e., by the maximum height of an element of L.

Theorem 6 *For every planar lattice $L = (X, <)$ with $|X| = n$, there is a plane straight-line drawing of the diagram such that the elements are represented by points on a set of at most $\sqrt{2n} - 1$ lines. Additionally*

- *each of the lines is either horizontal or vertical,*
- *each crossing point of a horizontal and a vertical line hosts an element of X.*

Proof. Let $\mathcal{A}, \mathcal{C}$ be an orthogonal pair of L such that $\mathcal{A}$ is a k-antichain, $\mathcal{C}$ an ℓ-chain, and $k + \ell \leq \sqrt{2n} - 1$. It follows from Corollary 2 that such a pair exists.

Since L has a fixed ordered realizer $[L_1, L_2]$, we can apply Lemma 1 to $\mathcal{A}$ and Lemma 2 to $\mathcal{C}$ to get an orthogonal pair $(A_1, \ldots, A_k), (C_1, \ldots, C_\ell)$ where the antichain family and the chain family are both canonical. Fix an order preserving map $h : X \to \mathbb{R}$ with the property that $h(x) = i$ for all $x \in A_i$. Such a map exists because the antichain family is canonical, i.e., $i < j$ and $x \in A_i$, $y \in A_j$ implies $x \not< y$.

In the following we will construct a drawing Γ of the diagram D_L of L such that each element $x \in X$ is represented by a point with y-coordinate $h(x)$, and in addition all elements of the chain C_i lie on a common vertical line $\mathbf{g}_i$ for $1 \leq i \leq \ell$. By Property 1 of orthogonal pairs, for each $x \in X$ there is an i such that $x \in A_i$ or a j such that $x \in C_j$ or both. Therfore, Γ will be a drawing such that the k horizontal lines $y = i$ with $i = 1, \ldots, k$ together with the ℓ vertical lines $\mathbf{g}_j$ with $j = 1, \ldots, \ell$ cover all the elements of X. Property 2 of orthogonal pairs implies the second extra property mentioned in the theorem.

If the number ℓ of chains is zero, then k equals the height of L and we get a drawing Γ with all the necessary properties from Proposition 4. Now let $\ell > 0$.

The chain family $C_1, \ldots, C_\ell$ is the canonical chain partition of the order induced on the set $X_{\mathcal{C}} = \bigcup\{C_i : i = 1 \ldots \ell\}$. Let $C_1^+, \ldots, C_\ell^+$ be the corresponding canonical chain covering of $X_{\mathcal{C}}$.

Let X_i for $1 \leq i \leq \ell$ be the set of all elements which are left of some element of C_i^+ in L, and let $X_{\ell+1} = X$. Since C_i^+ is a maximal chain it corresponds to a path from the zero element to to the one element in the diagram of the lattice, elements to the left of this path are in X_i which is defined via the more abstract 'left of' relation. Define S_i as the suborder of L induced by X_i. Also let $Y_i = X_{i+1} - X_i + C_i^+$ and let T_i be the suborder of L induced by Y_i. Note the following properties of these sets and orders:

- $X_i \cap C_j = \emptyset$ for $1 \leq i < j \leq \ell$.
- Each S_i is a planar sublattice of L, its right boundary chain is C_i^+.
- T_i is a planar sublattice of L, its left and right boundary chains are C_i^+ and C_{i+1}^+, respectively.

A drawing Γ_1 of S_1 with the right boundary chain being aligned vertically is obtained by applying Proposition 4 a reflection with a vertical axis to the diagram $D_L[X_1]$ and reflecting the resulting drawing again with a vertical axis.

We construct the drawing Γ of D_L in phases. In phase i we aim for a drawing Γ_{i+1} of S_{i+1} extending the given drawing Γ_i of S_i, i.e., we need to construct a drawing Λ_i of T_i such that

(1) The left boundary chain of Λ_i matches the right boundary chain of Γ_i.

(2) In Λ_i all elements of C_{i+1} are on a common vertical line $\mathbf{g}_{i+1}$.

The construction of Λ_i is done in three stages. First we extend C_i^+ to the right by adding 'ears'. Then we extend C_{i+1}^+ to the left by adding 'ears'. Finally we show that the left and the right part obtained from the first two stages can be combined to yield the drawing Λ_i.

To avoid extensive use of indices let $Y = Y_i$, $T = T_i$, $C^+ = C_i^+$, and let γ be a copy of the y-monotone polygonal right boundary of Γ_i, i.e., γ is a drawing of C. We initialize $\Lambda' = \gamma$.

A *left ear* of T is a face F in the diagram $D_L[Y]$ of T such that the left boundary of F is a subchain of the left boundary chain C^+ of $D_L[Y]$. The ear is *feasible* if the right boundary chain contains no element of C_{i+1}. Given a feasible ear we use the method from the proof of Proposition 4 to add F to γ. We represent the right boundary $z_0 < z_1 < \ldots < z_l$ excluding z_0 and z_l of F on a vertical line $\mathbf{g}$ by points $q_1, \ldots, q_{l-1}$ with y-coordinates as prescribed by h. The points q_0 and q_l representing z_0 and z_l respectively are already represented on γ. Then we place $\mathbf{g}$ at some distance β to the right of γ. The value of β has to be chosen large enough to ensure that edges (q_0, q_1) and (q_{l-1}, q_l) are drawn such that they do not interfere with γ. Let Λ' be the drawing augmented by the polygonal path $q_0, q_1, \ldots, q_{l-1}, q_l$ and let C^+ again refer to the right boundary chain γ of Λ'. Delete all elements of the left boundary of F except z_0 and z_l from Y and T. This *shelling of a left ear* from T is iterated until there remains no left feasible ear. Upon stopping we have a drawing Λ' which can be glued to the right side of Γ_i. Let γ' be the right boundary chain of Λ'.

Now let $C = C_{i+1}$. Initialize a new drawing Λ'' by placing the elements of C on a vertical line **g** at the heights prescribed by h and connect consecutive ones by an edge whenever the order relation is indeed a cover relation of L. The initial drawing may thus be disconnected and if so this will remain the case throughout this stage. We now consider right ears from T. A right ear of T corresponding to a face F is feasible if the left boundary chain of F contains no element of γ'. The left boundary chain of a feasible ear can be drawn as a y-monotone polygonal chain left of the left boundary γ'' of Λ''. Update γ'' to be the new left boundary of the augmented Λ'' and remove the elements of the ear from Y and T. The shelling of right ears from T is iterated until there remains no feasible right ear. Note that γ'' is y-monotone but it may consist of several components.

In the final stage we have to combine the drawings Λ', Λ'' into a single drawing. This is done by drawing the edges and chains which remain in T between the two boundary chains as straight segments between γ' and γ''. This will be possible because we can shift γ' and γ'' as far apart horizontally as necessary. Figure 6 gives an example.

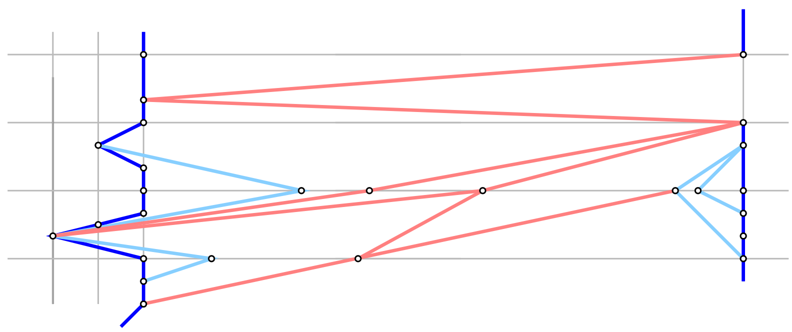

Figure 6: After having added ears (light blue) to the chains C_i and C_{i+1} (dark blue) the connecting edges and components are put between them (pink).

First we draw all the edges connecting the two chains. Let E be the set of edges connecting the left and right boundary chains of T. For each $e \in E$ the endpoints are represented by points $p \in \gamma'$ and $q \in \gamma''$. Let K_p be an open cone with apex p which intersects γ' only at p and contains a horizontal ray to the right and let K_q be an open cone with apex q which intersects γ'' only at q and contains a horizontal ray to the left. Let b_e be the minimal horizontal distance of γ' and γ'' such that $p \in K_q$ and $q \in K_p$. Let $\beta = \max(b_e : e \in E)$. If we place γ' and γ'' at horizontal distance β, then the edges of E can be drawn such that they do not interfere (introduce crossings) with γ' and γ''. We claim that there is no crossing of edges of E. Let $e = (p, q)$ and $e' = (p', q')$ be two drawn edges from E_i. Since they are edges of a planar diagram, they are cover edges and have endpoints on two chains. We can assume that $h(p) < h(p')$ which implies $h(q) < h(q')$. If the two edges cross, their crossing point is in $K_p \cap K_{p'}$. Let c be the leftmost point of $\overline{K_p} \cap \overline{K_{p'}}$ and note that $h(p) < h(c) < h(p')$. This implies that at the x-coordinate of c edge e' is above e. The crossing point is also contained in $K_q \cap K_{q'}$. Let b be the rightmost point of $\overline{K_q} \cap \overline{K_{q'}}$ and note that $h(q) < h(b) < h(q')$. This implies that at the x-coordinate of b edge e' is above e. It follows that edge e' is above e in the vertical strip defined by a and b, whence, e and e' are not crossing.

Placing Λ' and Λ'' such that β is the distance between their outer chains and drawing the edges of E yields a drawing Λ of a lattice. An important feature of the drawing is that if we move the two subdrawings Λ' and Λ'' further apart the drawing keeps the needed properties, i.e., the height of elements remains unaltered, vertices of a chain which should be vertically aligned remain vertically aligned, and the drawing is crossing-free.

Now assume that T contains elements which are not represented in Λ. Let B be a connected component of such elements where connectivity is with respect to D_L. All the elements of B have to be placed in a face F_B of Λ. Let δ' and δ'' be the left and right boundary path of F_B.

In the following we repeatedly select a component B and a chain C from B which is to be drawn in the corresponding face F_B of Λ such that the minimum and the maximum of C have connecting edges to the two sides of the boundary of F_B. Let us consider the case that in D_L the maximum of C has an outgoing edge to an element which is represented by a point $p \in \delta'$ and the minimum of C has an incoming edge from an element represented by $q \in \delta''$. We represent the elements of C as points on the prescribed heights on a line segment ζ with endpoints p and q. It may become necessary to stretch the face horizontally to be able to place C. In this case we stretch the whole drawing between γ' and γ'' with a uniform stretch factor. There may be additional edges between elements of C and elements on δ' and δ''. They can also be drawn without crossing when the distance of δ' and δ'' exceeds some value b.

Stretching the whole drawing between γ' and γ'' allows us to draw the segment ζ and additional edges inside of F_B because of the following invariant.

- For each face F of the drawing Λ and two points x and y from the boundary of F it holds that: if the segment x, y is not in the interior of F, then the parts of the boundary obstructing the segment x, y belong to γ' or γ''.

When including a chain C in the drawing Λ, we place the elements of C at the prescribed heights on a common line segment ζ. This ensures that each new element contributes convex corners in all incident faces. Hence, new elements can not obstruct a visibility within a face. Therefore, obstructing corners correspond to elements of γ' or γ'' and the invariant holds.

The case where the left side of F connects to the minimum and the right side to the maximum of C is symmetric to the previous. Now consider the case where maximum and minimum of the chain C connect to two elements p and q on the same side of F. Since γ' and γ'' do not admit ear extensions we know that not both of p and q belong to one of γ' and γ''. If the segment from p to q is obstructed, then the invariant ensures that with sufficient horizontal stretch the segment ζ connecting p and q will be inside F. Hence, chain C can be drawn and Λ can be extended.

When there remains no component B containing a chain C which can be included in the drawing using the above strategy, then either all elements of Y are drawn or we have the following: every component B only connects to elements of a line segment ζ_B.

In this situation B is kind of a big ear over ζ_B. We next describe how to draw B, but note, that doing this we will not maintain or need the invariant.

By construction all elements of ζ_B belong to a common chain C_B. Consider the union $B + C_B$ and note that this is a planar lattice L_B, moreover, C_B is either the left or the right boundary chain of L_B. Assume that C_B is the left boundary chain of L_B, the other case is symmetric. Now use Proposition 4 to get a drawing Λ_B of L_B with C_B aligned vertically. Using an affine transformation we can map Λ_B into Λ such that the line containing C_B in Λ_B is mapped to the line supporting the segment ζ_B. Since the elements of C_B are at their prescribed heights, their representing points in Λ_B are mapped to the representing points of Λ. The affine map also has to compress Λ_B horizontally so that it is placed in a narrow strip on the right side of ζ_B. This strip can be chosen narrow enough to make sure that all of B is mapped to the face of Λ where it belongs.

The constructed drawing Λ is a drawing Λ_i of T_i. Glueing Λ_i to Γ_i yields a drawing Γ_{i+1} of S_{i+1}. Eventually the drawing Γ_ℓ will be constructed. From there the drawing $\Gamma = \Gamma_{\ell+1}$ is obtained by adding some left ears. □

4 Transversal Structures on Few Lines

Theorem 7 *For every internally 4-connected inner triangulation of a 4-gon $G = (V, E)$ with n vertices there is a planar straight line drawing such that the vertices are represented by points on a set of at most $\sqrt{2n} - 1$ lines. Additionally*

- *each of the lines is either horizontal or vertical,*
- *each crossing point of a horizontal and a vertical line hosts a vertex.*

Proof. Fix a transversal structure of G and consider the red graph $G_R = (V, E_R^+)$. From Proposition 1 and (R) we know that G_R is bipolar and transitively reduced. This implies that there is a planar lattice $L = (V, <)$ such that a diagram of L is an upward drawing of G_R. The relation $<$ is defined as $v < v'$ if and only if there is a directed path from v to v' in G_R.

We would like to use Theorem 6 to draw G_R on $\sqrt{2n} - 1$ lines and then include the blue edges of the transversal structure in the drawing. This, however, may yield crossings. The good news is that by property (F) every blue edge connects the left and right side of a red face. This suggests that when, in the construction given in the proof of Theorem 6 a red face is closed, we can look at the blue edges and adapt the distance between the sides of the face to include the blue edges crossing free in the drawing.

To see that this is indeed possible we have to go through the cases of the proof. When adding a left feasible ear, i.e., when adding the right boundary of a face F, we draw all the blue edges corresponding to the face F. If there is an edge e from $p \in \gamma$ to $q \in \mathbf{g}$ define b_e as the minimal horizontal distance of γ and $\mathbf{g}$ such that $q \in K_p$. When placing $\mathbf{g}$ at a distance β from γ which exceeds all the values b_e, the blue edges can be drawn crossing free. When adding a right feasible ear the situation is symmetric.

Now let us consider the stage where a left and right drawing Λ' and Λ'' with boundary chains γ' and γ'' have to be combined. When drawing edges connecting the two chains we include all red and all blue edges with one end on γ' and one on γ'' and adapt the distance of the two chains accordingly. Then we continue the combination on the basis of the red edges. Only in the 'bad' case where a component B is added to a line segment ζ_B we have to be careful. First, when drawing L_B using Proposition 4 we also include the blue edges in the drawing. This only requires to choose the distances β as maxima over larger sets of values b_e. Second, when placing the drawing Λ_B in a narrow strip on the side of ζ_B we have to be careful that this does not obstruct a visibility from the left side of the face to the right side. Finally, all the remaining blue edges have to be drawn in the faces between γ' and γ''. Due to the invariant this is possible if we stretch the drawing between the two chains sufficiently. Figure 7 illustrates an intermediate step of such a drawing procedure. $\square$

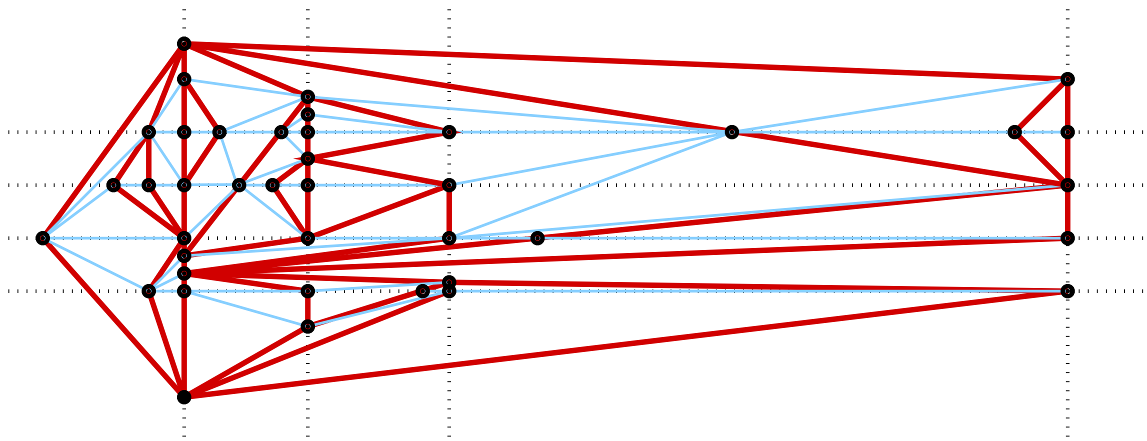

Figure 7: A partially drawn transversal structure. The figure shows a drawing of Γ_4, these are the vertices left of some element in C_4^+ together with the induced edges.

It remains to see how Theorem 1 follows from Theorem 7. Let G be a 4-connected triangulation and let G' be obtained from G by deleting one of the outer edges. Now G' is an internally 4-connected inner triangulation of a 4-gon. Label the outer vertices of G' such that the deleted edge is the edge s, t. Slightly stretching Theorem 7 we prescribe $h(s) = -\infty$ and $h(t) = \infty$, this yields a planar straight-line drawing Γ of G' such that the vertices except s and t are represented by points on a set of at most $\sqrt{2n} - 1$ lines and the edges connecting to s and t are vertical rays. Moreover with every edge v, s or v, t there is an open cone K containing the vertical ray, such that the point representing v is the apex of K and this is the only vertex contained in K. Now let $\mathbf{g}$ be a vertical line which is disjoint from Γ. On $\mathbf{g}$ we find a point p_s which is contained in all the upward cones and a point p_t contained in all the downward cones. Taking p_s and p_t as representatives for s and t we can tilt the rays and make them finite edges ending in p_s and p_t respectively, and in addition draw the edge p_s, p_t.

We conclude with a remark and two open problems.

- Our results are constructive and can be complemented with algorithms running in polynomial time.
- Is $\pi(G) \in O(\sqrt{n})$ for every planar graph G on n vertices?
- What size of a grid is needed for drawings of 4-connected plane graphs on $O(\sqrt{n})$ lines?

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