# On Computing the Hamiltonian Index of Graphs 

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#### Abstract

For an integer $r \geqslant 0$ the $r$-th iterated line $\operatorname{graph} L^{r}(G)$ of a graph $G$ is defined by: (i) $L^{0}(G)=G$ and (ii) $L^{r}(G)=L\left(L^{(r-1)}(G)\right)$ for $r>0$, where $L(G)$ denotes the line graph of $G$. The Hamiltonian Index $h(G)$ of $G$ is the smallest $r$ such that $L^{r}(G)$ has a Hamiltonian cycle [Chartrand, 1968]. Checking if $h(G)=k$ is NP-hard for any fixed integer $k \geqslant 0$ even for subcubic graphs $G$ [Ryjáček et al., 2011]. We study the parameterized complexity of this problem with the parameter treewidth, $\operatorname{tw}(\mathrm{G})$, and show that we can find $h(G)$ in $\operatorname{time}^{1} \mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{t w(G)}\right)$ where $\omega$ is the matrix multiplication exponent. Prior work on computing $h(G)$ includes various $\mathcal{O}^{\star}\left(2^{\mathcal{O}(t w(G))}\right)$-time algorithms for checking if $h(G)=0$ holds; i.e., whether $G$ has a Hamiltonian Cycle [Cygan et al., FOCS 2011; Bodlaender et al., Inform. Comput., 2015; Fomin et al., JACM 2016]; an $\mathcal{O}^{\star}\left(\mathrm{tw}(\mathrm{G})^{\mathcal{O}(\mathrm{tw}(\mathrm{G}))}\right)$-time algorithm for checking if $h(G)=1$ holds; i.e., whether $L(G)$ has a Hamiltonian Cycle [Lampis et al., Discrete Appl. Math., 2017]; and, most recently, an $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}(\mathrm{G})}\right)$-time algorithm for checking if $h(G)=1$ holds [Misra et al., CSR 2019]. Our algorithm for computing h(G) generalizes these results.


The NP-hard Eulerian Steiner Subgraph problem takes as input a graph G and a specified subset $K$ of terminal vertices of $G$ and asks if $G$ has an Eulerian ${ }^{2}$ subgraph $H$ containing all the terminals. A key ingredient of our algorithm for finding $h(G)$ is an algorithm which solves Eulerian Steiner Subgraph in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\operatorname{tw}(G)}\right)$ time. To the best of our knowledge this is the first FPT algorithm for Eulerian Steiner Subgraph. Prior work on the special case of finding a spanning Eulerian subgraph (i.e., with $\mathrm{K}=\mathrm{V}(\mathrm{G})$ ) includes a polynomial-time algorithm for series-parallel graphs [Richey et al., 1985] and an $\mathcal{O}^{\star}\left(2^{\mathcal{O}(\sqrt{n})}\right)$-time algorithm for planar graphs on $n$ vertices [Sau and Thilikos, 2010]. Our algorithm for Eulerian Steiner Subgraph generalizes both these results.

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## 1 Introduction

All graphs in this article are finite and undirected, and are without self-loops or multiple edges unless explicitly stated. We use $\mathbb{N}$ to denote the set of non-negative integers, and $\mathrm{V}(\mathrm{G}), \mathrm{E}(\mathrm{G})$, respectively, to denote the vertex and edge sets of graph G. A graph is Eulerian if it has a closed Eulerian trail, and Hamiltonian if it has a Hamiltonian cycl $\AA^{3}$ The vertex set of the line graph of a graph $G$-denoted $L(G)$-is the edge set $E(G)$ of $G$, and two vertices $e, f$ are adjacent in $L(G)$ if and only if the edges $e$ and $f$ share a vertex in $G$. A graph $H$ is said to be a line graph if there exists a graph $G$ such that $\mathrm{L}(\mathrm{G})=\mathrm{H}$. Line graphs are an extremely well studied class of graphs; we recall a few well-known properties (See, e.g.: [1, 7, 8, 33, 40]). The line graph operation is almost injective: if H is a line graph then there is a unique graph G such that $L(G)=H$, except when $H$ is the triangle $C_{3}$, in which case $G$ can either be $C_{3}$ or the star $K_{1,3}$ with three leaves. A graph $G$ is connected if and only if its line graph $L(G)$ is connected ${ }^{4}$ Let $P_{\ell}$ (respectively, $C_{\ell}$ ) denote the path (resp. cycle) with $\ell$ edges. Then for any $\ell \geqslant 1$ we have $L\left(P_{\ell}\right)=P_{\ell-1}$, and for any $\ell \geqslant 3$ we have $L\left(C_{\ell}\right)=C_{\ell}$. More generally, for any connected graph G which is not a path we have that $\mathrm{L}(\mathrm{G})$ is connected and has at least as many edges as G itself. This implies that starting with a non-empty connected graph G which is not a path and repeatedly applying the line graph operation will never lead to the empty graph. More precisely: Let r be a non-negative integer. The $r$-th iterated line graph $L^{r}(G)$ of $G$ is defined by: (i) $L^{0}(G)=G$, and (ii) $L^{r}(G)=L\left(L^{(r-1)}(G)\right.$ ) for $r>0$. If $G=P_{\ell}$ for a non-negative integer $\ell$ then $L^{\ell}(G)$ is $K_{1}$, the graph with one vertex and no edges, and $L^{r}(G)$ is the empty graph for all $r>\ell$. If $G$ is a connected graph which is not a path then $\mathrm{L}^{\mathrm{r}}(\mathrm{G})$ is nonempty for all $\mathrm{r} \geqslant 0[7$.

It was noticed early on that the operation of taking line graphs has interesting effects on the properties of the (line) graph being Eulerian or Hamiltonian. For instance, Chartrand 8 observed that: (i) if $G$ is Eulerian, then $L(G)$ is Eulerian; (ii) if $G$ is Eulerian, then $L(G)$ is Hamiltonian; and (iii) if G is Hamiltonian, then $\mathrm{L}(\mathrm{G})$ is Hamiltonian, and that the converse does not (always) hold in each case. Another example, again due to Chartrand [9]: If G is a connected graph which is not a path then exactly one of the following holds: (i) G is Eulerian, (ii) G is not Eulerian, but $L(G)$ is Eulerian, (iii) neither $G$ nor $L(G)$ is Eulerian, but $L^{2}(G)$ is Eulerian, and (iv) there is no integer $r \geqslant 0$ such that $L^{r}(G)$ is Eulerian. For a third example we look at two characterizations of graphs $G$ whose line graphs $\mathrm{L}(\mathrm{G})$ are Hamiltonian. An edge Hamiltonian path of a graph $G$ is any permutation $\Pi$ of the edge set $E(G)$ of $G$ such that every pair of consecutive edges in $\Pi$ has a vertex in common, and an edge Hamiltonian cycle of G is an edge Hamiltonian path of G in which the first and last edges also have a vertex in common.

- Theorem 1. The following are equivalent for a graph G:
- Its line graph $\mathrm{L}(\mathrm{G})$ is Hamiltonian
- G has an edge Hamiltonian cycle 9]
- G contains a closed trail T such that every edge in G has at least one end-point in $\mathrm{T}[20$ Given these results a natural question would be: what are the graphs $G$ such that $L^{r}(G)$ is Hamiltonian for some integer $\mathrm{r} \geqslant 0$ ? Chartrand found the - perhaps surprising-answer: all of them except for the obvious discards.
- Theorem 2. 9] If G is a connected graph on n vertices which is not a path, then $\mathrm{L}^{\mathrm{r}}(\mathrm{G})$ is Hamiltonian for all integers $\mathrm{r} \geqslant(\mathrm{n}-3)$.

[^1]This led Chartrand to define the Hamiltonian Index $h(G)$ of a connected graph $G$ which is not a path, to be the smallest non-negative integer $r$ such that $L^{r}(G)$ is Hamiltonian (9]. The Hamiltonian Index of graphs has since received a lot of attention from graph theorists, and a number of interesting results, especially on upper and lower bounds, are known. An early result by Chartrand and Wall [10], for instance, states that if the minimum degree of a graph $G$ is at least three then $h(G) \leqslant 2$ holds. See the references for a number of other interesting graph-theoretic results on the Hamiltonian Index $[7,9,10,19,26,42,44]$.

We now move on to the algorithmic question of computing $h(G)$, which is the main focus of this work. Checking if $h(G)=0$ holds is the same as checking if graph $G$ is Hamiltonian. This is long known to be NP-complete, even when the input graph is planar and has maximum degree at most 3 [18]. Checking if $h(G)=1$ holds is the same as checking if (i) $G$ is not Hamiltonian, and (ii) the line graph $\mathrm{L}(\mathrm{G})$ is Hamiltonian. Bertossi [2] showed that the latter problem is NP-complete, and Ryjáček et al. proved that this holds even if graph $G$ has maximum degree at most 3 37. Xiong and Liu 44 described a polynomial-time procedure which took a graph $G$ with $h(G) \geqslant 4$ as input, and output a graph $G^{\prime}$ such that $h(G)=h\left(G^{\prime}\right)+1$ holds. They conjectured that given an input graph $G$ with the guarantee that $h(G) \geqslant 2$ holds, it should be possible to compute $h(G)$ in polynomial time, since by their procedure it suffices to (eventually) check whether the index is 2 or 3 . Ryjáček et al. disproved this conjecture [37]; they showed that checking whether $h(G)=t$ is NP-complete for any fixed integer $t \geqslant 0$, even when the input graph $G$ has maximum degree at most 3 .

Our problems and results. In this work we take up the parameterized complexity analysis of the problem of computing the Hamiltonian Index. Briefly put, an instance of a parameterized problem is a pair ( $\mathrm{x}, \mathrm{k}$ ) where x is an instance of a classical problem and k is a (usually numerical) parameter which captures some aspect of $x$. A primary goal is to find a fixedparameter tractable (or FPT) algorithm for the problem, one which solves the instance in time $\mathcal{O}\left(f(k) \cdot|x|^{c}\right)$ where $f()$ is a function of the parameter $k$ alone, and $c$ is a constant independent of $x$ and $k$; this running time is abbreviated as $\mathcal{O}^{\star}(f(k))$. The design of FPT algorithms is a vibrant field of research; we refer the interested reader to standard textbooks 12,14 .

Since checking whether $h(G)=t$ is NP-complete for any fixed $t \geqslant 0$, the value $h(G)$ is not a sensible parameter for this problem. Indeed, if computing $h(G)$ were fixed-parameter tractable with $h(G)$ as the parameter then we could, for instance, check whether any graph $G$ is Hamiltonian $(h(G)=0)$ in polynomial time, which in turn would imply $P=N P$. $A$ similar comment applies to the maximum (or average) degree of the input graph, since the problem is NP-complete already for graphs of maximum degree 3. We choose the treewidth ${ }^{5}$ of the input graph G as our parameter. This is motivated by prior related work as well, as we describe below. Thus the main problem which we take up in this work is

Hamiltonian Index (HI) Parameter: tw
Input: A connected undirected graph $G=(\mathrm{V}, \mathrm{E})$ which is not a path, a tree decomposition $\mathcal{T}=\left(T,\left\{X_{t}\right\}_{t \in V(T)}\right)$ of $G$ of width $t w$, and $r \in \mathbb{N}$.
Question: Is $h(G) \leqslant r$ ?
Our main result is that this problem is fixed-parameter tractable. $\omega$ denotes the matrix multiplication exponent; it is known that $\omega<2.3727$ holds 41].

- Theorem 3. There is an algorithm which solves an instance ( $\mathcal{G}, \mathcal{T}, \mathrm{t} w, r$ ) of Hamiltonian Index in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw} w}\right)$ time.

[^2]From this and Theorem 2 we get

- Corollary 4. There is an algorithm which takes as input a graph G and a tree decomposition $\mathcal{T}$ of width tw of G as input, and outputs the Hamiltonian Index $\mathrm{h}(\mathrm{G})$ of G in $\mathcal{O}^{\star}((1+$ $\left.2^{(\omega+3)}\right)^{\mathrm{t} w}$ ) time.

We now describe a key ingredient of our solution which we believe to be interesting in its own right. The input to a Steiner subgraph problem consists of a graph G and a specified set K of terminal vertices of G , and the objective is to find a subgraph of $G$ which (i) contains all the terminals, and (ii) satisfies some other specified set of constraints, usually including connectivity constraints on the set K . The archetypal example is the Steiner Tree problem where the goal is to find a connected subgraph of G of the smallest size (number of edges) which contains all the terminals. Note that such a subgraph-if it exists-will be a tree, called a Steiner tree for the terminal set K. The non-terminal vertices in a Steiner tree, which are included for providing connectivity at small cost for the terminals, are called Steiner vertices. Steiner Tree and a number of its variants have been the subject of extensive research from graph-theoretical, algorithmic, theoretical, and applied points of view [3, 11, 15, 21, 23, 34.

A key part of our algorithm for computing $h(G)$ consists of solving:
Eulerian Steiner Subgraph (ESS)
Parameter: $t w$
Input: An undirected graph $G=(V, E)$, a set of "terminal" vertices $K \subseteq V$, and a tree decomposition $\mathcal{T}=\left(T,\left\{X_{t}\right\}_{t \in V(T)}\right)$ of $G$, of width $t w$.
Question: Does there exist an Eulerian subgraph $\mathrm{G}^{\prime}=\left(\mathrm{V}^{\prime}, \mathrm{E}^{\prime}\right)$ of G such that $\mathrm{K} \subseteq \mathrm{V}^{\prime}$ ?
We call such a subgraph $\mathrm{G}^{\prime}$ an Eulerian Steiner subgraph of G for the terminal set K .

- Theorem 5. There is an algorithm which solves an instance (G, K, $\mathcal{T}, \mathrm{tw}$ ) of Eulerian Steiner Subgraph in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\text {tw }}\right)$ time.

Related work. The parameterized complexity of computing $h(G)$ per se has not, to the best of our knowledge, been previously explored. The two special cases of checking if $h(G) \in\{0,1\}$ have been studied with the treewidth tw of the input graph G as the parameter; we now summarize the main existing results. Checking whether $h(G)=0$ holds-that is, whether $G$ is Hamiltonian - was long known to be solvable in $\mathcal{O}^{\star}\left(\mathrm{t} \mathcal{w}^{\mathcal{O}(\mathrm{tw})}\right)$ time. This was suspected to be the best possible till, in a breakthrough result in 2011, Cygan et al. 13] showed that this can be done in randomized $\mathcal{O}^{\star}\left(4^{\mathrm{tw}}\right)$ time. More recently, Bodlaender et al. [5] and Fomin et al. 17] showed, independently and using different techniques, that this can be done in deterministic $\mathcal{O}^{\star}\left(2^{\mathcal{O}(\mathrm{t} w)}\right)$ time.

Recall that a vertex cover of graph $G$ is any subset $S \subseteq V(G)$ such that every edge in $E(G)$ has at least one of its two endpoints in the set $S$. A subgraph $G^{\prime}$ of a graph $G$ is said to be a dominating Eulerian subgraph of $G$ if (i) $\mathrm{G}^{\prime}$ is Eulerian, and (ii) $\mathrm{V}\left(\mathrm{G}^{\prime}\right)$ contains a vertex cover of G. Note that - in conformance with the literature (e.g. [16, 29, 31, 39]) on this subject - the word "dominating" here denotes the existence of a vertex cover, and not of a dominating set. The input to the Dominating Eulerian Subgraph (DES) problem consists of a graph $G$ and a tree decomposition $\mathcal{T}$ of $G$ of width $t w$, and the question is whether G has a dominating Eulerian subgraph; the parameter is $t w$. The input to the Edge Hamiltonian Path (EHP) (respectively, Edge Hamiltonian Cycle (EHC)) problem consists of a graph $G$ and a tree decomposition $\mathcal{T}$ of $G$ of width $t w$, and the question is whether G has an edge Hamiltonian path (resp. cycle); the parameter is tw. Observe that a closed trail in graph G is an Eulerian subgraph of G. So Theorem 1 tells us that EHC is equivalent to DES.

The parameterized complexity of checking whether $h(G)=1$ holds was first taken up by Lampis et al. in 2014 [28, 29], albeit indirectly: they addressed EHC and EHP. They showed that EHP is FPT if and only if EHC is FPT, and that these problems (and hence DES) can be solved in $\mathcal{O}^{\star}\left(\mathrm{t} \boldsymbol{w}^{\mathcal{O}(\mathrm{t} w)}\right)$ time. Very recently Misra et al. 32 investigated an optimization variant of Edge Hamiltonian Path which they called Longest Edge-Linked Path (LELP). An edge-linked path is a sequence of edges in which every consecutive pair has a vertex in common. Given a graph $G, k \in \mathbb{N}$, and a tree decomposition $\mathcal{T}$ of $G$ of width $t w$ as input the LELP problem asks whether $G$ has an edge-linked path of length at least $k$. Note that setting $k=|E(G)|$ yields EHP as a special case. Misra et al. 32] gave an algorithm which solves LELP (and hence, EHP, EHC and DES) in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw} w}\right)$ time. This gives the current best algorithm ${ }^{6}$ for checking if $h(G)=1$ holds.

- Theorem 6. 32 There is an algorithm which solves an instance (G, T, tw) of Edge Hamiltonian Path (respectively, Edge Hamiltonian Cycle or Dominating Eulerian Subgraph ) in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}}\right)$ time.

To the best of our knowledge, ours is the first FPT algorithm for Eulerian Steiner Subgraph. A subgraph $H$ of a graph $G$ is a spanning subgraph of $G$ if $H$ contains every vertex of $G$. A graph $G$ is supereulerian if it has a spanning subgraph H which is Eulerian. We could not find references to the Eulerian Steiner Subgraph problem in the literature, but we did find quite a bit of existing work on the special case - obtained by setting $K=V(G)$ of checking if an input graph G is supereulerian [6, 27]. Pulleyblank observed already in 1979 that this latter problem is NP-complete even on graphs of maximum degree at most 3 [35]. This implies that Eulerian Steiner Subgraph is NP-complete as well. Richey et al. 36 showed in 1985 that the problem can be solved in polynomial time on series-parallel graphs. More recently, Sau and Thilikos showed in 2010 that the problem can be solved in $\mathcal{O}^{\star}\left(2^{\mathcal{O}(\sqrt{n})}\right)$ time on planar graphs with $\mathfrak{n}$ vertices [38]. Now consider the following parameterization:

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Spanning Eulerian Subgraph (SES) Parameter: tw
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Input: An undirected graph $G=(\mathrm{V}, \mathrm{E})$ and a tree decomposition $\mathcal{T}=\left(\mathrm{T},\left\{\mathrm{X}_{\mathrm{t}}\right\}_{\mathrm{t} \in \mathrm{V}(\mathrm{T})}\right)$ of
$G$, of width $t w$.
Question: Does G have a spanning Eulerian subgraph?
Setting $K=V(G)$ in Theorem 5 we get

- Corollary 7. There is an algorithm which solves an instance ( $\mathrm{G}, \mathcal{T}, \mathrm{tw}$ ) of Spanning Eulerian Subgraph in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{t} w}\right)$ time.

It is known that series-parallel graphs have treewidth at most 2 and are planar, and that planar graphs on $n$ vertices have treewidth $\mathcal{O}(\sqrt{n})[4$. Further, given a planar graph $G$ of treewidth $t$ we can, in polynomial time, output a tree decomposition of $G$ of width $\mathcal{O}(t)$ [24]. These facts together with Corollary 7 subsume the results of Richey et al. and Sau and Thilikos, respectively.

Organization of the rest of the paper. In Section 2 we collect together various definitions, observations and preliminary results which we use in the rest of the work. We prove Theorem 5 in Section 3 and our main result Theorem 3 in Section 4. We conclude in Section 5. An alternate proof of Theorem 6 is in Appendix A

[^3]
## 2 Preliminaries

We use $\operatorname{deg}_{\mathrm{G}}(v)$ to denote the degree of vertex $v$ in graph G. The union of graphs G and $H$, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For
 X. A walk in a graph $G$ is a sequence $\left(v_{0}, e_{1}, v_{1}, e_{2}, \ldots, e_{\ell}, v_{\ell}\right)$ of vertices $v_{i}$ and edges $e_{j}$ of $G$ such that for each $1 \leqslant \mathfrak{i} \leqslant n$ the edge $e_{i}$ has endpoints $v_{i-1}$ and $v_{i}$. A trail is a walk with no repeated edge, and a path is a trail with no repeated vertex. A walk is closed if its first and last vertices are the same. We consider the walk with one vertex and no edges to be closed. A tour is a closed trail. A cycle is a graph which consists of a path $\left(u_{1}, u_{2}, \ldots u_{n}\right)$ and the additional edge $\left\{u_{n}, u_{1}\right\}$. Note that a cycle contains no repeated vertex or edge. The length of a walk/trail/path/cycle is the number of edges present in it. A cycle (respectively, path) on $\ell$ vertices is denoted $C_{\ell}$ ( respectively, $P_{\ell}$ ). We say that a walk (or trail/path/tour/cycle) T contains, or passes through, a vertex $v$ (respectively, an edge $e$ ), if $v$ (respectively, $e$ ) is present in the sequence T. A spanning walk (or trail/path/tour/cycle) is one which passes through all vertices in the graph. A Hamiltonian path (respectively, Hamiltonian cycle) in a graph G is any spanning path (respectively, cycle) in G. An Eulerian tour in G is any spanning tour which, in addition, contains every edge of G. A graph is said to be Hamiltonian if it contains a Hamiltonian cycle, and Eulerian if it has an Eulerian tour. A graph is Eulerian if and only if it is connected and all its vertices have even degrees [40, Theorem 1.2.26].

A tree decomposition of a graph $G$ is a pair $\mathcal{T}=\left(T,\left\{X_{t}\right\}_{t \in V(T)}\right)$ where $T$ is a tree and every vertex $t$ of $T$ is assigned a subset $X_{t} \subseteq V(G)$ of the vertex set of $G$. Each such $X_{t}$ is called a bag, and the structure satisfies the following conditions:

1. Every vertex of $G$ is in at least one bag.
2. For every edge $u v$ in $G$ there is at least one node $t \in V(T)$ such that $\{u, v\} \subseteq X_{t}$.
3. For each vertex $v$ in $G$ the set $\left\{t \in \mathrm{~V}(\mathrm{~T}) ; v \in X_{\mathrm{t}}\right\}$ of all nodes whose bags contain $v$, form a connected subgraph (i.e, a sub-tree) of T .
The width of this tree decomposition is the maximum size of a bag, minus one. The treewidth of a graph G , denoted $\mathrm{tw}(\mathrm{G})$, is the minimum width of a tree decomposition of G . A nice tree decomposition of a graph G is a tree decomposition $\mathcal{T}=\left(\mathrm{T},\left\{\mathrm{X}_{\mathrm{t}}\right\}_{\mathrm{t} \in \mathrm{V}(\mathrm{T})}\right)$ with the following additional structure:
4. The tree T is rooted at a distinguished root node $\mathrm{r} \in \mathrm{V}(\mathrm{T})$.
5. The bags associated with the root node $r$ and with every leaf node are all empty.
6. Every non-leaf node is one of four types:
a. An introduce vertex node: This is a node $t \in \mathrm{~V}(\mathrm{~T})$ with exactly one child node $\mathrm{t}^{\prime}$ such that $\left(X_{t} \backslash X_{t^{\prime}}\right)=\{v\}$ for some vertex $v \in \mathrm{~V}(\mathrm{G})$; the vertex $v$ is introduced at node t .
b. An introduce edge node: This is a node $t \in \mathrm{~V}(\mathrm{~T})$ with exactly one child node $\mathrm{t}^{\prime}$ such that $X_{t}=X_{t^{\prime}}$. Further, the node $t$ is labelled with an edge $u v \in \mathrm{E}(\mathrm{G})$ such that $\{u, v\} \subseteq X_{t}$; the edge $u v$ is introduced at node $t$. Moreover, every edge in the graph G is introduced at exactly one introduce edge node in the entire tree decomposition.
c. A forget node: This is a node $t \in \mathrm{~V}(\mathrm{~T})$ with exactly one child node $\mathrm{t}^{\prime}$ such that $\left(X_{t^{\prime}} \backslash X_{t}\right)=\{v\}$ for some vertex $v \in \mathrm{~V}(\mathrm{G})$; the vertex $v$ is forgotten at node t .
d. A join node: This is a node $\mathrm{t} \in \mathrm{V}(\mathrm{T})$ with exactly two child nodes $\mathrm{t}_{1}, \mathrm{t}_{2}$ such that $X_{t}=X_{t_{1}}=X_{t_{2}}$.
For a node $t \in V(T)$ of the nice tree decomposition $\mathcal{T}$ we define (i) $T_{t}$ to be the subtree of $T$ which is rooted at $t$, (ii) $V_{t}$ to be the union of all the bags associated with nodes in $T_{t}$, (iii) $E_{t}$ to be the set of all edges introduced in $T_{t}$, and (iv) $G_{t}=\left(V_{t}, E_{t}\right)$ to be the subgraph of $G$ defined by $\mathrm{T}_{\mathrm{t}}$. Note that, in general, $\mathrm{G}_{\mathrm{t}}$ is not the subgraph of G induced by $\mathrm{V}_{\mathrm{t}}$.

- Definition 8 (Residual subgraph). Let $\mathrm{G}_{1}$ be a subgraph of G , let t be a node of $\mathcal{T}$, and let $\mathrm{Y}_{\mathrm{t}}=\left(\mathrm{V}_{\mathrm{t}} \backslash \mathrm{X}_{\mathrm{t}}\right)$. We define the residual subgraph of $\mathrm{G}_{1}$ with respect to t to be the graph $\mathrm{G}_{1}^{\mathrm{t}}=\left(\left(\mathrm{V}\left(\mathrm{G}_{1}\right) \backslash \mathrm{Y}_{\mathrm{t}}\right),\left(\mathrm{E}\left(\mathrm{G}_{1}\right) \backslash \mathrm{E}_{\mathrm{t}}\right)\right)$ obtained by deleting from $\mathrm{G}_{1}$ (i) all edges of the graph $\mathrm{G}_{\mathrm{t}}$ and (ii) all vertices of $\mathrm{G}_{\mathrm{t}}$ except those in bag $\mathrm{X}_{\mathrm{t}}$. More generally, we say that a subgraph $\mathrm{G}^{\prime}$ of G is a residual subgraph with respect to t if (i) $\mathrm{V}\left(\mathrm{G}^{\prime}\right) \cap \mathrm{Y}_{\mathrm{t}}=\emptyset$ and (ii) $\mathrm{E}\left(\mathrm{G}^{\prime}\right) \cap \mathrm{E}_{\mathrm{t}}=\emptyset$.

The next theorem lets us assume, without loss of generality, that any tree decomposition is a nice tree decomposition.

- Theorem 9. 25], 12, Section 7.3.2], 5, Proposition 2.2] There is an algorithm which, given a graph G and a tree decomposition $\mathfrak{T}=\left(\mathrm{T},\left\{\mathrm{X}_{\mathrm{t}}\right\}_{\mathrm{t} \in \mathrm{V}(\mathrm{T})}\right)$ of G of width $w$, computes a nice tree decomposition of G of width $w$ and with $\mathcal{O}(w \cdot|\mathrm{~V}(\mathrm{G})|)$ nodes, in time which is polynomial in $|\mathrm{V}(\mathrm{G})|+|\mathrm{V}(\mathrm{T})|+w$.

Let X be a finite set. A partition of X is any nonempty collection of pairwise disjoint nonempty subsets of $X$, whose union is $X$. We use $\Pi(X)$ to denote the set of all partitions of $X$. Each subset in a partition is called a block of the partition. For a partition $P$ of $X$ and an element $v \in X$ we use $\mathrm{P}(v)$ to denote the block of P to which $v$ belongs. We use $\mathrm{P}-v$ to denote the partition of $X \backslash\{v\}$ obtained by eliding $v$ from $P$ : if $P(v)=\{v\}$ then $P-v$ is the partition obtained by deleting block $\{v\}$ from $P$. Otherwise, $P-v$ is the partition obtained by deleting element $v$ from its block in $P$. Let $P, Q$ be partitions of $X$. We say that $Q$ is a refinement of $P$, denoted $Q \sqsubseteq P$, if every block of $Q$ is a subset of some block of $P$. Note that $P \sqsubseteq P$ holds for every partition $P$ of $X$. We use $P \sqcup Q$ to denote the unique partition $R$ of $X$-called the join of P and Q -such that (i) $\mathrm{P} \sqsubseteq \mathrm{R}$, (ii) $\mathrm{Q} \sqsubseteq \mathrm{R}$, and (iii) if both $\mathrm{P} \sqsubseteq \mathrm{R}^{\prime}$ and $\mathrm{Q} \sqsubseteq \mathrm{R}^{\prime}$ hold for any partition $R^{\prime}$ of $X$ then $R \sqsubseteq R^{\prime}$ holds as well. For a graph $G$, subset $X \subseteq V(G)$ of the vertex set of $G$, and partition $P$ of $X$ we say that $P$ is the partition of $X$ defined by $G$ if each block of $P$ consists of the set of all vertices of $X$ which belong to a distinct connected component of $G$. In particular, if $G$ is a connected graph then the partition of $X$ defined by $G$ is $P=\{X\}$.

- Theorem 10. 12, Section 11.2.2], [5] Let $\mathrm{P}, \mathrm{Q}$ be partitions of a finite set X , and let $\mathrm{G}_{\mathrm{P}}, \mathrm{G}_{\mathrm{Q}}$ be two graphs on vertex set X such that P is the partition of X defined by $\mathrm{G}_{\mathrm{P}}$ and Q is the partition of X defined by $\mathrm{G}_{\mathrm{Q}}$. Then $\mathrm{P} \sqcup \mathrm{Q}$ is the partition of X defined by the graph $\mathrm{G}_{\mathrm{P}} \cup \mathrm{G}_{\mathrm{Q}}$.

The effect of graph union on connectivity is correctly captured by the join operation of partitions, even when restricted to arbitrary subsets of vertices.

- Lemma 11. Let $\mathrm{P}, \mathrm{Q}$ be two partitions of a non-empty finite set X and let $\mathrm{H}_{\mathrm{P}}, \mathrm{H}_{\mathrm{Q}}$ be two graphs such that (i) $\mathrm{V}\left(\mathrm{H}_{\mathrm{P}}\right) \cap \mathrm{V}\left(\mathrm{H}_{\mathrm{Q}}\right)=\mathrm{X}$, (ii) $\mathrm{E}\left(\mathrm{H}_{\mathrm{P}}\right) \cap \mathrm{E}\left(\mathrm{H}_{\mathrm{Q}}\right)=\emptyset$, (iii) the vertex set of each component of $\mathrm{H}_{\mathrm{P}}$ and of $\mathrm{H}_{\mathrm{Q}}$ has a nonempty intersection with X , and (iv) P is the partition of X defined by $\mathrm{H}_{\mathrm{P}}$ and Q is the partition of X defined by $\mathrm{H}_{\mathrm{Q}}$. Let $\mathrm{H}=\mathrm{H}_{\mathrm{P}} \cup \mathrm{H}_{\mathrm{Q}}$. Then (i) $\mathrm{P} \sqcup \mathrm{Q}$ is the partition of X defined by graph H , and (ii) H is connected if and only if $\mathrm{P} \sqcup \mathrm{Q}=\{\mathrm{X}\}$.

Proof. Let $G_{P}, G_{Q}$ be two graphs on vertex set $X$ such that $P$ is the partition of $X$ defined by $G_{P}$ and $Q$ is the partition of $X$ defined by $G_{Q}$, and let $G=G_{P} \cup G_{Q}$. From Theorem 10 we know that $P \sqcup Q$ is the partition of $X$ defined by graph $G$. To show that $P \sqcup Q$ is the partition of $X$ defined by graph $H$, it is thus enough to show that two vertices from the set $X$ are in the same component of graph $H$ if and only if they are in the same component of graph G.

So let $x_{1}, \chi_{2}$ be two vertices in the set $X$ such that there is a path $\mathcal{P}$ in $H$ from $x_{1}$ to $x_{2}$. Since $\left.E(H)=E\left(H_{P}\right) \cup E\left(H_{Q}\right)\right)$ and $E\left(H_{P}\right) \cap E\left(H_{Q}\right)=\emptyset$, each edge in path $\mathcal{P}$ corresponds to an edge in the graph $H_{P}$ or to an edge in the graph $H_{Q}$. Call a maximal contiguous sequence of edges in $\mathcal{P}$ from any one of $\left\{\mathrm{H}_{\mathrm{P}}, \mathrm{H}_{\mathrm{Q}}\right\}$ a run. Equivalently: In graph H , give the colour red to each edge from the set $E\left(H_{P}\right)$ and the colour blue to each edge from the set $E\left(H_{Q}\right)$. A run in path $\mathcal{P}$ then consists of a maximal contiguous set of edges with the same colour. Note that the edges of any one run belong to a single connected component of one of the two graphs $\mathrm{H}_{\mathrm{P}}, \mathrm{H}_{\mathrm{Q}}$. Let t be the number of runs in path $\mathcal{P}$. We prove by induction on t that there is a path from $x_{1}$ to $x_{2}$ in graph $G$ as well.
Base case, $t=1$. In this case the entire path $\mathcal{P}$ consists of edges from exactly one of the two graphs $H_{P}$ and $H_{Q}$. If $\mathcal{P}$ is made up exclusively of edges from $H_{P}$ then $x_{1}$ and $x_{2}$ belong to the same connected component of $H_{P}$, and hence to the same block of partition P , and so there is a path from $x_{1}$ to $x_{2}$ in the graph $G_{p}$. Similarly, if $\mathcal{P}$ is made up exclusively of edges from $H_{Q}$ then there is a path from $x_{1}$ to $x_{2}$ in the graph $G_{Q}$. In either case, this path survives intact in graph G, and so there is a path from $x_{1}$ to $x_{2}$ in the graph $G$.
Inductive step, $t \geq 2$. In this case path $\mathcal{P}$ has at least two runs. Let $R_{1}$ be the first run in $\mathcal{P}$, starting from vertex $x_{1}$. Without loss of generality, suppose the edges of $R_{1}$ are all from graph $H_{P}$. Let $x$ be the last vertex in $\mathcal{P}$ which is incident with an edge in $R_{1}$. Then the edges of $R_{1}$ form a path from $x_{1}$ to $x$ in graph $H_{p}$. So the vertices $x_{1}$ and $x$ belong to the same connected component of graph $H_{P}$, and hence to the same block of partition $P$. It follows that vertices $x_{1}$ and $x$ belong to the same connected component of graph $G_{p}$, and hence there is a path, say $\mathcal{P}_{1}$, from $x_{1}$ to $x$ in graph $G_{p}$; this path survives intact in graph G.
Now since $t \geqslant 2$ we have that (i) $x$ is not the vertex $x_{2}$, and (ii) the other edge in $\mathcal{P}$ which is incident on $x$ is from graph $H_{Q}$. Since $V\left(H_{P}\right) \cap V\left(H_{Q}\right)=X$ we get that $x \in X$ holds. By the inductive hypothesis applied to the sub-path of $\mathcal{P}$ from $x$ to $x_{2}$ we get that there is a path, say $\mathcal{P}_{2}$, from $x$ to $x_{2}$ in graph $G$. The union of the paths $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ contains a path from $x_{1}$ to $x_{2}$ in the graph $G$.
This completes the induction.
Now we prove the reverse direction; the arguments are quite similar to those above. So let $x_{1}, x_{2}$ be two vertices in the set $X$ such that there is a path $\mathcal{P}$ in $G$ from $x_{1}$ to $x_{2}$. For each edge $u v$ in graph $G$, if $u v$ is an edge in the set $E\left(G_{P}\right)$ then give the colour red to edge $u v$. Give the colour blue to each remaining edge in $G$; the blue edges are all in $E\left(G_{Q}\right)$. Define a run in path $\mathcal{P}$ to consist of a maximal contiguous set of edges with the same colour. Note that the edges of any one run belong to a single connected component of one of the two graphs $G_{P}, G_{Q}$. Let $t$ be the number of runs in path $\mathcal{P}$. We prove by induction on $t$ that there is a path from $x_{1}$ to $x_{2}$ in graph H as well.
Base case, $\mathbf{t}=\mathbf{1}$. In this case the entire path $\mathcal{P}$ consists of edges from exactly one connected component of one of the two graphs $G_{P}, G_{Q}$. If $\mathcal{P}$ is made up exclusively of edges from $G_{P}$ then-since the partitions of $X$ defined by the two graphs $G_{p}$ and $H_{P}$ are identical-there is a path from $x_{1}$ to $x_{2}$ in the graph $H_{p}$. If $\mathcal{P}$ is made up exclusively of edges from $G_{Q}$ then-since the partitions of $X$ defined by the two graphs $G_{Q}$ and $H_{Q}$ are identical-there is a path from $x_{1}$ to $x_{2}$ in the graph $H_{Q}$. In either case, this path survives intact in graph $H$, and so there is a path from $x_{1}$ to $x_{2}$ in the graph $H$.
Inductive step, $t \geq 2$. In this case path $\mathcal{P}$ has at least two runs. Let $R_{1}$ be the first run in $\mathcal{P}$, starting from vertex $x_{1}$. Without loss of generality, suppose the edges of $R_{1}$ are all from graph $G_{p}$. Let $x$ be the last vertex in $\mathcal{P}$ which is incident with an edge in $R_{1}$. Then the edges of $R_{1}$ form a path from $x_{1}$ to $x$ in graph $G_{p}$. So the vertices $x_{1}$ and $x$ belong to
the same connected component of graph $G_{P}$, and hence to the same block of partition $P$. It follows that vertices $x_{1}$ and $x$ belong to the same connected component of graph $H_{P}$, and hence there is a path, say $\mathcal{P}_{1}$, from $x_{1}$ to $x$ in graph $H_{P}$; this path survives intact in graph H .
Now since $t \geqslant 2$ we have that (i) $x$ is not the vertex $x_{2}$, and (ii) the other edge in $\mathcal{P}$ which is incident on $x$ is from graph $G_{Q}$. Since $V\left(G_{Q}\right)=X$ we get that $x \in X$ holds. By the inductive hypothesis applied to the sub-path of $\mathcal{P}$ from $x$ to $x_{2}$ we get that there is a path, say $\mathcal{P}_{2}$, from $x$ to $x_{2}$ in graph $H$. The union of the paths $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ contains a path from $x_{1}$ to $x_{2}$ in the graph $H$.
This completes the proof that $\mathrm{P} \sqcup \mathrm{Q}$ is the partition of X defined by graph H .
Now we take up the second claim of the lemma. If graph H is connected then-since $X \subseteq V(H)$ holds - the partition of $X$ defined by $H$ is $\{X\}$. From the first part of the lemma we know that this partition is in fact $P \sqcup Q$. Thus $P \sqcup Q=\{X\}$ holds. In the reverse direction, suppose $P \sqcup Q=\{X\}$ holds. Since-from the first part of the lemma- $P \sqcup Q$ is the partition of $X$ defined by graph $H$ we get that there is a path in $H$ between every pair of vertices in $X$. Now since every component of $H_{P}$ and of $H_{Q}$ has a non-empty intersection with the set $X$, we get that there is a path between any two vertices of the graph $H=H_{p} \cup H_{Q}$. Thus graph H is connected as well. This completes the proof of the lemma.

Let $\mathcal{A} \subseteq \Pi(\mathrm{X}), \mathcal{B} \subseteq \Pi(\mathrm{X})$ be collections of partitions of set X . We say that $\mathcal{B}$ is a representative subset of $\mathcal{A}$ if (i) $\mathcal{B} \subseteq \mathcal{A}$, and (ii) for any two partitions $\mathrm{P} \in \mathcal{A}$ and $\mathrm{R} \in \Pi(\mathrm{X})$ with $\mathrm{P} \sqcup \mathrm{R}=\{\mathrm{X}\}$, there exists a partition $Q \in \mathcal{B}$ such that $Q \sqcup R=\{X\}$ holds. The next theorem lets us keep a very small subset of possible partitions of each bag in order to remember all relevant connectivity information, while doing dynamic programming over the bags of a tree decomposition.

- Theorem 12. [12, Theorem 11.11], [5. Theorem 3.7] There is an algorithm which, given a set of partitions $\mathcal{A} \subseteq \Pi(\mathrm{X})$ of a finite set X as input, runs in time $|\mathcal{A}| \cdot 2^{(\omega-1)(|\mathrm{X}|)} \cdot|\mathrm{X}|^{\mathcal{O}(1)}$ and outputs a representative subset $\mathcal{B} \subseteq \mathcal{A}$ of size at most $2^{|X|-1}$.


## 3 An FPT Algorithm for Eulerian Steiner Subgraph

In this section we prove Theorem 5 we describe an algorithm which takes an instance $(G, K, \mathcal{T}, \mathrm{t} w)$ of Eulerian Steiner Subgraph as input and tells in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{t} w}\right)$ time whether graph G has a subgraph which is (i) Eulerian, and (ii) contains every vertex in the terminal set K . As a first step our algorithm applies Theorem 9 to $\mathcal{T}$ to obtain a nice tree decomposition in polynomial time. So we assume, without loss of generality, that $\mathcal{T}$ is itself a nice tree decomposition of width tw . The rest of our algorithm for Eulerian Steiner SubGRAPH consists of doing dynamic programming (DP) over the bags of this nice tree decomposition, and is modelled after the algorithm of Bodlaender et al. 5] for Steiner Tree; see also the exposition of this algorithm in the textbook of Cygan et al. 12. Sections 7.3.3 and 11.2.2].

We make one further modification to the given nice tree decomposition $\mathfrak{T}$ : we pick an arbitrary terminal $v^{\star} \in \mathrm{K}$ and add it to every bag of $\mathcal{T}$; from now on we use $\mathcal{T}$ to refer to the resulting "nearly-nice" tree decomposition in which the bags at all the leaves and the root are equal to $\left\{v^{\star}\right\}$. Note that $v^{\star}$ is neither introduced nor forgotten at any bag of $\mathcal{T}$. This step increases the width of $\mathcal{T}$ by at most 1 and ensures that every bag of $\mathcal{T}$ contains at least one terminal vertex.

Recall that $G_{t}$ denotes the graph defined by the vertices $V_{t}$ and edges $E_{t}$ of $G$ which have been "seen" in the subtree $T_{t}$ of $\mathcal{T}$ rooted at a node $t$. If the graph $G$ has an Eulerian subgraph

## XX:10 On Computing the Hamiltonian Index of Graphs

$\mathrm{G}^{\prime}=\left(\mathrm{V}^{\prime}, \mathrm{E}^{\prime}\right)$ which contains all the terminals K then it interacts with the structures defined by node $t$ in the following way: The part of $\mathrm{G}^{\prime}$ contained in $\mathrm{G}_{\mathrm{t}}$ is a collection $\mathcal{C}=\left\{\mathrm{C}_{1}, \ldots, \mathrm{C}_{\ell}\right\}$ of pairwise vertex-disjoint connected subgraphs of $\mathrm{G}_{\mathrm{t}}$. This collection is never empty because the bag $X_{t}$ contains at least one terminal vertex, viz. $v^{\star}$. Indeed, since $G^{\prime}$ is connected we get that each element $\mathcal{C}_{i}$ of $\mathcal{C}$ has a non-empty intersection with $X_{t}$. Further, every terminal vertex in the set $K \cap V_{t}$ belongs to exactly one element of $\mathcal{C}$.

- Definition 13 (Valid partitions, witness for validity). For a bag $X_{t}$ and subsets $\mathrm{X} \subseteq X_{\mathrm{t}}$, $\mathrm{O} \subseteq \mathrm{X}$, we say that a partition $\mathrm{P}=\left\{\mathrm{X}^{1}, \mathrm{X}^{2}, \ldots \mathrm{X}^{\mathrm{p}}\right\}$ of X is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ if there exists a subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}=\left(\mathrm{V}_{\mathrm{t}}^{\prime}, \mathrm{E}_{\mathrm{t}}^{\prime}\right)$ of $\mathrm{G}_{\mathrm{t}}$ such that

1. $X_{t} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X}$.
2. $\mathrm{G}_{\mathrm{t}}^{\prime}$ has exactly p connected components $\mathrm{C}_{1}, \mathrm{C}_{2}, \ldots, \mathrm{C}_{\mathrm{p}}$ and for each $\mathrm{i} \in\{1,2, \ldots, \mathrm{p}\}$, $\mathrm{X}^{\mathrm{i}} \subseteq \mathrm{V}\left(\mathrm{C}_{\mathrm{i}}\right)$. That is, the vertex set of each connected component of $\mathrm{G}_{\mathrm{t}}^{\prime}$ has a non-empty intersection with set X , and P is the partition of X defined by the subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$.
3. Every terminal vertex from $\mathrm{K} \cap \mathrm{V}_{\mathrm{t}}$ is in $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$.
4. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .

Such a subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$ of $\mathrm{G}_{\mathrm{t}}$ is a witness for partition P being valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ or, in short: $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$.

Note that the fourth condition implies in particular that every vertex $v \in V_{t}^{\prime} \backslash X_{t}$ has an even degree in $G_{t}^{\prime}$. The intuition behind this definition is that (i) the subgraph $G_{t}^{\prime}$ of $G_{t}$ is the intersection of an (unknown) Eulerian Steiner subgraph G' of G with the "uncovered" subgraph $G_{t}$, (ii) the set $X \subseteq X_{t}$ is the subset of vertices of $G_{t}^{\prime}$ which could potentially gain new neighbours as we uncover the rest of the subgraph $\mathrm{G}^{\prime}$, and (iii) the set $\mathrm{O} \subseteq X$ is exactly the subset of vertices of $G_{t}^{\prime}$ which have odd degrees in the uncovered part, and hence will definitely gain new neighbours as we uncover the rest of $\mathrm{G}^{\prime}$. By the time we uncover all of $\mathrm{G}^{\prime}$ (e.g., at the root node of $\mathfrak{T}$ ) there will be (i) no vertices in the set O and (ii) exactly one set in the partition $P$.

- Definition 14 (Completion). For a bag $\mathrm{X}_{\mathrm{t}}$ and subsets $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ let P be a partition of X which is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Let H be a residual subgraph with respect to t such that $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$. We say that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H if there exists a subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$ of $\mathrm{G}_{\mathrm{t}}$ which is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$, such that the graph $\mathrm{G}_{\mathrm{t}}^{\prime} \cup \mathrm{H}$ is an Eulerian Steiner subgraph of G for the terminal set K . We say that $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a certificate for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completing H .
- Lemma 15. Let $(\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw})$ be an instance of Eulerian Steiner Subgraph. Let t be an arbitrary node of $\mathcal{T}$, let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, let P be a partition of X which is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ), and let H be a residual subgraph with respect to t with $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$. If $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H then the set of odd-degree vertices in H is exactly the set O .

Proof. Let $H_{o d d} \subseteq V(H)$ be the set of odd-degree vertices in $H$. Since ( $\left.(t, X, O), P\right)$ completes $H$ we know that there is a subgraph $G_{t}^{\prime}=\left(V_{t}^{\prime}, E_{t}^{\prime}\right)$ of $G_{t}$ which is a witness for $((t, X, O), P)$, such that the graph $G^{\star}=G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K. Since $G_{t}^{\prime}$ is a witness for $((t, X, O), P)$ we get that the set of odd-degree vertices in $G_{t}^{\prime}$ is exactly the set $O$. Since $H$ is a residual subgraph with respect to $t$ we have that $E_{t}^{\prime} \cap E(H)=\emptyset$. Thus the degree of any vertex $v$ in the graph $G^{\star}$ is the sum of its degrees in the two subgraphs $H$ and $G_{t}^{\prime}: \operatorname{deg}_{G^{*}}(v)=\operatorname{deg}_{H}(v)+\operatorname{deg}_{G_{t}^{\prime}}(v)$. And since $G^{\star}$ is Eulerian we have that $\operatorname{deg}_{G^{*}}(v)$ is even for every vertex $v \in \mathrm{~V}\left(\mathrm{G}^{\star}\right)$.

Now let $v \in \mathrm{H}_{\mathrm{odd}} \subseteq \mathrm{V}(\mathrm{H})$ be a vertex of odd degree in H . Then $v \in \mathrm{~V}\left(\mathrm{G}^{\star}\right)$ and we get that $\operatorname{deg}_{G_{\mathrm{t}}^{\prime}}(v)=\operatorname{deg}_{\mathrm{G}^{\star}}(v)-\operatorname{deg}_{\mathrm{H}}(v)$ is odd. Thus $v \in \mathrm{O}$, and so $\mathrm{H}_{\mathrm{odd}} \subseteq 0$.

Conversely, let $x \in O \subseteq V_{t}^{\prime}$ be a vertex of odd degree in $G_{t}^{\prime}$. Then $x \in V\left(G^{\star}\right)$ and we get that $\operatorname{deg}_{\mathrm{H}}(\mathrm{x})=\operatorname{deg}_{\mathrm{G}^{*}}(\mathrm{x})-\operatorname{deg}_{\mathrm{G}_{\mathrm{t}}^{\prime}}(\mathrm{x})$ is odd. Thus $\mathrm{x} \in \mathrm{H}_{\mathrm{odd}}$, and so $\mathrm{O} \subseteq \mathrm{H}_{\mathrm{odd}}$. Thus the set of odd-degree vertices in H is exactly the set O .

The next lemma tells us that it is safe to apply the representative set computation to collections of valid partitions.

- Lemma 16. Let ( $\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw}$ ) be an instance of Eulerian Steiner Subgraph, and let t be an arbitrary node of $\mathcal{T}$. Let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, and let $\mathcal{A}$ be a collection of partitions of X , each of which is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Let $\mathcal{B}$ be a representative subset of $\mathcal{A}$, and let H be an arbitrary residual subgraph of G with respect to t such that $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$ holds. If there is a partition $\mathrm{P} \in \mathcal{A}$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H then there is a partition $\mathrm{Q} \in \mathcal{B}$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H .

Proof. Suppose there is a partition $\mathrm{P} \in \mathcal{A}$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes the residual subgraph $H$. Then there exists a subgraph $G_{t}^{\prime}=\left(V_{t}^{\prime}, E_{t}^{\prime}\right)$ of $G_{t}-G_{t}^{\prime}$ being a witness for $((t, X, O), P)$-such that (i) $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$, (ii) $P$ is the partition of $X$ defined by $G_{t}^{\prime}$, (iii) every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$, (iv) the set of odd-degree vertices in $G_{t}^{\prime}$ is exactly the set $O$, and (v) the graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Observe that every terminal vertex in the set $K \backslash V_{t}$ is in the set $V(H)$. Let $R$ be the partition of the set $X$ defined by the residual subgraph $H$. Since the union of $G_{t}^{\prime}$ and $H$ is connected we get Lemma 11- that $P \sqcup R=\{X\}$ holds. Since $\mathcal{B}$ is a representative subset of $\mathcal{A}$ we get that there exists a partition $Q \in \mathcal{B}$ such that $Q \sqcup R=\{X\}$ holds. Since $\mathcal{B} \subseteq \mathcal{A}$ we have that the partition $Q$ of $X$ is valid for the combination $(t, X, O)$. So there exists a subgraph $G_{t}^{\star}=\left(V_{t}^{\star}, E_{t}^{\star}\right)$ of $G_{t}-G_{t}^{\star}$ being a witness for $((t, X, O), Q)$-such that (i) $X_{t} \cap V\left(G_{t}^{\star}\right)=X$, (ii) $Q$ is the partition of $X$ defined by $G_{t}^{\star}$, (iii) every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\star}\right)$, and (iv) the set of odd-degree vertices in $G_{t}^{\star}$ is exactly the set $O$. Now the graph $\mathrm{G}_{\mathrm{t}}^{\star} \cup \mathrm{H}$ :

1. Contains all the terminal vertices $K$, because every terminal vertex from $K \cap V_{t}$ is in $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\star}\right)$, and every terminal vertex in the set $\mathrm{K} \backslash \mathrm{V}_{\mathrm{t}}$ is in the set $\mathrm{V}(\mathrm{H})$.
2. Has all degrees even, because (i) the edge sets $E\left(G_{t}^{\star}\right)$ and $E(H)$ are disjoint, and (ii) the sets of odd-degree vertices in the two graphs $\mathrm{G}_{\mathrm{t}}^{\star}$ and H are identical-namely, the set O .
3. Is connected-by Lemma 11 -because $Q \sqcup R=\{X\}$ holds.

Thus the subgraph $G_{t}^{\star}$ of $G_{t}$ is a witness for $((t, X, O), Q)$ such that the graph $G_{t}^{\star} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K. Hence ( $(t, X, O), Q)$ completes the residual subgraph H .

- Lemma 17. Let $(\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw})$ be an instance of Eulerian Steiner Subgraph, let r be the root node of $\mathcal{T}$, and let $\boldsymbol{v}^{\star}$ be the terminal vertex which is present in every bag of $\mathcal{T}$. Then $(\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw})$ is a yes instance of Eulerian Steiner Subgraph if and only if the partition $\mathrm{P}=\left\{\left\{v^{\star}\right\}\right\}$ is valid for the combination $\left(\mathrm{r}, \mathrm{X}=\left\{v^{\star}\right\}, \mathrm{O}=\emptyset\right)$.

Proof. Let ( $\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw}$ ) be a yes instance of Eulerian Steiner Subgraph and let $\mathrm{G}^{\prime}$ be an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Then the terminal vertex $v^{\star}$ is in $V\left(G^{\prime}\right)$. Since $r$ is the root node of $\mathcal{T}$ we have that $X_{r}=\left\{\nu^{\star}\right\}, V_{r}=V(G)$ and $G_{r}=G$. We set $\mathrm{G}_{\mathrm{r}}^{\prime}=\mathrm{G}^{\prime}$. Then (i) $\mathrm{X}_{\mathrm{r}} \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)=\left\{\nu^{\star}\right\}=\mathrm{X}$, (ii) $\mathrm{G}_{\mathrm{r}}^{\prime}=\mathrm{G}^{\prime}$ has exactly one connected component $C_{1}=V\left(G^{\prime}\right)$ and the partition $P=\left\{\left\{\nu^{\star}\right\}\right\}$ of $X=\left\{\nu^{\star}\right\}$ is the partition of $X$ defined by $\mathrm{G}_{\mathrm{r}}^{\prime}$, (iii) every terminal vertex from $\mathrm{K} \cap \mathrm{V}_{\mathrm{r}}=\mathrm{K}$ is in $\mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)=\mathrm{V}\left(\mathrm{G}^{\prime}\right)$, and (iv) the set of odd-degree vertices in $\mathrm{G}_{\mathrm{r}}^{\prime}$ is exactly the empty set O . Thus the partition $\mathrm{P}=\left\{\left\{\nu^{\star}\right\}\right\}$ is valid for the combination $\left(r, X=\left\{v^{\star}\right\}, O=\emptyset\right)$. This completes the forward direction.

## XX:12 On Computing the Hamiltonian Index of Graphs

For the reverse direction, suppose the partition $P=\left\{\left\{\nu^{\star}\right\}\right\}$ is valid for the combination $\left(r, X=\left\{v^{\star}\right\}, O=\emptyset\right)$. Then by definition there exists a subgraph $G_{r}^{\prime}=\left(V_{r}^{\prime}, E_{r}^{\prime}\right)$ of $G_{r}=G$ such that (i) $X_{r} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)=\mathrm{X}=\left\{v^{\star}\right\}$, (ii) $\mathrm{G}_{\mathrm{r}}^{\prime}$ has exactly one connected component $\mathrm{C}_{1}=\mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)$, (iii) every terminal vertex from $K \cap V_{r}=K$ is in $V\left(G_{r}^{\prime}\right)$, and (iv) the set of odd-degree vertices in $G_{r}^{\prime}$ is exactly the empty set $O$. Thus $G_{r}^{\prime}$ is a connected subgraph of $G$ which contains every terminal vertex, and whose degrees are all even. But $\mathrm{G}_{\mathrm{r}}^{\prime}$ is then an Eulerian Steiner subgraph of $G$, and so ( $\mathcal{G}, \mathrm{K}, \mathcal{T}, \mathrm{t} w$ ) is a yes instance of Eulerian Steiner Subgraph.

- Lemma 18. Let $(\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw})$ be an instance of Eulerian Steiner Subgraph, let r be the root node of $\mathcal{T}$, and let $v^{\star}$ be the terminal vertex which is present in every bag of $\mathcal{T}$. Let $\mathrm{H}=\left(\left\{\nu^{\star}\right\}, \emptyset\right), \mathrm{X}=\left\{\nu^{\star}\right\}, \mathrm{O}=\emptyset$, and $\mathrm{P}=\left\{\left\{\nu^{\star}\right\}\right\}$. Then $(\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw})$ is a yes instance if and only if $((\mathrm{r}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H .

Proof. Note that $G_{r}=G$. It is easy to verify by inspection that $H$ is a residual subgraph with respect to $r$.

Let ( $\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw}$ ) be a yes instance of Eulerian Steiner Subgraph and let $\mathrm{G}^{\prime}$ be an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Then the terminal vertex $v^{\star}$ is in $V\left(G^{\prime}\right)$. From Lemma 17 we get that partition $P$ is valid for the combination ( $r, X, O$ ), and from the proof of Lemma 17 we get that the Eulerian Steiner subgraph $G^{\prime}$ is itself a witness for $((r, X, O), P)$. Now $\left(\left(V\left(G^{\prime}\right) \cup V(H)\right),\left(E\left(G^{\prime}\right) \cup E(H)\right)\right)=\left(V\left(G^{\prime}\right), E\left(G^{\prime}\right)\right)=G^{\prime}$, and so $G^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K. Thus ( $\left.(r, X, O), P\right)$ completes H .

The reverse direction is trivial: if ( $(\mathrm{r}, \mathrm{X}, \mathrm{O}), \mathrm{P}$ ) completes H then by definition there exists an Eulerian Steiner subgraph of $G$ for the terminal set K, and so (G, K, $\mathcal{T}, \mathrm{tw}$ ) is a yes instance.

A naïve implementation of our algorithm would consist of computing, for each node $t$ of the tree decomposition $\mathcal{T}$-starting at the leaves and working up towards the root-and subsets $\mathrm{O} \subseteq \mathrm{X} \subseteq X_{t}$, the set of all partitions $P$ which are valid for the combination ( $t, X, O$ ). At the root node $r$ the algorithm would apply Lemma 17 to decide the instance ( $G, K, \mathcal{T}, \mathrm{tw}$ ). Since a bag $X_{t}$ can have up to $t w+2$ elements (including the special terminal $v^{\star}$ ) the running time of this algorithm could have a factor of $t w^{t w}$ in it, since $X_{t}$ can have these many partitions. To avoid this we turn to the completion-based alternate characterization of yes instances-Lemma 18 and the fact-Lemma 16 that representative subset computations do not "forget" completion properties. After computing a set $\mathcal{A}$ of valid partitions for each combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ) we compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and throw away the remaining partitions $\mathcal{A} \backslash \mathcal{B}$. Thus the number of partitions which we need to remember for any combination $(t, X, O)$ never exceeds $2^{\mathrm{tw}}$. We now describe the steps of the DP algorithm for each type of node in $\mathcal{T}$. We use $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$ to denote the set of valid partitions for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ) which we store in the DP table for node t .

Leaf node t : In this case $\mathrm{X}_{\mathrm{t}}=\left\{\nu^{\star}\right\}$. Set $\mathrm{VP}\left[\mathrm{t},\left\{\nu^{\star}\right\},\left\{\nu^{\star}\right\}\right]=\emptyset, \mathrm{VP}\left[\mathrm{t},\left\{\nu^{\star}\right\}, \emptyset\right]=\left\{\left\{\left\{\nu^{\star}\right\}\right\}\right\}$, and $\mathrm{VP}[\mathrm{t}, \emptyset, \emptyset]=\{\emptyset\}$.
Introduce vertex node $t$ : Let $t^{\prime}$ be the child node of $t$, and let $v$ be the vertex introduced at t . Then $v \notin X_{\mathrm{t}^{\prime}}$ and $X_{\mathrm{t}}=X_{\mathrm{t}^{\prime}} \cup\{v\}$. For each $X \subseteq X_{\mathrm{t}}$ and $\mathrm{O} \subseteq X$,

1. If $v$ is a terminal vertex, then

- if $v \notin \mathrm{X}$ or if $v \in \mathrm{O}$ then set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\emptyset$
= if $v \in(X \backslash O)$ then for each partition $P^{\prime}$ in $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$, add the partition $P=\left(P^{\prime} \cup\{\{v\}\}\right)$ to the set $V P[t, X, O]$

2. If $v$ is not a terminal vertex, then
= if $v \in O$ then set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\emptyset$
$=$ if $v \in(X \backslash O)$ then for each partition $\mathrm{P}^{\prime}$ in $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \backslash\{v\}, \mathrm{O}\right]$, add the partition $P=P^{\prime} \cup\{\{v\}\}$ to the set VP[t, X, O]

- if $v \notin \mathrm{X}$ then set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$

3. Set $\mathcal{A}=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathcal{B}$. Introduce edge node $t$ : Let $t^{\prime}$ be the child node of $t$, and let $u v$ be the edge introduced at
t. Then $X_{t}=X_{t^{\prime}}$ and $u v \in\left(E\left(G_{t}\right) \backslash E\left(G_{t^{\prime}}\right)\right)$. For each $X \subseteq X_{t}$ and $O \subseteq X$,
4. Set $V P[t, X, O]=V P\left[t^{\prime}, X, O\right]$.
5. If $\{u, v\} \subseteq X$ then:
a. Construct a set of candidate partitions $\mathcal{P}$ as follows. Initialize $\mathcal{P}=\emptyset$.
$=$ if $\{u, v\} \subseteq O$ then add all the partitions in $V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$ to $\mathcal{P}$.
= if $\{u, v\} \cap O=\{u\}$ then add all the partitions in $V P\left[t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right]$ to $\mathcal{P}$.
$=$ if $\{u, v\} \cap O=\{v\}$ then add all the partitions in VP[t', X, $(O \backslash\{v\}) \cup\{u\}]$ to $\mathcal{P}$.
$=$ if $\{u, v\} \cap O=\emptyset$ then add all the partitions in $V P\left[t^{\prime}, X, O \cup\{u, v\}\right]$ to $\mathcal{P}$.
b. For each candidate partition $P^{\prime} \in \mathcal{P}$, if vertices $u, v$ are in different blocks of $P^{\prime}$-say $u \in \mathrm{P}_{\mathfrak{u}}^{\prime}, v \in \mathrm{P}_{v}^{\prime} ; \mathrm{P}_{\mathfrak{u}}^{\prime} \neq \mathrm{P}_{v}^{\prime}$-then merge these two blocks of $\mathrm{P}^{\prime}$ to obtain P . That is, set $P=\left(P^{\prime} \backslash\left\{P_{u}^{\prime}, P_{v}^{\prime}\right\}\right) \cup\left(P_{u}^{\prime} \cup P_{v}^{\prime}\right)$. Now set $\mathcal{P}=\left(\mathcal{P} \backslash\left\{P^{\prime}\right\}\right) \cup P$.
c. Add all of $\mathcal{P}$ to the list VP[t, X, O].
6. Set $\mathcal{A}=V P[t, X, O]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $V P[t, X, O]=\mathcal{B}$. Forget node $t$ : Let $t^{\prime}$ be the child node of $t$, and let $v$ be the vertex forgotten at $t$. Then $v \in X_{t^{\prime}}$ and $X_{t}=X_{t^{\prime}} \backslash\{v\}$. Recall that $P(v)$ is the block of partition $P$ which contains element $v$, and that $\mathrm{P}-v$ is the partition obtained by eliding $v$ from P . For each $\mathrm{X} \subseteq X_{\mathrm{t}}$ and $\mathrm{O} \subseteq \mathrm{X}$,
7. Set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\left\{\mathrm{P}^{\prime}-v ; \mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right],\left|\mathrm{P}^{\prime}(v)\right|>1\right\}$.
8. If $v$ is not a terminal vertex then set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}] \cup \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$.
9. Set $\mathcal{A}=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathcal{B}$. Join node $t$ : Let $t_{1}, t_{2}$ be the children of $t$. Then $X_{t}=X_{t_{1}}=X_{t_{2}}$. For each $X \subseteq X_{t}, O \subseteq X$ :
10. Set $V P[t, X, O]=\emptyset$
11. For each $\mathrm{O}_{1} \subseteq \mathrm{O}$ and $\hat{O} \subseteq(\mathrm{X} \backslash \mathrm{O})$ :
a. Let $\mathrm{O}_{2}=\mathrm{O} \backslash \mathrm{O}_{1}$.
b. For each pair of partitions $P_{1} \in V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], P_{2} \in V P\left[t_{2}, X, O_{2} \cup \hat{O}\right]$, add their join $P_{1} \sqcup P_{2}$ to the set VP $[t, X, O]$.
12. Set $\mathcal{A}=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $\mathrm{V}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathcal{B}$.

We now show that this DP correctly computes a solution in the stated time bound. We assume that the tree decomposition in the input instance is modified as described earlier. We prove the correctness of the algorithm by induction on the structure of this tree decomposition $\mathcal{T}$. The key insight in the proof is that the processing at every node in $\mathcal{T}$ preserves the following

## Correctness Criteria

Let t be a node of $\mathcal{T}$, let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, and let $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$ be the set of partitions computed by the DP for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ).

1. Soundness: Every partition $P \in V P[t, X, O]$ is valid for the combination ( $t, X, O$ ).
2. Completeness: For any residual subgraph $H$ with respect to $t$ with $V(H) \cap X_{t}=X$, if there exists a partition $P$ of $X$ such that $((t, X, O), P)$ completes $H$ then the set $V P[t, X, O]$ contains a partition $Q$ of $X$ such that $((t, X, O), Q)$ completes $H$. Note that - the two partitions $P, Q$ must both be valid for the combination ( $t, X, O$ ); and,

## XX:14 On Computing the Hamiltonian Index of Graphs

- $Q$ can potentially be the same partition as $P$.

The processing at each of the non-leaf nodes computes a representative subset as a final step. This step does not negate the correctness criteria.
$\triangleright$ Observation 19. Let t be a node of $\mathcal{T}$, let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, and let $\mathcal{A}$ be a set of partitions which satisfies the correctness criteria for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). Let $\mathcal{B}$ be a representative subset of $\mathcal{A}$. Then $\mathcal{B}$ satisfies the correctness criteria for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ).

Proof. Since $\mathcal{B} \subseteq \mathcal{A}$ holds we get that $\mathcal{B}$ satisfies the soundness criterion. From Lemma 16 we get that $\mathcal{B}$ satisfies the completeness criterion as well.

- Lemma 20. Let t be a leaf node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Here $X_{t}=\left\{\nu^{\star}\right\}$. Note that the graph $G_{t}$ consists of (i) the one vertex $\nu^{\star}$, and (ii) no edges. We verify the conditions for all the three possible cases:

- $\mathrm{X}=\left\{v^{\star}\right\}, \mathrm{O}=\left\{v^{\star}\right\}$. The algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously. Observe that there is no subgraph $G_{t^{\prime}}$ of $G_{t}$ in which vertex $v^{\star}$ has an odd degree. This means that there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which the fourth condition in the definition of a valid partition- Definition 13 holds. Thus there is no partition which is valid for the combination $(t, X, O)$. Hence the completeness criterion holds vacuously as well.
- $\mathrm{X}=\left\{v^{\star}\right\}, \mathrm{O}=\emptyset$. The algorithm sets $\mathcal{A}=\left\{\left\{\left\{v^{\star}\right\}\right\}\right\}$. It is easy to verify by inspection that the subgraph $G_{t^{\prime}}=G_{t}$ of $G_{t}$ is a witness for the partition $\left\{\left\{\nu^{\star}\right\}\right\}$ being valid for the combination ( $t, X, O$ ). Hence the soundness criterion holds.
Since $X$ is the set $\left\{\nu^{\star}\right\}$, the only valid partition for the combination $(t, X, O)$ is $\left\{\left\{\nu^{\star}\right\}\right\}$. Hence the completeness criterion holds trivially.
- $X=\emptyset, O=\emptyset$. The algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously.

Since $v^{\star} \in V_{t}$ is a terminal vertex and $X=\emptyset$ holds, there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which both the conditions (1) and (3) of the definition of a valid partitionDefinition 13 -hold simultaneously. Thus there is no partition which is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). Hence the completeness criterion holds vacuously as well.

- Lemma 21. Let t be an introduce vertex node of the tree decomposition $\mathfrak{T}$ and let $\mathrm{X} \subseteq$ $\mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ) satisfies the correctness criteria.

Proof. Let $t^{\prime}$ be the child node of $t$, and let $v$ be the vertex introduced at $t$. Then $v \notin X_{t^{\prime}}$ and $X_{t}=X_{t^{\prime}} \cup\{v\}$ hold. Note that no edges incident with $v$ have been introduced so far; so we have that $\operatorname{deg}_{G_{t}}(v)=0$ holds. We analyze each choice made by the algorithm:

1. If $v \in O$ holds then the algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously. Since $\operatorname{deg}_{G_{t}}(v)=0$ holds, there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which the fourth condition of the definition of a valid partition- Definition 13 -holds. Thus there is no partition which is valid for the combination $(t, X, O)$. Hence the completeness condition holds vacuously as well.
2. If $v \in(X \backslash O)$ holds then the algorithm takes each partition $P^{\prime}$ in $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$ and adds the partition $\mathrm{P}=\left(\mathrm{P}^{\prime} \cup\{\{\nu\}\}\right)$ to the set $\mathcal{A}$. By inductive assumption we have that the set VP[ $\left.t^{\prime}, X \backslash\{v\}, O\right]$ of partitions is sound and complete for the combination $\left(t^{\prime}, X \backslash\{v\}, O\right)$. Let $\mathrm{P}=\left(\mathrm{P}^{\prime} \cup\{\{v\}\}\right)$ be an arbitrary partition in the set $\mathcal{A}$, where $\mathrm{P}^{\prime}$ is a partition from the set $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$. Then the partition $P^{\prime}$ is valid for the combination $\left(t^{\prime}, X \backslash\{v\}, O\right)$, and
so there exists a subgraph $H$ of the graph $G_{t^{\prime}}$ such that $H$ is a witness for $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), P^{\prime}\right)$. It is easy to verify by inspection that the graph $G_{t}^{\prime}=(V(H) \cup\{v\}, E(H))$ is a subgraph of $G_{t}$ which satisfies all the four conditions of Definition 13 for being a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the soundness condition holds for the set $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H . We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H . Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X}$.
b. $\mathrm{G}_{\mathrm{t}}^{\prime}$ has exactly p connected components $\mathrm{C}_{1}, \mathrm{C}_{2}, \ldots, \mathrm{C}_{\mathrm{p}}$ and for each $\mathfrak{i} \in\{1,2, \ldots, \mathrm{p}\}$, $X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Since $\operatorname{deg}_{G_{t}}(v)=0$ holds, we get that $\operatorname{deg}_{G_{t}^{\prime}}(v)=0$ holds as well. Thus vertex $v$ forms a connected component by itself in graph $\mathrm{G}_{\mathrm{t}}^{\prime}$. Without loss of generality, let this component by $C_{p}$. Then we get that $X^{p}=V\left(C_{p}\right)=\{v\}$, and that $P^{\prime}=\left\{X^{1}, X^{2}, \ldots X^{(p-1)}\right\}$ is a partition of the set $X \backslash\{v\}$.
Since $v \in X$ and $V(H) \cap X_{t}=X$ hold, and since the graph $G_{t}^{\prime} \cup H$ is Eulerian, we get that vertex $v$ has a positive even degree in graph $H$. Since $H$ is a residual subgraph with respect to $t$ we have that (i) $V(H) \cap\left(V_{t} \backslash X_{t}\right)=\emptyset$ and (ii) $E(H) \cap E_{t}=\emptyset$ hold. Since $X_{t}=X_{t^{\prime}} \cup\{v\}$ holds, we get that $V_{t^{\prime}}=V_{t} \backslash\{v\}$ and hence $V_{t^{\prime}} \backslash X_{t^{\prime}}=V_{t} \backslash X_{t}$ holds. Hence $V(H) \cap\left(V_{t^{\prime}} \backslash X_{t^{\prime}}\right)=\emptyset$ holds. Further, since $E_{t^{\prime}} \subseteq E_{t}$ holds we get that $E(H) \cap E_{t^{\prime}}=\emptyset$ holds as well. Thus H is a residual subgraph with respect to node $\mathrm{t}^{\prime}$ which (i) contains vertex $v$ and (ii) satisfies $V(H) \cap X_{t^{\prime}}=(X \backslash\{v\})$.
Now let $G_{t^{\prime}}^{\prime}$ be the graph obtained from $G_{t}^{\prime}$ by deleting vertex $v$. Then $G_{t^{\prime}}^{\prime}$ is a subgraph of $G_{t^{\prime}}$, and it is straightforward to verify that the following hold:
a. $\mathrm{X}_{\mathrm{t}^{\prime}} \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}\right)=(\mathrm{X} \backslash\{v\})$.
b. $G_{t}^{\prime}$, has exactly $p-1$ connected components $C_{1}, C_{2}, \ldots, C_{(p-1)}$ and for each $i \in$ $\{1,2, \ldots, p-1\}, X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t^{\prime}}$ is in $V\left(G_{t^{\prime}}^{\prime}\right)$.
d. The set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$ is exactly the set $O$.
e. The graph $G_{t}^{\prime}, \cup H$ is identical to the graph $G_{t}^{\prime} \cup H$, and hence is an Eulerian Steiner subgraph of G for the terminal set K .
Thus $H$ is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=(X \backslash\{v\})$, and $P^{\prime}=\left\{X^{1}, X^{2}, \ldots X^{(p-1)}\right\}$ is a partition of $X \backslash\{v\}$ such that $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), P^{\prime}\right)$ completes $H$. From the inductive assumption we know that the set $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$ contains a partition $Q^{\prime}=\left\{Y^{1}, Y^{2}, \ldots Y^{q}\right\}$ of $X \backslash\{v\}$ such that $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), Q^{\prime}\right)$ completes $H$. So there is a subgraph $G_{t^{\prime}}^{\prime \prime}$ of $G_{t^{\prime}}-a$ witness for $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), Q^{\prime}\right)$-such that the following hold:
a. $X_{t^{\prime}} \cap V\left(G_{t^{\prime}}^{\prime \prime}\right)=X \backslash\{v\}$.
b. $G_{t^{\prime}}^{\prime \prime}$ has exactly $q$ connected components $D_{1}, D_{2}, \ldots, D_{q}$ and for each $i \in\{1,2, \ldots, q\}$, $Y^{i} \subseteq V\left(D_{i}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t^{\prime}}$ is in $V\left(G_{t^{\prime}}^{\prime \prime}\right)$.
d. The set of odd-degree vertices in $G_{t^{\prime}}^{\prime \prime}$ is exactly the set $O$.
e. The graph $G_{t^{\prime}}^{\prime \prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

## XX:16 On Computing the Hamiltonian Index of Graphs

Now the algorithm adds the partition $Q=Q^{\prime} \cup\{\{\nu\}\}==\left\{Y^{1}, Y^{2}, \ldots Y^{q},\{v\}\right\}$ of set $X$ to the set $\mathcal{A}$. It is straightforward to verify that the graph $\hat{\mathrm{G}}_{\mathrm{t}}=\left(\mathrm{V}\left(\mathrm{G}_{\mathfrak{t}^{\prime}}^{\prime \prime}\right) \cup\{v\}, \mathrm{E}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right)\right)$ is a subgraph of graph $G_{t}$ for which the following hold:
a. $X_{t} \cap \mathrm{~V}\left(\hat{G}_{t}\right)=X$.
b. $\hat{\mathrm{G}}_{\mathrm{t}}$ has exactly $\mathrm{q}+1$ connected components $\mathrm{D}_{1}, \mathrm{D}_{2}, \ldots, \mathrm{D}_{\mathrm{q}}, \mathrm{D}_{\mathrm{q}+1}=(\{v\}, \emptyset)$ and for each $i \in\{1,2, \ldots, q+1\}, \mathrm{Y}^{i} \subseteq \mathrm{~V}\left(\mathrm{D}_{\mathrm{i}}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(\hat{G}_{t}\right)$.
d. The set of odd-degree vertices in $\hat{G}_{t}$ is exactly the set $O$.
e. The graph $\hat{G}_{t} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Thus $\mathcal{A}$ contains a partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H , as was required to be shown for completeness.
3. If $v$ is a terminal vertex and $v \notin \mathrm{X}$ holds then the algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously.
Since $v \in V_{t}$ is a terminal vertex and $v \notin X$ holds, there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which both the conditions (1) and (3) of the definition of a valid partitionDefinition 13 -hold simultaneously. Thus there is no partition which is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). Hence the completeness condition holds vacuously as well.
4. If $v$ is not a terminal vertex and $v \notin \mathrm{X}$ holds then the algorithm sets $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$. It is straightforward to verify using Definitions 8,13 and 14 that:
$=$ a partition $P$ of set $X$ is valid for the combination $(t, X, O)$ if and only if it is valid for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ );
= a subgraph of $G_{t}$ is a witness for $((t, X, O), P)$ if and only if it is (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for ( $\left.\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$;

- a graph $H$ is a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$ if and only if $H$ is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=X$; and,
= for any residual subgraph $H$ with respect to $t$ with $V(H) \cap X_{t}=X$ and any partition $P$ of $\mathrm{X},((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H if and only if $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ completes H .
By the inductive assumption we have that the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ of partitions is sound and complete for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ ). It follows from the above equivalences that the set $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is sound and complete for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.
- Lemma 22. Let t be an introduce edge node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq$ $\mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the $D P$ for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $t^{\prime}$ be the child node of $t$, and let $u v$ be the edge introduced at $t$. Then $X_{t}=X_{t^{\prime}}$, $V_{t}=V_{t^{\prime}}$ and $u v \in\left(E\left(G_{t}\right) \backslash E\left(G_{t^{\prime}}\right)\right)$. The algorithm initializes $\mathcal{A}=V P\left[t^{\prime}, X, O\right]$. By the inductive assumption we have that every partition $\mathrm{P}^{\prime} \in \mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is valid for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ ). Note that while edge uv is available for use in constructing a witness for $((t, X, O), P)$, it is not mandatory to use this edge in any such witness. Applying this observation, it is straightforward to verify that if a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P^{\prime}\right)$ then it is also (i) a subgraph of $G_{t}$, and (ii) a witness for $\left((t, X, O), P^{\prime}\right)$. Thus all partitions in $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ are valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ).

The algorithm adds zero or more partitions to $\mathcal{A}$ depending on how the set $\{u, v\}$ intersects the sets X and O . We analyze each choice made by the algorithm:

1. If $u \notin X$ or $v \notin X$ holds then the algorithm does not make further changes to $\mathcal{A}$ : it sets $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$. Since (i) the criteria for validity-Definition 13 are based only on graphs whose intersection with $X_{t}$ is exactly the set $X$, and (ii) the new edge $u v$ does not have both end points in this set, it is intuitively clear that the relevant set of valid
partitions should not change in this case. Formally, it is straightforward to verify using Definitions 813 and 14 that:

- a partition $P$ of set $X$ is valid for the combination $(t, X, O)$ if and only if it is valid for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ );
- a subgraph of $G_{t}$ is a witness for $((t, X, O), P)$ if and only if it is (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X, O\right), P\right)$;
- a graph $H$ is a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$ if and only if $H$ is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=X$; and,
= for any residual subgraph $H$ with respect to $t$ with $V(H) \cap X_{t}=X$ and any partition $P$ of $\mathrm{X},((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H if and only if $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ completes H .
By the inductive assumption we have that the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ of partitions is sound and complete for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ ). It follows from the above equivalences that the set $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is sound and complete for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.

2. If $\{u, v\} \subseteq O$ then for each partition $P^{\prime} \in V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$,
= If vertices $u, v$ are in the same block of $P^{\prime}$ then the algorithm adds $P=P^{\prime}$ to the set $\mathcal{A}$.

- If vertices $u, v$ are in different blocks of $\mathrm{P}^{\prime}$ then the algorithm merges these two blocks of $\mathrm{P}^{\prime}$ and adds the resulting partition P -with one fewer block than $\mathrm{P}^{\prime}$ - to the set $\mathcal{A}$.
In either case, by the inductive assumption we have that partition $P^{\prime}$ is valid for the combination ( $t^{\prime}, X, O \backslash\{u, v\}$ ). Let $G_{t^{\prime}}^{\prime \prime}$ be (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), P^{\prime}\right)$, and let $G_{t}^{\prime}=\left(V\left(G_{t^{\prime}}^{\prime \prime}\right), E\left(G_{t^{\prime}}^{\prime \prime}\right) \cup\{u v\}\right)$ be the graph obtained from $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}$ by adding the edge $\boldsymbol{u v}$. Then $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a subgraph of $\mathrm{G}_{\mathrm{t}}$. Vertices $\boldsymbol{u}, v$ have even degrees in $G_{t^{\prime}}^{\prime \prime}$, and hence they have odd degrees in $G_{t}^{\prime}$. It is straightforward to verify that $G_{t}^{\prime}$ is a witness for $((t, X, O), P)$. Thus the addition of partition P to $\mathcal{A}$ preserves the soundness of $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H . Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X}$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $\mathfrak{i} \in\{1,2, \ldots, p\}$, $X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K.

Note that by the definition of a residual subgraph, graph $H$ (i) does not contain edge $u v$, and (ii) is a residual subgraph with respect to node $t^{\prime}$ as well. We consider two cases.

- Suppose edge $\mathfrak{u v}$ is not present in graph $G_{t}^{\prime}$. Then it is straightforward to verify that $G_{t}^{\prime}$ is a witness for $\left(\left(\mathbf{t}^{\prime}, X, O\right), P\right)$ as well. By the inductive hypothesis there exists some partition $Q$ of $X$ in the set $V P\left[t^{\prime}, X, O\right]$ such that $\left(\left(t^{\prime}, X, O\right), Q\right)$ completes $H$. So there exists a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ which is a certificate for $\left(\left(t^{\prime}, X, O\right), Q\right)$ completing $H$. It is straightforward to verify that $G_{t^{\prime}}^{\prime}$ is a certificate for $((t, X, O), Q)$ completing $H$ as well. The algorithm adds partition Q to the set $\mathcal{A}$ during the initialization, so the completeness criterion is satisfied in this case.
- Suppose edge $u v$ is present in graph $G_{t}^{\prime}$. Let $H^{\prime}=(V(H),(E(H) \cup\{u v\}))$ be the graph obtained by adding edge $u v$ to graph $H$, and let $G_{t^{\prime}}^{\prime}=\left(V\left(G_{t}^{\prime}\right),\left(E\left(G_{t}^{\prime}\right) \backslash\{u v\}\right)\right)$ be the graph obtained by deleting edge $u v$ from graph $G_{t}^{\prime}$. Then it is straightforward to verify that (i) the set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$ is exactly the set $O \backslash\{u, v\}$, (ii)


## XX:18 On Computing the Hamiltonian Index of Graphs

$H^{\prime}$ is a residual subgraph for node $t^{\prime}$, and (iii) $G_{t^{\prime}}^{\prime}$ is a subgraph of $G_{t^{\prime}}$ such that the graph $G_{t}^{\prime} \cup H^{\prime}=G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Let $P^{\prime}$ be the partition of $X$ defined by graph $G_{t^{\prime}}^{\prime}$. Then $G_{t^{\prime}}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), P^{\prime}\right)$ such that the union of $G_{t^{\prime}}^{\prime}$ and the residual subgraph $H^{\prime}$ of $t^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. That is, ( $\left.\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O} \backslash\{u, v\}\right), \mathrm{P}^{\prime}\right)$ completes $\mathrm{H}^{\prime}$. So by the inductive assumption there exists some partition $\mathrm{Q}^{\prime}$ of X in the set $V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$ such that $\left((t, X, O \backslash\{u, v\}), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{G}^{\prime}$ is a witness for $\left((t, X, O \backslash\{u, v\}), Q^{\prime}\right)$ and (ii) $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Note that $Q^{\prime}$ is the partition of set $X$ defined by the graph $\hat{G}^{\prime}$. Suppose both $u$ and $v$ are in the same block of partition $Q^{\prime}$. Then adding the edge $u v$ to $\hat{G}^{\prime}$ does not change the partition of $X$ defined by $\hat{\mathrm{G}}^{\prime}$. It follows that the graph $\hat{\mathrm{G}}=\left(\mathrm{V}\left(\hat{\mathrm{G}}^{\prime}\right), \mathrm{E}\left(\hat{\mathrm{G}}^{\prime}\right) \cup\{\mathbf{u} v\}\right)$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $\left(\left(t, X, O, Q^{\prime}\right)\right.$ and (ii) $\hat{G} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K. Thus ( $\left(t, X, O, Q^{\prime}\right)$ completes the residual subgraph $H$. Now notice that our algorithm adds the partition $Q^{\prime}$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case.
In the remaining case, vertices $u$ and $v$ are in distinct blocks of partition $Q^{\prime}$. Let $Q$ be the partition obtained from $Q^{\prime}$ by merging together the two blocks to which vertices $u$ and $v$ belong, respectively, and leaving the other blocks as they are. Let $\hat{G}$ be defined as in the previous paragraph. Then the partition of $X$ defined by $\hat{G}$ is $Q$. It follows that $\hat{G}$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $((t, X, O, Q)$ and (ii) $\hat{G} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Thus $((t, X, O, Q)$ completes the residual subgraph H . Now notice that our algorithm adds the partition Q to the set $\mathcal{A}$. Thus the completeness criterion holds in this case as well.
3. If $\{u, v\} \cap O=\{u\}$ then for each partition $P^{\prime} \in V P\left[t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right]$,

- If vertices $u, v$ are in the same block of $\mathrm{P}^{\prime}$ then the algorithm adds $\mathrm{P}=\mathrm{P}^{\prime}$ to the set $\mathcal{A}$.
- If vertices $u, v$ are in different blocks of $\mathrm{P}^{\prime}$ then the algorithm merges these two blocks of $\mathrm{P}^{\prime}$ and adds the resulting partition P -with one fewer block than $\mathrm{P}^{\prime}$-to the set $\mathcal{A}$. In either case, by the inductive assumption we have that partition $P^{\prime}$ is valid for the combination $\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right)$. Let $G_{t^{\prime}}^{\prime \prime}$ be (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right), P^{\prime}\right)$, and let $G_{t}^{\prime}=\left(V\left(G_{t^{\prime}}^{\prime \prime}\right), E\left(G_{t^{\prime}}^{\prime \prime}\right) \cup\{u v\}\right)$ be the graph obtained from $G_{t^{\prime}}^{\prime \prime}$ by adding the edge $u v$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$. In $G_{t^{\prime}}^{\prime \prime}$ the degree of vertex $u$ is even, and the degree of vertex $v$ is odd. So in $G_{t}^{\prime}$ vertex $u$ has an odd degree, and vertex $v$ has an even degree. It is straightforward to verify that $G_{t}^{\prime}$ is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the addition of partition P to $\mathcal{A}$ preserves the soundness of $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that $((t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$ - a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $i \in\{1,2, \ldots, p\}$, $X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Note that by the definition of a residual subgraph, graph H (i) does not contain edge $u v$, and (ii) is a residual subgraph with respect to node $t^{\prime}$ as well. We consider two cases.
- Suppose edge $u v$ is not present in graph $G_{t}^{\prime}$. Then it is straightforward to verify that $G_{t}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P\right)$ as well. By the inductive hypothesis there exists some partition Q of X in the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H . This same partition Q is present in the set $\mathcal{A}$ as well.
- Suppose edge $u \boldsymbol{v}$ is present in graph $G_{t}^{\prime}$. Let $H^{\prime}=(V(H),(E(H) \cup\{u v\}))$ be the graph obtained by adding edge $u v$ to graph $H$, and let $G_{t^{\prime}}^{\prime}=\left(V\left(G_{t}^{\prime}\right),\left(E\left(G_{t}^{\prime}\right) \backslash\{u v\}\right)\right)$ be the graph obtained by deleting edge $u v$ from graph $G_{t}^{\prime}$. Then it is straightforward to verify that (i) the set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$, is exactly the set $(O \backslash\{u\}) \cup\{v\}$, (ii) $H^{\prime}$ is a residual subgraph for node $t^{\prime}$, and (iii) $G_{t^{\prime}}^{\prime}$, is a subgraph of $G_{t^{\prime}}$ such that the graph $G_{t}^{\prime} \cup H^{\prime}=G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Let $P^{\prime}$ be the partition of $X$ defined by graph $G_{t^{\prime}}^{\prime}$. Then $G_{t^{\prime}}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right), \mathrm{P}^{\prime}\right)$ such that the union of $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}$ and the residual subgraph $H^{\prime}$ of $t^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set K. That is, $\left(\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right), \mathrm{P}^{\prime}\right)$ completes $\mathrm{H}^{\prime}$. So by the inductive assumption there exists some partition $Q^{\prime}$ of $X$ in the set $V P\left[t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right]$ such that $((t, X,(O \backslash\{u\}) \cup$ $\{v\}), Q^{\prime}$ ) completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{G}^{\prime}$ is a witness for $\left((t, X,(O \backslash\{u\}) \cup\{v\}), Q^{\prime}\right)$ and (ii) $\hat{\mathrm{G}}^{\prime} \cup H^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.
Note that $\mathrm{Q}^{\prime}$ is the partition of set $X$ defined by the graph $\hat{\mathrm{G}}^{\prime}$, and that the set of odd-degree vertices in $\hat{G}^{\prime}$ is exactly the set $(O \backslash\{u\}) \cup\{v\}$. Suppose both $u$ and $v$ are in the same block of partition $Q^{\prime}$. Then adding the edge $u v$ to $\hat{G}^{\prime}$ (i) does not change the partition of $X$ defined by $\hat{\mathrm{G}}^{\prime}$, and (ii) does change the set of odd-degree vertices to O. It follows that the graph $\hat{G}=\left(V\left(\hat{G}^{\prime}\right), E\left(\hat{G}^{\prime}\right) \cup\{u v\}\right)$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $\left(\left(t, X, O, Q^{\prime}\right)\right.$ and (ii) $\hat{G} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Thus ( $\left(t, X, O, Q^{\prime}\right)$ completes the residual subgraph H. Now notice that our algorithm adds the partition $\mathrm{Q}^{\prime}$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case.
In the remaining case, vertices $u$ and $v$ are in distinct blocks of partition $\mathrm{Q}^{\prime}$. Let $Q$ be the partition obtained from $Q^{\prime}$ by merging together the two blocks to which vertices $u$ and $v$ belong, respectively, and leaving the other blocks as they are. Let $\hat{G}$ be defined as in the previous paragraph. Then the partition of $X$ defined by $\hat{G}$ is $Q$. It follows that $\hat{G}$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $((t, X, O, Q)$ and (ii) $\hat{G} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Thus ( $(t, X, O, Q)$ completes the residual subgraph H . Now notice that our algorithm adds the partition Q to the set $\mathcal{A}$. Thus the completeness criterion holds in this case as well.

4. The case when $\{u, v\} \cap O=\{v\}$ is symmetrical to the previous case, so we leave out the arguments for this case.
5. If $\{u, v\} \cap O=\emptyset$ then for each partition $P^{\prime} \in V P\left[t^{\prime}, X, O \cup\{u, v\}\right]$,
$=$ If vertices $u, v$ are in the same block of $\mathrm{P}^{\prime}$ then the algorithm adds $\mathrm{P}=\mathrm{P}^{\prime}$ to the set $\mathcal{A}$.

- If vertices $u, v$ are in different blocks of $\mathrm{P}^{\prime}$ then the algorithm merges these two blocks of $\mathrm{P}^{\prime}$ and adds the resulting partition P -with one fewer block than $\mathrm{P}^{\prime}$ - to the set $\mathcal{A}$.
In either case, by the inductive assumption we have that partition $P^{\prime}$ is valid for the combination $\left(t^{\prime}, X, O \cup\{u, v\}\right)$. Let $G_{t^{\prime}}^{\prime \prime}$ be (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X, O \cup\{u, v\}\right), P^{\prime}\right)$, and let $G_{t}^{\prime}=\left(V\left(G_{t^{\prime}}^{\prime \prime}\right), E\left(G_{t^{\prime}}^{\prime \prime}\right) \cup\{u v\}\right)$ be the graph obtained from $G_{t^{\prime}}^{\prime \prime}$ by adding the edge $u v$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$. Vertices $u, v$ have odd degrees in $G^{\prime} \prime \prime$, and hence they have even degrees in $G_{t}^{\prime}$. It is straightforward to verify that $G_{t}^{\prime}$ is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the addition of partition P to $\mathcal{A}$ preserves the soundness of $\mathcal{A}$.

Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that $((t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$ - a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $\mathfrak{i} \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Note that by the definition of a residual subgraph, graph $H$ (i) does not contain edge $u v$, and (ii) is a residual subgraph with respect to node $\mathrm{t}^{\prime}$ as well. We consider two cases.
$=$ Suppose edge $u v$ is not present in graph $G_{t}^{\prime}$. Then it is straightforward to verify that $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a witness for $\left(\left(\mathbf{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ as well. By the inductive hypothesis there exists some partition $Q$ of $X$ in the set $V P\left[t^{\prime}, X, O\right]$ such that $((t, X, O), Q)$ completes $H$. This same partition Q is present in the set $\mathcal{A}$ as well.

- Suppose edge $u v$ is present in graph $G_{t}^{\prime}$. Let $H^{\prime}=(V(H),(E(H) \cup\{u v\}))$ be the graph obtained by adding edge $u v$ to graph $H$, and let $G_{t^{\prime}}^{\prime}=\left(V\left(G_{t}^{\prime}\right),\left(E\left(G_{t}^{\prime}\right) \backslash\{u v\}\right)\right)$ be the graph obtained by deleting edge $u v$ from graph $G_{t}^{\prime}$. Then it is straightforward to verify that (i) the set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$ is exactly the set $\mathrm{O} \backslash\{u, v\}$, (ii) $H^{\prime}$ is a residual subgraph for node $t^{\prime}$, and (iii) $G_{t^{\prime}}^{\prime}$, is a subgraph of $G_{t^{\prime}}$ such that the graph $G_{t^{\prime}}^{\prime} \cup H^{\prime}=G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Let $P^{\prime}$ be the partition of $X$ defined by graph $G_{t^{\prime}}^{\prime}$. Then $G_{t^{\prime}}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), \mathrm{P}^{\prime}\right)$ such that the union of $\mathrm{G}^{\prime}$, and the residual subgraph $\mathrm{H}^{\prime}$ of $\mathrm{t}^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. That is, $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), P^{\prime}\right)$ completes $\mathrm{H}^{\prime}$. So by the inductive assumption there exists some partition $\mathrm{Q}^{\prime}$ of X in the set $V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$ such that $\left((t, X, O \backslash\{u, v\}), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{G}^{\prime}$ is a witness for $\left((t, X, O \backslash\{u, v\}), Q^{\prime}\right)$ and (ii) $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Note that $Q^{\prime}$ is the partition of set $X$ defined by the graph $\hat{G}^{\prime}$, and that the set of odd-degree vertices in $\hat{\mathbf{G}}^{\prime}$ is exactly the set $\mathbf{O} \backslash\{u, v\}$. Suppose both $u$ and $v$ are in the same block of partition $\mathrm{Q}^{\prime}$. Then adding the edge $u v$ to $\hat{\mathrm{G}}^{\prime}$ (i) does not change the partition of $X$ defined by $\hat{\mathrm{G}}^{\prime}$, and (ii) does change the set of odd-degree vertices to O . It follows that the graph $\hat{\mathrm{G}}=\left(\mathrm{V}\left(\hat{\mathrm{G}}^{\prime}\right), \mathrm{E}\left(\hat{\mathrm{G}}^{\prime}\right) \cup\{u v\}\right)$ is a subgraph of $\mathrm{G}_{\mathrm{t}}$ such that (i) $\hat{G}$ is a witness for $\left(\left(t, X, O, Q^{\prime}\right)\right.$ and (ii) $\hat{G} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K . Thus ( $\left(\mathrm{t}, \mathrm{X}, \mathrm{O}, \mathrm{Q}^{\prime}\right)$ completes the residual subgraph H. Now notice that our algorithm adds the partition $\mathrm{Q}^{\prime}$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case.
In the remaining case, vertices $u$ and $v$ are in distinct blocks of partition $Q^{\prime}$. Let $Q$ be the partition obtained from $Q^{\prime}$ by merging together the two blocks to which vertices $u$ and $v$ belong, respectively, and leaving the other blocks as they are. Let $\hat{G}$ be defined as in the previous paragraph. Then the partition of $X$ defined by $\hat{G}$ is $Q$. It follows that $\hat{G}$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $((t, X, O, Q)$ and (ii) $\hat{G} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Thus $((t, X, O, Q)$ completes the residual subgraph H . Now notice that our algorithm adds the partition Q to the set $\mathcal{A}$. Thus the completeness criterion holds in this case as well.

- Lemma 23. Let t be a forget node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $\mathrm{t}^{\prime}$ be the child node of t , and let $v$ be the vertex forgotten at t . Then $v \in \mathrm{X}_{\mathrm{t}^{\prime}}$ and $X_{t}=X_{t^{\prime}} \backslash\{v\}$, and $v \notin O$ hold. Recall that $\mathrm{P}(v)$ is the block of partition P which contains element $v$ and that $\mathrm{P}-v$ is the partition obtained by eliding $v$ from P . The algorithm initializes $\mathcal{A}=\left\{\mathrm{P}^{\prime}-v ; \mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right],\left|\mathrm{P}^{\prime}(v)\right|>1\right\}$. By the inductive assumption we have that every partition $P^{\prime} \in V P\left[t^{\prime}, X \cup\{v\}, O\right]$ is valid for the combination $\left(t^{\prime}, X \cup\{v\}, O\right)$. Note that (i) the graph $G_{t^{\prime}}$ is identical to the graph $G_{t}$, and (ii) for any subgraph $H$ of $G_{t^{\prime}}=G_{t},\left(V(H) \cap X_{t^{\prime}}\right)=X \cup\{v\}$ implies $\left(V(H) \cap X_{t}\right)=X$. It follows that if every connected component of a graph $H$ contains at least two vertices from the set $X \cup\{v\}$ then every connected component of H contains at least one vertex from set X . Using these observations it is straightforward to verify that if a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ is a witness for $\left(\left(t^{\prime}, X \cup\{v\}, O\right), P^{\prime}\right)$ where $v \notin O$ and $\left|\mathrm{P}^{\prime}(v)\right|>1$ hold, then it is also (i) a subgraph of $\mathrm{G}_{\mathrm{t}}$, and (ii) a witness for $\left((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P}^{\prime}-v\right)$. Thus for each partition $\mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right],\left|\mathrm{P}^{\prime}(v)\right|>1$ the partition $P^{\prime}-v$ is valid for the combination $(t, X, O)$. Hence directly after the initialization, all partitions in the set $\mathcal{A}$ are valid for $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.

The algorithm adds zero or more partitions to $\mathcal{A}$ depending on whether vertex $v$ is a terminal or not. We analyze each choice made by the algorithm:

1. If $v$ is a terminal vertex then the algorithm does not make further changes to $\mathcal{A}$. We have shown above that this set $\mathcal{A}$ satisfies the validity criterion. We now argue that it satisfies the completeness criterion as well.
So let $H$ be a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition Q of X such that $((t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $i \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

From the definition of a residual subgraph we know that $v \notin \mathrm{~V}(\mathrm{H})$ holds, and since $v$ is a terminal vertex, from condition (c) above we get that $v \in \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ holds. Without loss of generality, let it be the case that $v \in C_{p}$ holds. Since $X_{t^{\prime}}=X_{t} \cup\{v\}$ we get that $X_{t^{\prime}} \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X} \cup\{v\}$ holds. Let $\mathrm{H}^{\prime}=(\mathrm{V}(\mathrm{H}) \cup\{v\}, \mathrm{E}(\mathrm{H}))$ be the graph obtained by adding vertex $v$ (and no extra edges) to graph $H$. Then it is straightforward to verify that (i) $\mathrm{H}^{\prime}$ is a residual subgraph with respect to $\mathrm{t}^{\prime}$ with $\mathrm{V}\left(\mathrm{H}^{\prime}\right) \cap \mathrm{X}_{\mathrm{t}^{\prime}}=\mathrm{X} \cup\{v\}$, (ii) the graph $G_{t}^{\prime}$ is a witness for the partition $P^{\prime}=\left\{X^{1}, X^{2}, \ldots\left(X^{p} \cup\{v\}\right)\right\}$ of $X \cup\{v\}$ being valid for the combination ( $t^{\prime}, X \cup\{v\}, O$ ), and (iii) the graph $G_{t}^{\prime} \cup H^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. That is, $\left(\left(t^{\prime}, X \cup\{v\}, O\right), P^{\prime}\right)$ completes $H^{\prime}$.
By the inductive assumption there exists some partition $\mathrm{Q}^{\prime}$ of $\mathrm{X} \cup\{v\}$ in the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\right.$ $\{v\}, O]\}$ such that $\left(\left(t^{\prime}, X \cup\{v\}, O\right), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{G}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X \cup\{v\}, O\right), Q^{\prime}\right)$ and (ii) $\hat{G}^{\prime} \cup H^{\prime}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Note that $X_{t} \cap V\left(\hat{G}^{\prime}\right)=X$ holds.
Since $v$ had degree zero in graph $\mathrm{H}^{\prime}$ we get that $v$ has a positive even degree in $\hat{\mathrm{G}}^{\prime}$. From the definition of a witness for validity Definition 13 we get that $Q^{\prime}$ is the partition
of the set $X \cup\{v\}$ defined by the graph $\hat{\mathrm{G}}^{\prime}$. Let $\mathrm{Q}_{H^{\prime}}$ be the partition of the set $\mathrm{X} \cup\{v\}$ defined by the graph $H^{\prime}$. Since $\operatorname{deg}_{H^{\prime}}(v)=0$ holds we get that vertex $v$ appears in a block of size one - namely, $\{v\}$-in $\mathrm{Q}_{\mathrm{H}^{\prime}}$. If $\{v\}$ is a block of $\mathrm{Q}^{\prime}$ as well, then $\{v\}$ will also be a block in their join $\mathrm{Q}_{H^{\prime}} \sqcup \mathrm{Q}^{\prime}$. But the union of graphs $\mathrm{H}^{\prime}$ and $\hat{\mathrm{G}}^{\prime}$ is connected and so from Lemma 11 we know that $\mathrm{Q}_{\mathrm{H}^{\prime}} \sqcup \mathrm{Q}^{\prime}=\{\{\mathrm{X} \cup\{v\}\}\}$. Thus $\{v\}$ is not a block of $\mathrm{Q}_{\mathrm{H}^{\prime}} \sqcup \mathrm{Q}^{\prime}$, or of $\mathrm{Q}^{\prime}$. So there exists a vertex $v^{\prime} \in X$ such that $v, v^{\prime}$ are in the same block of $\mathrm{Q}^{\prime}$. In particular, this implies that the partition $\mathrm{Q}=\mathrm{Q}^{\prime}-v$, which is the partition of set X defined by graph $\hat{G}^{\prime}$, has exactly as many blocks as has the partition $Q^{\prime}$ of $X \cup\{v\}$.
Putting these together we get that the subgraph $\hat{G}^{\prime}$ of $G_{t}$ is a witness for $((t, X, O), Q=$ $\mathrm{Q}^{\prime}-v$ ). Now since graph H can be obtained from graph $\mathrm{H}^{\prime}$ by deleting vertex $v$, we get that the graphs $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ and $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}$ are identical. In particular, the latter is an Eulerian Steiner subgraph of $G$ for the terminal set K . Thus ( $(\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes the residual graph H . Since the algorithm adds partition Q to the set $\mathcal{A}$, we get that $\mathcal{A}$ satisfies the completeness criterion.
2. If $v$ is not a terminal vertex then the algorithm adds all the partitions from $V P\left[t^{\prime}, X, O\right]$ to $\mathcal{A}$. By the inductive assumption we have that every partition $\mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is valid for the combination $\left(t^{\prime}, X, O\right)$. It is once again straightforward to verify that if a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P^{\prime}\right)$ then it is also (i) a subgraph of $G_{t}$, and (ii) a witness for $\left((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P}^{\prime}\right)$. Thus each partition $\mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). Hence all partitions added to the set $\mathcal{A}$ in this step are valid for ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ).
We now argue that the set $\mathcal{A}$ satisfies the completeness criterion. So let H be a residual subgraph with respect to $t$ with $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}-a$ witness for $((t, X, O), P)$ such that the following hold:
a. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $\mathfrak{i} \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set K.

Suppose graph $\mathrm{G}_{\mathrm{t}}^{\prime}$ does not contain vertex $v$. Then it is easy to verify that H is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=X$, and that graph $G_{t}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P\right)$ such that the union of graphs $H$ and $G_{t}^{\prime}$ is an Eulerian Steiner subgraph of G for the terminal set K . That is, $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ completes H . By inductive assumption there exists a partition $Q \in V P\left[t^{\prime}, X, O\right]$ such that $\left(\left(t^{\prime}, X, O\right), Q\right)$ completes $H$. Since the algorithm adds this partition Q to $\mathcal{A}$ we get that $\mathcal{A}$ satisfies the completeness criterion in this case.
Now suppose graph $G_{t}^{\prime}$ contains vertex $v$. The analysis from the case where $v$ was a terminal and was thus forced to be in graph $\mathrm{G}_{\mathrm{t}}^{\prime}$, applies verbatim in this case. Note that the set $\mathcal{A}$ in the present case is a superset of the set $\mathcal{A}$ computed in that case. Thus we get that the current set $\mathcal{A}$ satisfies the completeness criterion.

- Lemma 24. Let t be a join node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq X_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the $D P$ for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $t_{1}, t_{2}$ be the children of $t$. Then $X_{t}=X_{t_{1}}=X_{t_{2}}$. Note that $V\left(G_{t}\right)=V\left(G_{t_{1}}\right) \cup$ $V\left(G_{t_{2}}\right)$ and $E\left(G_{t}\right)=E\left(G_{t_{1}}\right) \cup E\left(G_{t_{2}}\right)$ hold, and so graph $G_{t}$ is the union of graphs $G_{t_{1}}$ and $G_{t_{2}}$. Further, since each edge in the graph is introduced at exactly one bag in $\mathcal{T}$ we get that $E\left(G_{t_{1}}\right) \cap E\left(G_{t_{2}}\right)=\emptyset$ holds. Moreover, $V\left(G_{t_{1}}\right) \cap V\left(G_{t_{2}}\right)=X_{t}$ holds as well. The algorithm initializes $\mathcal{A}$ to the empty set. For each way of dividing set O into two disjoint subsets $\mathrm{O}_{1}, \mathrm{O}_{2}$ (one of which could be empty) and for each subset $\hat{O}$ (which could also be empty) of the set $X \backslash O$, the algorithm picks a number of pairs ( $P_{1}, P_{2}$ ) of partitions and adds their joins $P_{1} \sqcup P_{2}$ to the set $\mathcal{A}$. We first show that the partition $P_{1} \sqcup P_{2}$ is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ), for each choice of pairs $\left(\mathrm{P}_{1}, \mathrm{P}_{2}\right)$ made by the algorithm.

So let $P_{1} \in V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], P_{2} \in V P\left[t_{2}, X, O_{2} \cup \hat{O}\right]$. By the inductive hypothesis we get that $P_{1}$ is valid for the combination $\left(t_{1}, X, O_{1} \cup \hat{O}\right)$ and $P_{2}$ is valid for the combination $\left(t_{2}, X, O_{2} \cup \hat{O}\right)$. So there exist subgraphs $G_{t_{1}}^{\prime}=\left(V_{t_{1}}^{\prime}, E_{t_{1}}^{\prime}\right)$ of $G_{t_{1}}$ and $G_{t_{2}}^{\prime}=\left(V_{t_{2}}^{\prime}, E_{t_{2}}^{\prime}\right)$ of $G_{t_{2}}$ such that

1. $\mathrm{X}_{\mathrm{t}} \cap \mathrm{V}_{\mathrm{t}_{1}}^{\prime}=\mathrm{X}=\mathrm{X}_{\mathrm{t}} \cap \mathrm{V}_{\mathrm{t}_{2}}^{\prime}$;
2. The vertex set of each connected component of $G_{t_{1}}^{\prime}$ and of $G_{t_{2}}^{\prime}$ has a non-empty intersection with set $X$. Moreover, $P_{1}$ is the partition of $X$ defined by the subgraph $G_{t_{1}}^{\prime}$ and $P_{2}$ is the partition of $X$ defined by the subgraph $G_{t_{2}}^{\prime}$;
3. Every terminal vertex from $K \cap V\left(G_{t_{1}}\right)$ is in $V_{t_{1}}^{\prime}$ and every terminal vertex from $K \cap V\left(G_{t_{2}}\right)$ is in $V_{\mathrm{t}_{2}}^{\prime}$; and,
4. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}_{1}}^{\prime}$ is exactly the set $\mathrm{O}_{1} \cup \hat{O}$ and the set of odd-degree vertices in $G_{t_{2}}^{\prime}$ is exactly the set $\mathrm{O}_{2} \cup \hat{O}$.
Let $G_{t}^{\prime}=G_{t_{1}}^{\prime} \cup G_{t_{2}}^{\prime}$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$, and
5. Since $X_{t} \cap V_{t_{1}}^{\prime}=X=X_{t} \cap V_{t_{2}}^{\prime}$ holds we have that $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$ holds as well;
6. The vertex set of each connected component of $G_{t}^{\prime}$ has a non-empty intersection with set $X$. Moreover, from Lemma 11 we get that $P_{1} \sqcup P_{2}$ is the partition of $X$ defined by the subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$;
7. Every terminal vertex from the set $K \cap V\left(G_{t}\right)$ is in $V\left(G_{t}^{\prime}\right)$; and,
8. Since $E\left(G_{t_{1}}\right) \cap E\left(G_{t_{2}}\right)=\emptyset$ holds we get that the degree of any vertex $v$ in graph $G_{t}^{\prime}$ is the sum of its degrees in the two graphs $G_{t_{1}}^{\prime}$ and $G_{t_{2}}^{\prime}$. Since (i) the set of odd-degree vertices in graph $G_{\mathfrak{t}_{1}}^{\prime}$ is exactly the set $\mathrm{O}_{1} \cup \hat{O}$, (ii) the set of odd-degree vertices in graph $\mathrm{G}_{\mathrm{t}_{2}}^{\prime}$ is exactly the set $\mathrm{O}_{2} \cup \hat{\mathrm{O}}$, and (iii) O is the disjoint union of sets $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$, we get that the set of odd-degree vertices in graph $G_{t}^{\prime}$ is exactly the set $O$.
Thus graph $G_{t}^{\prime}$ is a witness for partition $P_{1} \sqcup P_{2}$ being valid for the combination ( $t, X, O$ ), and so partition $P_{1} \sqcup P_{2} \in \mathcal{A}$ is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). This proves that collection $\mathcal{A}$ satisfies the soundness criterion.

We now argue that the set $\mathcal{A}$ satisfies the completeness criterion. So let H be a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$, for which there exists a partition $P=$ $\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that $((t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:

1. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
2. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $i \in\{1,2, \ldots, p\}$, $X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
3. Every terminal vertex from $K \cap V_{t}$ is in $V\left(G_{t}^{\prime}\right)$.
4. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
5. The graph $G_{t}^{\prime} \cup H$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Let $\mathrm{G}_{1}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}_{1}}\right), \mathrm{E}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{E}\left(\mathrm{G}_{\mathrm{t}_{1}}\right)\right)$ and $\mathrm{G}_{2}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}_{2}}\right), \mathrm{E}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{E}\left(\mathrm{G}_{\mathrm{t}_{2}}\right)\right)$ be, respectively, the subgraphs of $G_{t}^{\prime}$ defined by the subtrees of $\mathcal{T}$ rooted at nodes $t_{1}$ and $t_{2}$, respectively. Then $\mathrm{G}_{\mathrm{t}}^{\prime}=\mathrm{G}_{1} \cup \mathrm{G}_{2}, \mathrm{~V}\left(\mathrm{G}_{1}\right) \cap \mathrm{X}_{\mathrm{t}_{1}}=\mathrm{V}\left(\mathrm{G}_{2}\right) \cap \mathrm{X}_{\mathrm{t}_{2}}=\mathrm{V}\left(\mathrm{G}_{1}\right) \cap \mathrm{V}\left(\mathrm{G}_{2}\right)=\mathrm{X}$, and $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\emptyset$ all hold. Let $\tilde{O}_{1}, \tilde{O_{2}}$ be the sets of vertices of odd degree in graphs $G_{1}, G_{2}$, respectively. Since graph $\left(H \cup G_{1}\right) \cup G_{2}$ is Eulerian and since $V\left(H \cup G_{1}\right) \cap V\left(G_{2}\right)=X$ holds, we get that (i) $\tilde{O_{2}} \subseteq X$ holds, and (ii) every connected component of graph $G_{2}$ contains at least one vertex from set $X$. By symmetric reasoning we get that (i) $\tilde{O}_{1} \subseteq X$ holds, and (ii) every connected component of graph $\mathrm{G}_{1}$ contains at least one vertex from set X . Let $\mathrm{O}_{2}=\tilde{\mathrm{O}}_{2} \cap \mathrm{O}$ and $\hat{\mathrm{O}}=\tilde{\mathrm{O}}_{2} \backslash \mathrm{O}$. Then $\tilde{\mathrm{O}}_{2}=\mathrm{O}_{2} \cup \hat{\mathrm{O}}$. Define $\mathrm{O}_{1}=\mathrm{O} \backslash \mathrm{O}_{2}$. Since (i) the set of odd-degree vertices in graph $G_{t}^{\prime}$ is exactly the set $O$, and (ii) $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\emptyset$ holds, we get that the set of odd-degree vertices in graph $\mathrm{G}_{1}$ is $\tilde{\mathrm{O}}_{1}=\left(\mathrm{O} \backslash \mathrm{O}_{2}\right) \cup \hat{O}=\mathrm{O}_{1} \cup \hat{O}$.

Let $Q_{2}$ be the partition of set $X$ defined by graph $G_{2}$, and let $R_{1}=H \cup G_{1}$. It is straightforward to verify the following: (i) $R_{1}$ is a residual subgraph with respect to node $t_{2}$ with $V\left(R_{1}\right) \cap X_{t_{2}}=X$; (ii) graph $G_{2}$ is a witness for partition $Q_{2}$ being valid for the combination $\left(t_{2}, X, \tilde{O}_{2}\right)$, and (iii) $G_{2}$ is a certificate for ( $\left.\left(t_{2}, X, \tilde{O}_{2}\right), Q_{2}\right)$ completing the residual graph $R_{1}$. By the inductive assumption there is a partition $P_{2}$ of $X$ in the set $\mathrm{VP}\left[\mathrm{t}_{2}, \mathrm{X}, \mathrm{O}_{2} \cup \hat{O}\right]$ such that $\left(\left(\mathrm{t}_{2}, \mathrm{X}, \mathrm{O}_{2} \cup \hat{O}\right), \mathrm{P}_{2}\right)$ completes the residual graph $\mathrm{R}_{1}$. Let $H_{2}$ be a certificate for $\left(\left(t_{2}, X, O_{2} \cup \hat{O}\right), P_{2}\right)$ completing $R_{1}$. Note that $H_{2}$ is a subgraph of $G_{t_{2}}$, and that $R_{1} \cup H_{2}=\left(H \cup G_{1}\right) \cup H_{2}$ is an Eulerian Steiner subgraph of $G$ for the terminal set K.

Let $Q_{1}$ be the partition of set $X$ defined by graph $G_{1}$, and let $R_{2}=H \cup H_{2}$. From Lemma 15 we get that the set of odd-degree vertices of the residual subgraph H is exactly the set O, and from Definitions 13 and 14 we get that the set of odd-degree vertices of graph $\mathrm{H}_{2}$ is the set $\mathrm{O}_{2} \cup \hat{O}$. From the definition of a residual subgraph we get that $E(H) \cap E\left(H_{2}\right)=\emptyset$ holds. It follows that the set of odd-degree vertices of graph $R_{2}$ is $\left(O \backslash O_{2}\right) \cup \hat{O}=O_{1} \cup \hat{O}$, which is exactly the set of odd-degree vertices of graph $\mathrm{G}_{1}$.

It is now straightforward to verify the following: (i) $R_{2}$ is a residual subgraph with respect to node $t_{1}$ with $V\left(R_{2}\right) \cap X_{t_{1}}=X$; (ii) graph $G_{1}$ is a witness for partition $Q_{1}$ being valid for the combination $\left(t_{1}, X, O_{1} \cup \hat{O}\right)$, and (iii) $G_{1}$ is a certificate for ( $\left.\left(t_{1}, X, O_{1} \cup \hat{O}\right), Q_{1}\right)$ completing the residual graph $R_{2}$. By the inductive assumption there is a partition $P_{1}$ of $X$ in the set $\mathrm{VP}\left[\mathrm{t}_{1}, \mathrm{X}, \mathrm{O}_{1} \cup \hat{O}\right]$ such that $\left(\left(\mathrm{t}_{1}, \mathrm{X}, \mathrm{O}_{1} \cup \hat{O}\right), \mathrm{P}_{1}\right)$ completes the residual graph $\mathrm{R}_{2}$. Let $\mathrm{H}_{1}$ be a certificate for $\left(\left(t_{1}, X, O_{1} \cup \hat{O}\right), P_{1}\right)$ completing $R_{2}$. Note that $H_{1}$ is a subgraph of $G_{t_{1}}$, and that $R_{2} \cup H_{1}=\left(H \cup H_{2}\right) \cup H_{1}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$.

Let $\hat{H}=H_{1} \cup H_{2}$. Then $\hat{H}$ is a subgraph of $G_{t}$, and

1. Since $X_{t} \cap V\left(H_{1}\right)=X=X_{t} \cap V\left(H_{2}\right)$ holds we have that $X_{t} \cap V(\hat{H})=X$ holds as well;
2. The vertex set of each connected component of $\hat{H}$ has a non-empty intersection with set $X$. Moreover, from Lemma 11 we get that $P_{1} \sqcup P_{2}$ is the partition of $X$ defined by the subgraph $\hat{H}$;
3. Every terminal vertex from the set $K \cap V\left(G_{t}\right)$ is in $V(\hat{H})$; and,
4. Since $E\left(G_{t_{1}}\right) \cap E\left(G_{t_{2}}\right)=\emptyset$ holds we get that the degree of any vertex $v$ in graph $\hat{H}$ is the sum of its degrees in the two graphs $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$. Since (i) the set of odd-degree vertices in graph $\mathrm{H}_{1}$ is exactly the set $\mathrm{O}_{1} \cup \hat{\mathrm{O}}$, (ii) the set of odd-degree vertices in graph $\mathrm{H}_{2}$ is exactly the set $\mathrm{O}_{2} \cup \hat{\mathrm{O}}$, and (iii) O is the disjoint union of sets $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$, we get that the set of odd-degree vertices in graph $\hat{\mathrm{H}}$ is exactly the set O .
Graph $\hat{H}$ is thus a witness for partition $P_{1} \sqcup P_{2}$ of $X$ being valid for the combination ( $t, X, O$ ), and $H \cup \hat{H}$ is an Eulerian Steiner subgraph of $G$ for the terminal set $K$. Thus $\left((t, X, O), P_{1} \sqcup P_{2}\right)$ completes $H$. Since the algorithm adds partition $\mathrm{P}_{1} \sqcup \mathrm{P}_{2}$ to the set $\mathcal{A}$ we get that $\mathcal{A}$ satisfies the completeness criterion.

We can now prove

- Theorem 5. There is an algorithm which solves an instance ( $\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw}$ ) of Eulerian Steiner Subgraph in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\text {tw }}\right)$ time.

Proof. We first modify $\mathcal{T}$ to make it a "nearly-nice" tree decomposition rooted at r as described at the start of this section. We then execute the dynamic programming steps described above on $\mathcal{T}$. We return yes if the element $\left\{\left\{v^{\star}\right\}\right\}$ is present in the set $\mathrm{VP}[\mathrm{r}, \mathrm{X}=$ $\left.\left\{v^{\star}\right\}, \mathrm{O}=\emptyset\right]$ computed by the DP, and no otherwise.

From Lemma 18 we know that ( $\mathrm{G}, \mathrm{K}, \mathcal{T}, \mathrm{tw}$ ) is a yes instance of Eulerian Steiner SUbGRAPH if and only if the combination ( $\left.\left(\mathrm{r}, \mathrm{X}=\left\{\nu^{\star}\right\}, \mathrm{O}=\emptyset\right), \mathrm{P}=\left\{\left\{\nu^{\star}\right\}\right\}\right)$ completes the residual graph $\mathrm{H}=\left(\left\{\nu^{\star}\right\}, \emptyset\right)$. By induction on the structure of the tree decomposition $\mathcal{T}$ and using Observation 19 and Lemmas 20, 21, 22, 23, and 24 we get that the set $\mathrm{VP}[\mathrm{r}, \mathrm{X}=$ $\left.\left\{v^{\star}\right\}, \mathrm{O}=\emptyset\right]$ computed by the algorithm satisfies the correctness criteria. And since $\left\{\left\{v^{\star}\right\}\right\}$ is the unique partition of set $\left\{v^{\star}\right\}$ we get that the set $\mathrm{VP}\left[\mathrm{r}, \mathrm{X}=\left\{\nu^{\star}\right\}, \mathrm{O}=\emptyset\right]$ computed by the algorithm will contain the partition $\left\{\left\{\nu^{\star}\right\}\right\}$ if and only if ( $G, K, \mathcal{T}, \mathrm{tw}$ ) is a yes instance of Eulerian Steiner Subgraph.

Note that we compute representative subsets as the last step in the computation at each bag. So we get, while performing computations at an intermediate node $t$, that the number of partitions in any set VP[t $\left.t^{\prime}, X^{\prime}, \cdot\right]$ for any child node $t^{\prime}$ of $t$ and subset $X^{\prime}$ of $X_{t^{\prime}}$ is at most $2^{\left(\left|X^{\prime}\right|-1\right)}$ (See Theorem 12. We use Theorem 10 to perform various operations on one or two partitions - such as adding a block to a partition, merging two blocks of a partition, eliding an element from a partition, or computing the join of two partitions-in polynomial time.

The computation at each leaf node of $\mathcal{T}$ can be done in constant time. For an introduce vertex node or an introduce edge node or a forget node $t$ and a fixed pair of subsets $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, the computation of set $\mathcal{A}$ involves-in the worst case-spending polynomial time for each partition $\mathrm{P}^{\prime}$ in some set $V P\left[\mathrm{t}^{\prime}, \mathrm{X}^{\prime} \subseteq X, \cdot\right]$. Since the number of partitions in this latter set is at most $2^{\left(\left|X^{\prime}\right|-1\right)} \leqslant 2^{(|X|-1)}$ we get that the set $\mathcal{A}$ can be computed in $\mathcal{O}^{\star}\left(2^{(|X|-1)}\right)$ time, and that the set $\mathcal{B}$ can be computed-see Theorem 12 in $\mathcal{O}^{\star}\left(2^{(|X|-1)} \cdot 2^{(\omega-1) \cdot|X|}\right)=$ $\mathcal{O}^{\star}\left(2^{\omega \cdot|\mathrm{X}|}\right)$ time. Since the number of ways of choosing the subset $O \subseteq X$ is $2^{|\mathrm{X\mid}|}$ the entire computation at an introduce vertex, introduce edge, or forget node $t$ can be done in time

$$
\begin{aligned}
\sum_{|X|=0}^{\left|X_{t}\right|}\binom{\left|X_{t}\right|}{|X|} 2^{|X|} \mathcal{O}^{\star}\left(2^{\omega \cdot|X|}\right) & =\mathcal{O}^{\star}\left(\sum_{|X|=0}^{t w+1}\binom{t w+1}{|X|} 2^{(\omega+1)|X|}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2^{(\omega+1)}\right)^{(t w+1)}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2 \cdot 2^{\omega}\right)^{\mathrm{t} w}\right)
\end{aligned}
$$

For a join node $t$ and a fixed subset $X \subseteq X_{t}$ we guess three pairwise disjoint subsets $\hat{O}, O_{1}, O_{2}$ of $X$ in time $4^{|X|}$. For each guess we go over all partitions $P_{1} \in V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], P_{2} \in$ $\mathrm{VP}\left[\mathrm{t}_{2}, \mathrm{X}, \mathrm{O}_{2} \cup \hat{\mathrm{O}}\right]$ and add their join $\mathrm{P}_{1} \sqcup \mathrm{P}_{2}$ to the set $\mathcal{A}$. Since the number of partitions in each of the two sets $V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], V P\left[t_{2}, X, O_{2} \cup \hat{O}\right]$ is at most $2^{(|X|-1)}$, the size of set $\mathcal{A}$ is at most $2^{(2|X|-2)}$. The entire computation at the join node can be done in time

$$
\begin{aligned}
\sum_{|X|=0}^{\left|X_{\mathrm{t}}\right|}\binom{\left|\mathrm{X}_{\mathrm{t}}\right|}{|\mathrm{X}|} 4^{|\mathrm{X}|}\left(2^{(2|\mathrm{X}|-2)}+\mathcal{O}^{\star}\left(2^{(2|\mathrm{X}|-2)} \cdot 2^{(\omega-1) \cdot|\mathrm{X}|}\right)\right) & =\mathcal{O}^{\star}\left(\sum_{|\mathrm{X}|=0}^{\mathrm{t} w+1}\binom{\mathrm{t} w+1}{|\mathrm{X}|} 2^{4|\mathrm{X}|-2+\omega|\mathrm{X}|-|\mathrm{X}|}\right) \\
& =\mathcal{O}^{\star}\left(\sum_{|\mathrm{X}|=0}^{\mathrm{tw+1}}\binom{\mathrm{t} w+1}{|\mathrm{X}|} 2^{(\omega+3)|\mathrm{X}|}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{(\mathrm{t} w+1)}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{t} w}\right) .
\end{aligned}
$$

The entire DP over $\mathcal{T}$ can thus be done in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}}\right)$ time.

## 4 Finding the Hamiltonian Index

In this section we prove Theorem 3 we describe an algorithm which takes an instance ( $G, \mathcal{T}, \mathrm{tw}, \mathrm{r}$ ) of Hamiltonian Index as input and outputs in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}}\right)$ time whether graph $G$ has Hamiltonian Index at most $r$. If $r \geqslant(|V(G)|-3)$ holds then our algorithm returns yes. If $\mathrm{r}<(|\mathrm{V}(\mathrm{G})|-3)$ then it checks, for each $\mathfrak{i}=0,1, \ldots, \mathrm{r}$ in increasing order, whether $h(G)=i$ holds. From Theorem 2 we know that this procedure correctly solves Hamiltonian Index. We now describe how we check if $h(G)=i$ holds for increasing values of $i$. For $i=0$ we apply an algorithm of Bodlaender et al., and for $i=1$ we leverage a classical result of Harary and Nash-Williams.

- Theorem 25. 5 There is an algorithm which takes a graph G and a tree decomposition of G of width tw as input, runs in $\mathcal{O}^{\star}\left(\left(5+2^{(\omega+2) / 2}\right)^{\mathrm{tw}}\right)$ time, and tells whether G is Hamiltonian.
- Theorem 26. 20 Let G be a connected graph with at least three edges. Then $\mathrm{L}(\mathrm{G})$ is Hamiltonian if and only if G has a dominating Eulerian subgraph.

For checking if $h(G) \in\{2,3\}$ holds we make use of a structural result of Hong et al. 22 . For a connected subgraph $H$ of graph $G$ the contraction $G / H$ is the graph obtained from $G$ by replacing all of $\mathrm{V}(\mathrm{H})$ with a single vertex $\nu_{\mathrm{H}}$ and adding edges between $\nu_{\mathrm{H}}$ and $\mathrm{V}(\mathrm{G}) \backslash \mathrm{V}(\mathrm{H})$ such that the number of edges in $\mathrm{G} / \mathrm{H}$ between $\nu_{\mathrm{H}}$ and any vertex $v \in \mathrm{~V}(\mathrm{G}) \backslash \mathrm{V}(\mathrm{H})$ is equal to the number of edges in $G$ with one end point at $v$ and the other in $V(H)$. Note that the graph $\mathrm{G} / \mathrm{H}$ is, in general, a multigraph with multiedges incident on $\nu_{\mathrm{H}}$. Let $\mathrm{V}_{2}$ be the set of all vertices of $G$ of degree two, and let $\hat{V}=V(G) \backslash V_{2}$. A lane of $G$ is either (i) a path whose end-vertices are in $\hat{V}$ and internal vertices (if any) are in $V_{2}$, or (ii) a cycle which contains exactly one vertex from $\hat{V}$. The length of a lane is the number of edges in the lane. An end-lane is a lane which has a degree-one vertex of $G$ as an end-vertex.

For $\mathfrak{i} \in\{2,3\}$ let $U_{i}$ be the union of lanes of length less than $i$. Let $C_{1}^{i}, C_{2}^{i}, \ldots, C_{p_{i}}^{i}$ be the connected components of $G[\hat{V}] \cup U_{i}$. Then each $C_{j}^{i}$ consists of components of $G[\hat{V}]$ connected by lanes of length less than $i$. Let $\mathrm{H}^{(i)}$ be the graph obtained from $G$ by contracting each of the connected subgraphs $C_{1}^{i}, C_{2}^{i}, \ldots, C_{p_{i}}^{i}$ to a distinct vertex. Let $D_{j}^{i}$ denote the vertex of $H^{(i)}$ obtained by contracting subgraph $C_{j}^{i}$ of $G$. Let $\tilde{H}^{(i)}$ be the graph obtained from $H^{(i)}$ by these steps:

1. Delete all lanes beginning and ending at the same vertex $D_{j}^{i}$.
2. If there are two vertices $\mathrm{D}_{\mathrm{j}}^{i}$, $\mathrm{D}_{\mathrm{k}}^{i}$ in $\mathrm{H}^{(i)}$ which are connected by $\ell_{1}$ lanes of length at least $\mathfrak{i}+2$ and $\ell_{2}$ lanes of length $\mathfrak{i}$ or $\mathfrak{i}+1$ such that $\ell_{1}+\ell_{2} \geqslant 3$ holds, then delete an arbitrary subset of these lanes such that there remain $\ell_{3}$ lanes with length at least $\mathfrak{i}+2$ and $\ell_{4}$ lanes of length $\mathfrak{i}$ or $\mathfrak{i}+1$, where

$$
\left(\ell_{3}, \ell_{4}\right)= \begin{cases}(2,0) & \text { if } \ell_{1} \text { is even and } \ell_{2}=0 \\ (1,0) & \text { if } \ell_{1} \text { is odd and } \ell_{2}=0 \\ (1,1) & \text { if } \ell_{2}=1 \\ (0,2) & \text { if } \ell_{2} \geqslant 2\end{cases}
$$

3. Delete all end-lanes of length $\mathfrak{i}$, and replace each lane of length $\mathfrak{i}$ or $\mathfrak{i}+1$ by a single edge.

- Theorem 27. [22, See Theorem 3] Let G be a connected graph with $\mathrm{h}(\mathrm{G}) \geqslant 2$ and with at least one vertex of degree at least three, and let $\tilde{\mathrm{H}}^{(2)}, \tilde{\mathrm{H}}^{(3)}$ be graphs constructed from G as described above. Then
- $\mathrm{h}(\mathrm{G})=2$ if and only if $\tilde{\mathrm{H}}^{(2)}$ has a spanning Eulerian subgraph; and
- $\mathrm{h}(\mathrm{G})=3$ if and only if $\mathrm{h}(\mathrm{G}) \neq 2$ and $\tilde{\mathrm{H}}^{(3)}$ has a spanning Eulerian subgraph.

For checking if $h(G)=i$ holds for $i \in\{4,5, \ldots\}$ we appeal to a reduction due to Xiong and Liu 44]. Let $\mathcal{L}=\left\{\mathrm{L}_{1}, \mathrm{~L}_{2}, \ldots, \mathrm{~L}_{\mathrm{t}}\right\}$ be a set of lanes (called branches in 44]) in G, each of length at least 2. A contraction of $G$ by $\mathcal{L}$, denoted $G / / \mathcal{L}$, is a graph obtained from $G$ by contracting one edge of each lane in $\mathcal{L}$. Note that $G / / \mathcal{L}$ is not, in general, unique.

- Theorem 28. [44, Theorem 20] Let G be a connected graph with $\mathrm{h}(\mathrm{G}) \geqslant 4$ and let $\mathcal{L}$ be the set of all lanes of length at least 2 in G . Then $\mathrm{h}(\mathrm{G})=\mathrm{h}(\mathrm{G} / / \mathcal{L})+1$.

We can now prove

- Theorem3. There is an algorithm which solves an instance ( $\mathrm{G}, \mathcal{T}, \mathrm{tw}, \mathrm{r}$ ) of Hamiltonian INDEX in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{t w}\right)$ time.
Proof. We first apply Theorem 25 to check if G is Hamiltonian. If G is Hamiltonian then we return yes. If G is not Hamiltonian and $\mathrm{r}=0$ holds then we return no. Otherwise we apply Theorem 6 and Theorem 26 to check if $\mathrm{L}(\mathrm{G})$ is Hamiltonian. If $\mathrm{L}(\mathrm{G})$ is Hamiltonian then we return yes. If $\mathrm{L}(\mathrm{G})$ is not Hamiltonian and $\mathrm{r}=1$ holds then we return no.

At this point we know-since $G$ is connected, is not a path, and is not Hamiltonian-that $G$ has at least one vertex of degree at least three, and that $h(G) \geqslant 2$ holds. We construct the graph $\tilde{\mathrm{H}}^{(2)}$ of Theorem 27 and use Corollary 7 to check if $\tilde{\mathrm{H}}^{(2)}$ has a spanning Eulerian subgraph. If it does then we return yes. If it does not and $r=2$ holds then we return no. Otherwise we construct the graph $\tilde{\mathrm{H}}^{(3)}$ of Theorem 27 and use Corollary 7 to check if $\tilde{\mathrm{H}}^{(3)}$ has a spanning Eulerian subgraph. If it does then we return yes. If it does not and $r=3$ holds then we return no.

At this point we know that $h(G) \geqslant 4$ holds. We compute the set $\mathcal{L}$ of all lanes of $G$ of length at least 2 , and a contraction $\mathrm{G}^{\prime}=\mathrm{G} / / \mathcal{L}$. We construct a tree decomposition $\mathfrak{T}^{\prime}$ of $\mathrm{G}^{\prime}$ from $\mathcal{T}$ as follows: For each edge $x y$ of $G$ which is contracted to get $G^{\prime}$, we introduce a new vertex $v_{x y}$ to each bag of $\mathcal{T}$ which contains at least one of $\{x, y\}$. We now delete vertices $x$ and $y$ from all bags. It is easy to verify that the resulting structure $\mathcal{T}^{\prime}$ is a tree decomposition of $\mathrm{G}^{\prime}$, of width $\mathrm{t} w^{\prime} \leqslant \mathrm{tw}$. We now recursively invoke the algorithm on the instance $\left(\mathrm{G}^{\prime}, \mathfrak{T}^{\prime}, \mathrm{tw} \boldsymbol{w}^{\prime},(\mathrm{r}-1)\right.$ ) and return its return value (yes or no).

The correctness of this algorithm follows from Theorem 25, Theorem 6, Theorem 26, Theorem 27, Corollary 7 and Theorem 28. As for the running time, checking Hamiltonicity takes $\mathcal{O}^{\star}\left(\left(5+2^{(\omega+2) / 2}\right)^{\mathrm{tw}}\right)$ time Theorem 25). Checking if $\mathrm{L}(\mathrm{G})$ is Hamiltonian takes $\mathcal{O}^{\star}((1+$ $\left.2^{(\omega+3)}\right)^{t w}$ ) time Theorem 6 Theorem 26). The graphs $\tilde{\mathrm{H}}^{(2)}$ and $\tilde{\mathrm{H}}^{(3)}$ of Theorem 27 can each be constructed in polynomial time, and checking if each has a spanning Eulerian subgraph takes $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\text {tw }}\right)$ time Corollary 7 . The graph $\mathrm{G}^{\prime}$ and its tree decomposition $\mathcal{T}^{\prime}$ of width $t w^{\prime}$ can be constructed in polynomial time. Given that $5+2^{(\omega+2) / 2}<1+2^{(\omega+3)}$ and $\mathrm{t} w^{\prime} \leqslant \mathrm{t} w$ hold, we get that the running time of the algorithm satisfies the recurrence $T(r)=\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{t w}\right)+T(r-1)$. Since we recurse only if $r<|V(G)|-3$ holds we get that the recurrence resolves to $T(r)=\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{t w}\right)$.

## 5 Conclusion

The Hamiltonian $\operatorname{Index} h(G)$ of a graph G is a generalization of the notion of Hamiltonicity. It was introduced by Chartrand in 1968, and has received a lot of attention from graph
theorists over the years. It is known to be NP-hard to check if $h(G)=t$ holds for any fixed integer $t \geqslant 0$, even for subcubic graphs $G$. We initiate the parameterized complexity analysis of the problem of finding $h(G)$ with the treewidth $t w(G)$ of $G$ as the parameter. We show that this problem is FPT and can be solved in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}(\mathrm{G})}\right)$ time. This running time matches that of the current fastest algorithm, due to Misra et al. [32], for checking if $h(G)=1$ holds. We also derive an algorithm of our own, with the same running time, for checking if $h(G)=1$ holds. A key ingredient of our solution for finding $h(G)$ is an algorithm which solves the Eulerian Steiner Subgraph problem in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}(\mathrm{G})}\right)$ time. This is - to the best of our knowledge - the first FPT algorithm for this problem, and it subsumes known algorithms for the special case of Spanning Eulerian Subgraph in series-parallel graphs and planar graphs. We note in passing that it is not clear that the algorithm of Misra et al. for solving LELP can be adapted to check for larger values of $h(G)$. We believe that our FPT result on Eulerian Steiner Subgraph could turn out to be useful for solving other problems as well.

Two different approaches to checking if $h(G)=1$ holds-Misra et al.'s approach via LELP and our solution using Dominating Eulerian Subgraph-both run in $\mathcal{O}^{\star}((1+$ $\left.\left.2^{(\omega+3)}\right)^{t w(G)}\right)$ time. Does this suggest the existence of a matching lower bound, or can this be improved? More generally, can $h(G)$ be found in the same FPT running time as it takes to check if G is Hamiltonian (currently: $\mathcal{O}^{\star}\left(\left(5+2^{(\omega+2) / 2}\right)^{\mathrm{tw}(\mathrm{G})}\right)$ due to Bodlaender et al.)? Since $\operatorname{tw}(G) \leqslant(|V(G)|-1)$ our algorithm implies an $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{|V(G)|}\right)$-time exact exponential algorithm for finding $h(G)$. We ask if this can be improved, as a first step, to the classical $\mathcal{O}^{\star}\left(2^{|V(G)|}\right)$ bound for Hamiltonicity.

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## XX:30 On Computing the Hamiltonian Index of Graphs

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## A An FPT Algorithm for Dominating Eulerian Subgraph

In this section we derive an alternate algorithm for

## Dominating Eulerian Subgraph (DES) <br> Parameter: t $w$

Input: An undirected graph $G=(V, E)$ and a tree decomposition $\mathcal{T}=\left(T,\left\{X_{t}\right\}_{t \in V(T)}\right)$ of G , of width tw .
Question: Does there exist an Eulerian subgraph $G^{\prime}$ of $G$ such that $V\left(G^{\prime}\right)$ contains a vertex cover of G?
Following established terminology we call such a subgraph $\mathrm{G}^{\prime}$ a dominating Eulerian subgraph of G. Note that the word "dominating" here denotes the existence of a vertex cover, and not of a dominating set.

- Theorem 29. There is an algorithm which solves an instance ( $G, \mathcal{T}, \mathrm{tw}$ ) of Dominating Eulerian Subgraph in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{t w}\right)$ time.

We describe an algorithm which takes an instance ( $G, \mathcal{T}, \mathrm{tw}$ ) of Dominating Eulerian Subgraph as input and tells in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\text {tw }}\right)$ time whether graph $G$ has a subgraph which is Eulerian, and whose vertex set is a vertex cover of G . The algorithm is a DP over a tree decomposition, very similar to the one in Section 3. As before we assume that $\mathcal{T}$ is a nice tree decomposition. Our proofs are simplified if we assume that we know of a vertex $\nu^{\star}$ which is definitely part of the unknown dominating Eulerian subgraph which we are trying to find. Note that for any edge $x y$ of $G$ at least one of the two vertices $\{x, y\}$ must be part of any dominating Eulerian subgraph of G. So one of the two choices $v^{\star}=x$ and $v^{\star}=y$ will satisfy our requirement on $v^{\star}$. Hence we assume, without loss of generality, that we have picked a correct choice for $v^{\star}$. We add the vertex $v^{\star}$ to every bag of $\mathcal{T}$; from now on we use $\mathcal{T}$ to refer to the resulting "nearly-nice" tree decomposition in which the bags at all the leaves and the root are equal to $\left\{v^{\star}\right\}$.

If the graph $G$ has a dominating Eulerian subgraph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ then it interacts with the structures defined by node $t$ in the following way: The part of $G^{\prime}$ contained in $G_{t}$ is a collection $\mathcal{C}=\left\{\mathrm{C}_{1}, \ldots, \mathrm{C}_{\ell}\right\}$ of pairwise vertex-disjoint connected subgraphs of $\mathrm{G}_{\mathrm{t}}$ where each element $\mathcal{C}_{i}$ of $\mathcal{C}$ has a non-empty intersection with $X_{t}$. Further, the union of the vertex sets of the components in $\mathcal{C}$ forms a vertex cover of graph $G_{t}$.

- Definition 30 (Valid partitions, witness for validity for Dominating Eulerian Subgraph). For a bag $\mathrm{X}_{\mathrm{t}}$ and subsets $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, we say that a partition $\mathrm{P}=\left\{\mathrm{X}^{1}, \mathrm{X}^{2}, \ldots \mathrm{X}^{\mathrm{p}}\right\}$ of X is valid for the combination $(t, X, O)$ if there exists a subgraph $G_{t}^{\prime}=\left(V_{t}^{\prime}, E_{t}^{\prime}\right)$ of $G_{t}$ such that

1. $X_{t} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X}$.
2. $\mathrm{G}_{\mathrm{t}}^{\prime}$ has exactly p connected components $\mathrm{C}_{1}, \mathrm{C}_{2}, \ldots, \mathrm{C}_{\mathrm{p}}$ and for each $\mathrm{i} \in\{1,2, \ldots, \mathrm{p}\}$, $\mathrm{X}^{\mathrm{i}} \subseteq \mathrm{V}\left(\mathrm{C}_{\mathrm{i}}\right)$. That is, the vertex set of each connected component of $\mathrm{G}_{\mathrm{t}}^{\prime}$ has a non-empty intersection with set X , and P is the partition of X defined by the subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$.
3. $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ holds, and $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.
4. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .

Such a subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$ of $\mathrm{G}_{\mathrm{t}}$ is a witness for partition P being valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ or, in short: $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$.

- Definition 31 (Completion for Dominating Eulerian Subgraph). For a bag $\mathrm{X}_{\mathrm{t}}$ and subsets $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ let P be a partition of X which is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Let H be a residual subgraph with respect to t such that $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$. We say that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H if there exists a subgraph $\mathrm{G}_{\mathrm{t}}^{\prime}$ of $\mathrm{G}_{\mathrm{t}}$ which is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$, such that the graph $\mathrm{G}_{\mathfrak{t}}^{\prime} \cup \mathrm{H}$ is a dominating Eulerian subgraph of G . We say that $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a certificate for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completing H .
$\triangleright$ Observation 32. If $((t, X, O), P)$ completes $H$ then every edge in the set $E(G) \backslash E\left(G_{t}\right)$ has at least one end-point in the vertex set $\mathrm{V}(\mathrm{H})$.
- Lemma 33. Let $(\mathrm{G}, \mathcal{T}, \mathrm{tw})$ be an instance of Dominating Eulerian Subgraph. Let t be an arbitrary node of $\mathcal{T}$, let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, let P be a partition of X which is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$, and let H be a residual subgraph with respect to t with $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$. If $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H then the set of odd-degree vertices in H is exactly the set O .

Proof. Let $H_{\text {odd }} \subseteq \mathrm{V}(\mathrm{H})$ be the set of odd-degree vertices in H . Since ( $\left.(\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P}\right)$ completes $H$ we know that there is a subgraph $G_{t}^{\prime}=\left(V_{t}^{\prime}, E_{t}^{\prime}\right)$ of $G_{t}$ which is a witness for $((t, X, O), P)$, such that the graph $G^{\star}=G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$. Since $G_{t}^{\prime}$ is a witness for $((t, X, O), P)$ we get that the set of odd-degree vertices in $G_{t}^{\prime}$ is exactly the set $O$. Since $H$ is a residual subgraph with respect to $t$ we have that $E_{t}^{\prime} \cap E(H)=\emptyset$. Thus the degree of any vertex $v$ in the graph $\mathrm{G}^{\star}$ is the sum of its degrees in the two subgraphs H and $\mathrm{G}_{\mathrm{t}}^{\prime}: \operatorname{deg}_{\mathrm{G}^{\star}}(v)=\operatorname{deg}_{\mathrm{H}}(v)+\operatorname{deg}_{\mathrm{G}_{\mathrm{t}}^{\prime}}(v)$. And since $\mathrm{G}^{\star}$ is Eulerian we have that $\operatorname{deg}_{\mathrm{G}^{\star}}(v)$ is even for every vertex $v \in \mathrm{~V}\left(\mathrm{G}^{*}\right)$.

Now let $v \in \mathrm{H}_{\mathrm{odd}} \subseteq \mathrm{V}(\mathrm{H})$ be a vertex of odd degree in H . Then $v \in \mathrm{~V}\left(\mathrm{G}^{\star}\right)$ and we get that $\operatorname{deg}_{G_{\mathrm{t}}^{\prime}}(v)=\operatorname{deg}_{\mathrm{G}^{*}}(v)-\operatorname{deg}_{\mathrm{H}}(v)$ is odd. Thus $v \in \mathrm{O}$, and so $\mathrm{H}_{\mathrm{odd}} \subseteq \mathrm{O}$. Conversely, let $x \in O \subseteq V_{t}^{\prime}$ be a vertex of odd degree in $G_{t}^{\prime}$. Then $x \in V\left(G^{\star}\right)$ and we get that $\operatorname{deg}_{\mathrm{H}}(\mathrm{x})=\operatorname{deg}_{\mathrm{G}^{*}}(\mathrm{x})-\operatorname{deg}_{\mathrm{G}_{\mathrm{t}}^{\prime}}(\mathrm{x})$ is odd. Thus $\mathrm{x} \in \mathrm{H}_{\mathrm{odd}}$, and so $\mathrm{O} \subseteq \mathrm{H}_{\mathrm{odd}}$. Thus the set of odd-degree vertices in H is exactly the set O .

The next lemma tells us that it is safe to apply the representative set computation to collections of valid partitions.

- Lemma 34. Let $(\mathrm{G}, \mathcal{T}, \mathrm{tw})$ be an instance of Dominating Eulerian Subgraph, and let t be an arbitrary node of $\mathcal{T}$. Let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, and let $\mathcal{A}$ be a collection of partitions of X , each of which is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Let $\mathcal{B}$ be a representative subset of $\mathcal{A}$, and let H be an arbitrary residual subgraph of G with respect to t such that $\mathrm{V}(\mathrm{H}) \cap \mathrm{X}_{\mathrm{t}}=\mathrm{X}$ holds. If there is a partition $\mathrm{P} \in \mathcal{A}$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H then there is a partition $\mathrm{Q} \in \mathcal{B}$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H .

Proof. Suppose there is a partition $\mathrm{P} \in \mathcal{A}$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes the residual subgraph $H$. Then there exists a subgraph $G_{t}^{\prime}=\left(V_{t}^{\prime}, E_{t}^{\prime}\right)$ of $G_{t}-G_{t}^{\prime}$ being a witness for $((t, X, O), P)$ such that (i) $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$, (ii) $P$ is the partition of $X$ defined by $G_{t}^{\prime}$, (iii) $V\left(G_{t}^{\prime}\right)$ is a vertex cover of of $G_{t}$, (iv) the set of odd-degree vertices in $G_{t}^{\prime}$ is exactly the set O, and (v) the graph $G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of G. Observe that every edge in $E(G) \backslash E\left(G_{t}\right)$ has at least one end-point in the set $V(H)$. Let $R$ be the partition of the set $X$ defined by the residual subgraph $H$. Since the union of $G_{t}^{\prime}$ and $H$ is connected we get-Lemma 11 that $P \sqcup R=\{X\}$ holds. Since $\mathcal{B}$ is a representative subset of $\mathcal{A}$ we get that there exists a partition $Q \in \mathcal{B}$ such that $Q \sqcup R=\{X\}$ holds. Since $\mathcal{B} \subseteq \mathcal{A}$ we have that the partition $Q$ of $X$ is valid for the combination ( $t, X, O$ ). So there exists a subgraph $G_{t}^{\star}=\left(V_{t}^{\star}, E_{t}^{\star}\right)$ of $G_{t}-G_{t}^{\star}$ being a witness for $((t, X, O), Q)$ such that (i) $X_{t} \cap V\left(G_{t}^{\star}\right)=X$,
(ii) $Q$ is the partition of $X$ defined by $G_{t}^{\star}$, (iii) $V\left(G_{t}^{\star}\right)$ is a vertex cover of $G_{t}$ with $v^{\star} \in V\left(G_{t}^{\star}\right)$ , and (iv) the set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\star}$ is exactly the set O . Now:

1. The vertex set of the graph $G_{t}^{\star} \cup H$ is a vertex cover of graph $G$. This follows from Observation 32 since $V\left(G_{t}^{\star}\right)$ is a vertex cover of $G_{t}$.
2. The graph $G_{t}^{\star} \cup H$ has all degrees even, because (i) the edge sets $E\left(G_{t}^{\star}\right)$ and $E(H)$ are disjoint, and (ii) the sets of odd-degree vertices in the two graphs $\mathrm{G}_{\mathrm{t}}^{\star}$ and H are identical-namely, the set O .
3. $G_{t}^{\star} \cup H$ is connected-by Lemma 11-because $Q \sqcup R=\{X\}$ holds.

Thus the subgraph $G_{t}^{\star}$ of $G_{t}$ is a witness for $((t, X, O), Q)$ such that the graph $G_{t}^{\star} \cup H$ is a dominating Eulerian subgraph of G. Hence ( $(t, X, O), Q)$ completes $H$.

- Lemma 35. Let ( $\mathrm{G}, \mathcal{T}, \mathrm{tw}$ ) be an instance of Dominating Eulerian Subgraph, let r be the root node of $\mathfrak{T}$, and let $v^{\star}$ be the vertex which is present in every bag of $\mathcal{T}$. Then ( $\mathrm{G}, \mathfrak{T}, \mathrm{tw}$ ) is a yes instance of Dominating Eulerian Subgraph if and only if the partition $P=\left\{\left\{\nu^{\star}\right\}\right\}$ is valid for the combination $\left(\mathrm{r}, \mathrm{X}=\left\{\nu^{\star}\right\}, \mathrm{O}=\emptyset\right.$ ).

Proof. Let $(\mathrm{G}, \mathcal{T}, \mathrm{tw})$ be a yes instance of Dominating Eulerian Subgraph and let $\mathrm{G}^{\prime}$ be a dominating Eulerian subgraph of G. By assumption vertex $v^{\star}$ is in $V\left(G^{\prime}\right)$. Since $r$ is the root node of $\mathcal{T}$ we have that $X_{r}=\left\{\nu^{\star}\right\}, V_{r}=V(G)$ and $G_{r}=G$. We set $G_{r}^{\prime}=G^{\prime}$. Then (i) $\mathrm{X}_{\mathrm{r}} \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)=\left\{\nu^{\star}\right\}=\mathrm{X}$, (ii) $\mathrm{G}_{\mathrm{r}}^{\prime}=\mathrm{G}^{\prime}$ has exactly one connected component $\mathrm{C}_{1}=\mathrm{V}\left(\mathrm{G}^{\prime}\right)$ and the partition $P=\left\{\left\{\nu^{\star}\right\}\right\}$ of $X=\left\{v^{\star}\right\}$ is the partition of $X$ defined by $G_{r}^{\prime}$, (iii) $V\left(G_{r}^{\prime}\right)=V\left(G^{\prime}\right)$ is a vertex cover of graph $G$ with $\nu^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)$, and (iv) the set of odd-degree vertices in $\mathrm{G}_{\mathrm{r}}^{\prime}$ is exactly the empty set. Thus the partition $\mathrm{P}=\left\{\left\{v^{\star}\right\}\right\}$ is valid for the combination $\left(r, X=\left\{v^{\star}\right\}, O=\emptyset\right)$. This completes the forward direction.

For the reverse direction, suppose the partition $P=\left\{\left\{\nu^{\star}\right\}\right\}$ is valid for the combination $\left(r, X=\left\{v^{\star}\right\}, O=\emptyset\right)$. Then by definition there exists a subgraph $G_{r}^{\prime}=\left(V_{r}^{\prime}, E_{r}^{\prime}\right)$ of $G_{r}=G$ such that (i) $X_{r} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)=\mathrm{X}=\left\{\nu^{\star}\right\}$, (ii) $\mathrm{G}_{\mathrm{r}}^{\prime}$ has exactly one connected component $\mathrm{C}_{1}=\mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)$, (iii) $V\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)$ is a vertex cover of graph G with $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{r}}^{\prime}\right)$, and (iv) the set of odd-degree vertices in $G_{r}^{\prime}$ is exactly the empty set $O$. Thus $G_{r}^{\prime}$ is a connected subgraph of $G$ whose vertex set is a vertex cover of $G$, and whose degrees are all even. But $G_{r}^{\prime}$ is then a dominating Eulerian subgraph of $G$, and so $(G, \mathcal{T}, \mathrm{tw})$ is a yes instance of Dominating Eulerian Subgraph.

- Lemma 36. Let ( $\mathrm{G}, \mathfrak{T}, \mathrm{tw}$ ) be an instance of Dominating Eulerian Subgraph, let r be the root node of $\mathcal{T}$, and let $v^{\star}$ be the vertex which is present in every bag of $\mathcal{T}$. Let $\mathrm{H}=\left(\left\{v^{\star}\right\}, \emptyset\right), \mathrm{X}=\left\{v^{\star}\right\}, \mathrm{O}=\emptyset$, and $\mathrm{P}=\left\{\left\{\nu^{\star}\right\}\right\}$. Then $(\mathrm{G}, \mathcal{T}, \mathrm{tw})$ is a yes instance if and only if $((\mathrm{r}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H .

Proof. Note that $G_{r}=G$. It is easy to verify by inspection that $H$ is a residual subgraph with respect to $r$.

Let $(G, \mathcal{T}, \mathrm{tw})$ be a yes instance of Dominating Eulerian Subgraph and let $\mathrm{G}^{\prime}$ be a dominating Eulerian subgraph of G. By assumption vertex $v^{\star}$ is in $V\left(\mathrm{G}^{\prime}\right)$. From Lemma 35 we get that partition $P$ is valid for the combination ( $r, X, O$ ), and from the proof of Lemma 35 we get that the dominating Eulerian subgraph $G^{\prime}$ is itself a witness for $((r, X, O), P)$. Now $\left(\left(V\left(G^{\prime}\right) \cup V(H)\right),\left(E\left(G^{\prime}\right) \cup E(H)\right)\right)=\left(V\left(G^{\prime}\right), E\left(G^{\prime}\right)\right)=G^{\prime}$, and so $G^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$. Thus ( $(\mathrm{r}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H .

The reverse direction is trivial: if $((r, X, O), P)$ completes $H$ then by definition there exists a dominating Eulerian subgraph of $G$, and so $(G, \mathcal{T}, \mathrm{tw})$ is a yes instance.

As in Section 3 we use the completion-based alternate characterization of yes instancesLemma 36 and representative subset computations-Lemma 34-to speed up our DP. We now describe the steps of the DP algorithm for each type of node in $\mathcal{T}$. We use VP[t, X, O] to denote the set of valid partitions for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ) which we store in the DP table for node $t$.
Leaf node t : In this case $\mathrm{X}_{\mathrm{t}}=\left\{v^{\star}\right\}$. Set $\mathrm{VP}\left[\mathrm{t},\left\{\nu^{\star}\right\},\left\{v^{\star}\right\}\right]=\emptyset, \mathrm{VP}\left[\mathrm{t},\left\{\nu^{\star}\right\}, \emptyset\right]=\left\{\left\{\left\{v^{\star}\right\}\right\}\right\}$, and $V P[t, \emptyset, \emptyset]=\{\emptyset\}$.
Introduce vertex node $t$ : Let $t^{\prime}$ be the child node of $t$, and let $v$ be the vertex introduced at t . Then $v \notin X_{\mathrm{t}^{\prime}}$ and $X_{\mathrm{t}}=X_{\mathrm{t}^{\prime}} \cup\{v\}$. For each $\mathrm{X} \subseteq X_{\mathrm{t}}$ and $\mathrm{O} \subseteq X$,

1. if $v \in \mathrm{O}$ then set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\emptyset$
2. if $v \in(X \backslash O)$ then for each partition $P^{\prime}$ in $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$, add the partition $P=P^{\prime} \cup\{\{\nu\}\}$ to the set VP[t, X, O]
3. if $v \notin X$ then set $V P[t, X, O]=V P\left[t^{\prime}, X, O\right]$
4. Set $\mathcal{A}=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathcal{B}$.

Introduce edge node $t$ : Let $t^{\prime}$ be the child node of $t$, and let $u v$ be the edge introduced at
t. Then $X_{t}=X_{t^{\prime}}$ and $u v \in\left(E\left(G_{t}\right) \backslash E\left(G_{t^{\prime}}\right)\right)$. For each $X \subseteq X_{t}$ and $O \subseteq X$,

1. If $\{u, v\} \cap X=\emptyset$ then set $V P[t, X, O]=\emptyset$; else set $V P[t, X, O]=V P\left[t^{\prime}, X, O\right]$.
2. If $\{u, v\} \subseteq X$ then:
a. Construct a set of candidate partitions $\mathcal{P}$ as follows. Initialize $\mathcal{P}=\emptyset$.
$=$ if $\{u, v\} \subseteq O$ then add all the partitions in $V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$ to $\mathcal{P}$.
$=$ if $\{u, v\} \cap O=\{u\}$ then add all the partitions in $V P\left[t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right]$ to $\mathcal{P}$. = if $\{u, v\} \cap O=\{v\}$ then add all the partitions in $V P\left[t^{\prime}, X,(O \backslash\{v\}) \cup\{u\}\right]$ to $\mathcal{P}$. $=$ if $\{u, v\} \cap O=\emptyset$ then add all the partitions in $V P\left[t^{\prime}, X, O \cup\{u, v\}\right]$ to $\mathcal{P}$.
b. For each candidate partition $P^{\prime} \in \mathcal{P}$, if vertices $u, v$ are in different blocks of $P^{\prime}$-say $u \in \mathrm{P}_{\mathfrak{u}}^{\prime}, v \in \mathrm{P}_{v}^{\prime} ; \mathrm{P}_{\mathfrak{u}}^{\prime} \neq \mathrm{P}_{v}^{\prime}$ - then merge these two blocks of $\mathrm{P}^{\prime}$ to obtain P . That is, set $\mathrm{P}=\left(\mathrm{P}^{\prime} \backslash\left\{\mathrm{P}_{\mathfrak{u}}^{\prime}, \mathrm{P}_{v}^{\prime}\right\}\right) \cup\left(\mathrm{P}_{\mathfrak{u}}^{\prime} \cup \mathrm{P}_{v}^{\prime}\right)$. Now set $\mathcal{P}=\left(\mathcal{P} \backslash\left\{\mathrm{P}^{\prime}\right\}\right) \cup \mathrm{P}$.
c. Add all of $\mathcal{P}$ to the list $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$.
3. Set $\mathcal{A}=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathcal{B}$. Forget node $t$ : Let $t^{\prime}$ be the child node of $t$, and let $v$ be the vertex forgotten at $t$. Then $v \in X_{t^{\prime}}$ and $X_{t}=X_{t^{\prime}} \backslash\{v\}$. Recall that $\mathrm{P}(v)$ is the block of partition P which contains element $v$, and that $\mathrm{P}-v$ is the partition obtained by eliding $v$ from $P$. For each $X \subseteq X_{t}$ and $\mathrm{O} \subseteq \mathrm{X}$,
4. Set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\left\{\mathrm{P}^{\prime}-v ; \mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right],\left|\mathrm{P}^{\prime}(v)\right|>1\right\} \cup \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$.
5. Set $\mathcal{A}=V P[t, X, O]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $V P[t, X, O]=\mathcal{B}$.

Join node $t$ : Let $t_{1}, t_{2}$ be the children of $t$. Then $X_{t}=X_{t_{1}}=X_{t_{2}}$. For each $X \subseteq X_{t}, O \subseteq X$ :

1. Set $\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\emptyset$
2. For each $\mathrm{O}_{1} \subseteq \mathrm{O}$ and $\hat{O} \subseteq(X \backslash O)$ :
a. Let $\mathrm{O}_{2}=\mathrm{O} \backslash \mathrm{O}_{1}$.
b. For each pair of partitions $\mathrm{P}_{1} \in \mathrm{VP}\left[\mathrm{t}_{1}, \mathrm{X}, \mathrm{O}_{1} \cup \hat{\mathrm{O}}\right], \mathrm{P}_{2} \in \mathrm{VP}\left[\mathrm{t}_{2}, \mathrm{X}, \mathrm{O}_{2} \cup \hat{\mathrm{O}}\right]$, add their join $P_{1} \sqcup P_{2}$ to the set $V P[t, X, O]$.
3. Set $\mathcal{A}=\mathrm{VP}[\mathrm{t}, \mathrm{X}, \mathrm{O}]$. Compute a representative subset $\mathcal{B} \subseteq \mathcal{A}$ and set $\mathrm{V} P[\mathrm{t}, \mathrm{X}, \mathrm{O}]=\mathcal{B}$.

As before, we prove by induction on the structure of $\mathcal{T}$ that every node in $\mathcal{T}$ preserves the Correctness Criteria (see page 13). The processing at each of the non-leaf nodes computes a representative subset as a final step. This step does not negate the correctness criteria.
$\triangleright$ Observation 37. Let t be a node of $\mathcal{T}$, let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$, and let $\mathcal{A}$ be a set of partitions which satisfies the correctness criteria for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Let $\mathcal{B}$ be a representative subset of $\mathcal{A}$. Then $\mathcal{B}$ satisfies the correctness criteria for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ).

Proof. Since $\mathcal{B} \subseteq \mathcal{A}$ holds we get that $\mathcal{B}$ satisfies the soundness criterion. From Lemma 34 we get that $\mathcal{B}$ satisfies the completeness criterion as well.

- Lemma 38. Let t be a leaf node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq X_{\mathrm{t}}, \mathrm{O} \subseteq X$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Here $X_{t}=\left\{v^{\star}\right\}$. Note that the graph $G_{t}$ consists of (i) the one vertex $v^{\star}$, and (ii) no edges. We verify the conditions for all the three possible cases:

- $X=\left\{v^{\star}\right\}, \mathrm{O}=\left\{v^{\star}\right\}$. The algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously. Observe that there is no subgraph $G_{t^{\prime}}$ of $G_{t}$ in which vertex $v^{\star}$ has an odd degree. This means that there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which the fourth condition in the definition of a valid partition-Definition 30-holds. Thus there is no partition which is valid for the combination $(t, X, O)$. Hence the completeness criterion holds vacuously as well.
- $\mathrm{X}=\left\{\nu^{\star}\right\}, \mathrm{O}=\emptyset$. The algorithm sets $\mathcal{A}=\left\{\left\{\left\{\nu^{\star}\right\}\right\}\right\}$. It is easy to verify by inspection that the subgraph $G_{t^{\prime}}=G_{t}$ of $G_{t}$ is a witness for the partition $\left\{\left\{v^{\star}\right\}\right\}$ being valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). Hence the soundness criterion holds.
Since $X$ is the set $\left\{\nu^{\star}\right\}$, the only valid partition for the combination $(t, X, O)$ is $\left\{\left\{\nu^{\star}\right\}\right\}$. Hence the completeness criterion holds trivially.
- X $=\emptyset, \mathrm{O}=\emptyset$. The algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously.

Since $X=\emptyset$ holds there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which both the conditions (1) and (3) of the definition of a valid partition-Definition 30 hold simultaneously. Thus there is no partition which is valid for the combination ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ). Hence the completeness criterion holds vacuously as well.

- Lemma 39. Let t be an introduce vertex node of the tree decomposition $\mathfrak{T}$ and let $\mathrm{X} \subseteq$ $\mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $\mathrm{t}^{\prime}$ be the child node of t , and let $v$ be the vertex introduced at t . Then $v \notin X_{\mathrm{t}^{\prime}}$ and $X_{t}=X_{t^{\prime}} \cup\{v\}$ hold. Note that no edges incident with $v$ have been introduced so far; so we have that $\operatorname{deg}_{G_{t}}(v)=0$ holds. We analyze each choice made by the algorithm:

1. If $v \in O$ holds then the algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously. Since $\operatorname{deg}_{G_{t}}(v)=0$ holds, there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which the fourth condition of the definition of a valid partition-Definition 30-holds. Thus there is no partition which is valid for the combination ( $t, X, O$ ). Hence the completeness condition holds vacuously as well.
2. If $v \in(X \backslash O)$ holds then the algorithm takes each partition $P^{\prime}$ in $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$ and adds the partition $\mathrm{P}=\left(\mathrm{P}^{\prime} \cup\{\{v\}\}\right)$ to the set $\mathcal{A}$. By inductive assumption we have that the set $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$ of partitions is sound and complete for the combination ( $t^{\prime}, X \backslash\{v\}, O$ ). Let $\mathrm{P}=\left(\mathrm{P}^{\prime} \cup\{\{\nu\}\}\right)$ be an arbitrary partition in the set $\mathcal{A}$, where $\mathrm{P}^{\prime}$ is a partition from the set $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$. Then the partition $P^{\prime}$ is valid for the combination $\left(t^{\prime}, X \backslash\{v\}, O\right)$, and so there exists a subgraph $H$ of the graph $G_{t^{\prime}}$ such that $H$ is a witness for $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), P^{\prime}\right)$. It is easy to verify by inspection that the graph $G_{t}^{\prime}=(V(H) \cup\{v\}, E(H))$ is a subgraph of $G_{t}$ which satisfies all the four conditions of Definition 30 for being a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the soundness condition holds for the set $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H . Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $\mathfrak{i} \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. $v^{\star} \in V\left(G_{t}^{\prime}\right)$ holds, and $V\left(G_{t}^{\prime}\right)$ is a vertex cover of graph $G_{t}$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$.

Since $\operatorname{deg}_{G_{\mathrm{t}}}(v)=0$ holds, we get that $\operatorname{deg}_{G_{\mathrm{t}}^{\prime}}(v)=0$ holds as well. Thus vertex $v$ forms a connected component by itself in graph $\mathrm{G}_{\mathrm{t}}^{\prime}$. Without loss of generality, let this component by $C_{p}$. Then we get that $X^{p}=V\left(C_{p}\right)=\{v\}$, and that $P^{\prime}=\left\{X^{1}, X^{2}, \ldots X^{(p-1)}\right\}$ is a partition of the set $X \backslash\{v\}$.
Since $v \in X$ and $V(H) \cap X_{t}=X$ hold, and since the graph $G_{t}^{\prime} \cup H$ is Eulerian, we get that vertex $v$ has a positive even degree in graph $H$. Since $H$ is a residual subgraph with respect to $t$ we have that (i) $V(H) \cap\left(V_{t} \backslash X_{t}\right)=\emptyset$ and (ii) $E(H) \cap E_{t}=\emptyset$ hold. Since $X_{t}=X_{t^{\prime}} \cup\{v\}$ holds, we get that $V_{t^{\prime}}=V_{t} \backslash\{v\}$ and hence $V_{t^{\prime}} \backslash X_{t^{\prime}}=V_{t} \backslash X_{t}$ holds. Hence $V(H) \cap\left(V_{t^{\prime}} \backslash X_{t^{\prime}}\right)=\emptyset$ holds. Further, since $E_{t^{\prime}} \subseteq E_{t}$ holds we get that $E(H) \cap E_{t^{\prime}}=\emptyset$ holds as well. Thus H is a residual subgraph with respect to node $\mathrm{t}^{\prime}$ which (i) contains vertex $v$ and (ii) satisfies $V(H) \cap X_{t^{\prime}}=(X \backslash\{v\})$.
Now let $G_{t}^{\prime}$, be the graph obtained from $G_{t}^{\prime}$ by deleting vertex $v$. Then $G_{t}^{\prime}$, is a subgraph of $G_{t^{\prime}}$, and it is straightforward to verify that the following hold:
a. $X_{t^{\prime}} \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}\right)=(\mathrm{X} \backslash\{v\})$.
b. $G_{t^{\prime}}^{\prime}$ has exactly $p-1$ connected components $C_{1}, C_{2}, \ldots, C_{(p-1)}$ and for each $i \in$ $\{1,2, \ldots, p-1\}, X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
c. $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}\right)$ holds, and $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}^{\prime}}$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}$ is exactly the set O .
e. The graph $G_{t^{\prime}}^{\prime} \cup H$ is identical to the graph $G_{t}^{\prime} \cup H$, and hence is a dominating Eulerian subgraph of G.
Thus $H$ is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=(X \backslash\{v\})$, and $P^{\prime}=\left\{X^{1}, X^{2}, \ldots X^{(p-1)}\right\}$ is a partition of $X \backslash\{v\}$ such that $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), P^{\prime}\right)$ completes $H$. From the inductive assumption we know that the set $V P\left[t^{\prime}, X \backslash\{v\}, O\right]$ contains a partition $Q^{\prime}=\left\{Y^{1}, Y^{2}, \ldots Y^{q}\right\}$ of $X \backslash\{v\}$ such that $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), Q^{\prime}\right)$ completes $H$. So there is a subgraph $G_{t^{\prime}}^{\prime \prime}$ of $G_{t^{\prime}}-$ a witness for $\left(\left(t^{\prime}, X \backslash\{v\}, O\right), Q^{\prime}\right)$-such that the following hold:
a. $X_{t^{\prime}} \cap V\left(G_{t^{\prime}}^{\prime \prime}\right)=X \backslash\{v\}$.
b. $G_{t^{\prime}}^{\prime \prime}$ has exactly $q$ connected components $D_{1}, D_{2}, \ldots, D_{q}$ and for each $i \in\{1,2, \ldots, q\}$, $Y^{i} \subseteq V\left(D_{i}\right)$ holds.
c. $v^{\star} \in V\left(G_{t^{\prime}}^{\prime \prime}\right)$ holds, and $V\left(G_{t^{\prime}}^{\prime \prime}\right)$ is a vertex cover of graph $G_{t^{\prime}}$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}$ is exactly the set O .
e. The graph $G_{t^{\prime}}^{\prime \prime} \cup H$ is a dominating Eulerian subgraph of $G$.

Now the algorithm adds the partition $Q=Q^{\prime} \cup\{\{v\}\}==\left\{Y^{1}, \gamma^{2}, \ldots Y^{q},\{v\}\right\}$ of set $X$ to the set $\mathcal{A}$. It is straightforward to verify that the graph $\hat{\mathrm{G}}_{\mathrm{t}}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right) \cup\{v\}, \mathrm{E}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right)\right)$ is a subgraph of graph $G_{t}$ for which the following hold:
a. $X_{t} \cap \mathrm{~V}\left(\hat{G}_{t}\right)=X$.
b. $\hat{\mathrm{G}}_{\mathrm{t}}$ has exactly $\mathrm{q}+1$ connected components $\mathrm{D}_{1}, \mathrm{D}_{2}, \ldots, \mathrm{D}_{\mathrm{q}}, \mathrm{D}_{\mathrm{q}+1}=(\{v\}, \emptyset)$ and for each $\mathfrak{i} \in\{1,2, \ldots, q+1\}, Y^{\mathfrak{i}} \subseteq \mathrm{V}\left(\mathrm{D}_{\mathfrak{i}}\right)$ holds.
c. $v^{\star} \in \mathrm{V}\left(\hat{\mathrm{G}}_{\mathrm{t}}\right)$ holds, and $\mathrm{V}\left(\hat{\mathrm{G}}_{\mathrm{t}}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.
d. The set of odd-degree vertices in $\hat{G}_{t}$ is exactly the set O .
e. The graph $\hat{\mathrm{G}}_{\mathrm{t}} \cup \mathrm{H}$ is a dominating Eulerian subgraph of G.

Thus $\mathcal{A}$ contains a partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H , as was required to be shown for completeness.
3. If $v \notin \mathrm{X}$ holds then the algorithm sets $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$. It is straightforward to verify using Definitions 8, 30, and 31 that:

- a partition $P$ of set $X$ is valid for the combination ( $t, X, O$ ) if and only if it is valid for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ );
= a subgraph of $G_{t}$ is a witness for $((t, X, O), P)$ if and only if it is (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X, O\right), P\right)$;
- a graph $H$ is a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$ if and only if $H$ is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=X$; and,
- for any residual subgraph $H$ with respect to $t$ with $V(H) \cap X_{t}=X$ and any partition $P$ of $\mathrm{X},((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H if and only if $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ completes H .
By the inductive assumption we have that the set $V P\left[t^{\prime}, X, O\right]$ of partitions is sound and complete for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ ). It follows from the above equivalences that the set $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is sound and complete for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.

Lemma 40. Let t be an introduce edge node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq$ $\mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $t^{\prime}$ be the child node of $t$, and let $u v$ be the edge introduced at $t$. Then $X_{t}=X_{t^{\prime}}$, $V_{t}=V_{t^{\prime}}$ and $u v \in\left(E\left(G_{t}\right) \backslash E\left(G_{t^{\prime}}\right)\right)$. If $\{u, v\} \cap X=\emptyset$ holds then the algorithm sets $\mathcal{A}=\emptyset$. The soundness criterion holds vacuously. Since edge $u v$ is present in graph $G_{t}$ there can exist no subgraph $G_{t^{\prime}}$ of $G_{t}$ for which the first and third conditions of the definition of a valid partition-Definition 30-hold simultaneously. Thus there is no partition which is valid for the combination ( $t, X, O$ ). Hence the completeness condition holds vacuously as well.

If $\{u, v\} \cap X \neq \emptyset$ holds then the algorithm initializes $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$. By the inductive assumption we have that every partition $\mathrm{P}^{\prime} \in \mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is valid for the combination $\left(t^{\prime}, X, O\right)$. Note that while edge $u v$ is available for use in constructing a witness for $((t, X, O), P)$, it is not mandatory to use this edge in any such witness. Applying this observation, it is straightforward to verify that if a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P^{\prime}\right)$ then it is also (i) a subgraph of $G_{t}$, and (ii) a witness for $\left((t, X, O), P^{\prime}\right)$. Thus all partitions in $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ are valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.

The algorithm adds zero or more partitions to $\mathcal{A}$ depending on how the set $\{u, v\}$ intersects the sets X and O . We analyze each choice made by the algorithm:

1. If $u \notin X$ or $v \notin X$ holds then the algorithm does not make further changes to $\mathcal{A}$ : it sets $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$. Since (i) the criteria for validity Definition 30 are based only on graphs whose intersection with $X_{t}$ is exactly the set $X$, and (ii) the new edge $u v$ does not have both end points in this set, it is intuitively clear that the relevant set of valid partitions should not change in this case. Formally, it is straightforward to verify using Definitions 8,30 and 31 that:

- a partition $P$ of set $X$ is valid for the combination $(t, X, O)$ if and only if it is valid for the combination ( $t^{\prime}, X, O$ );
= a subgraph of $G_{t}$ is a witness for $((t, X, O), P)$ if and only if it is (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for ( $\left.\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$;
- a graph $H$ is a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$ if and only if $H$ is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=X$; and,
- for any residual subgraph $H$ with respect to $t$ with $V(H) \cap X_{t}=X$ and any partition $P$ of $\mathrm{X},((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$ completes H if and only if $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ completes H .
By the inductive assumption we have that the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ of partitions is sound and complete for the combination ( $\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}$ ). It follows from the above equivalences that the set $\mathcal{A}=\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is sound and complete for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.

2. If $\{u, v\} \subseteq O$ then for each partition $P^{\prime} \in V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$,
= If vertices $u, v$ are in the same block of $\mathrm{P}^{\prime}$ then the algorithm adds $\mathrm{P}=\mathrm{P}^{\prime}$ to the set $\mathcal{A}$.
= If vertices $u, v$ are in different blocks of $P^{\prime}$ then the algorithm merges these two blocks of $\mathrm{P}^{\prime}$ and adds the resulting partition P -with one fewer block than $\mathrm{P}^{\prime}$ - to the set $\mathcal{A}$.

In either case, by the inductive assumption we have that partition $P^{\prime}$ is valid for the combination $\left(t^{\prime}, X, O \backslash\{u, v\}\right)$. Let $G_{t^{\prime}}^{\prime \prime}$ be (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), \mathrm{P}^{\prime}\right)$, and let $\mathrm{G}_{\mathrm{t}}^{\prime}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right), \mathrm{E}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right) \cup\{u v\}\right)$ be the graph obtained from $G_{t}^{\prime \prime}$ by adding the edge $u v$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$. Vertices $u, v$ have even degrees in $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}$, and hence they have odd degrees in $\mathrm{G}_{\mathrm{t}}^{\prime}$. It is straightforward to verify that $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the addition of partition P to $\mathcal{A}$ preserves the soundness of $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that $((t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$ - a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $i \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. $v^{\star} \in V\left(G_{t}^{\prime}\right)$ holds, and $V\left(G_{t}^{\prime}\right)$ is a vertex cover of graph $G_{t}$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of G.

Note that by the definition of a residual subgraph, graph H (i) does not contain edge $u v$, and (ii) is a residual subgraph with respect to node $t^{\prime}$ as well. We consider two cases.

- Suppose edge $u v$ is not present in graph $G_{t}^{\prime}$. Then it is straightforward to verify that $G_{t}^{\prime}$ is a witness for $\left(\left(\mathbf{t}^{\prime}, X, O\right), P\right)$ as well. By the inductive hypothesis there exists some partition Q of X in the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ such that $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{Q}\right)$ completes H . So there exists a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ which is a certificate for ( $\left.\left(t^{\prime}, X, O\right), Q\right)$ completing $H$. It is straightforward to verify that $G_{t^{\prime}}^{\prime}$ is a certificate for $((t, X, O), Q)$ completing $H$ as well. The algorithm adds partition Q to the set $\mathcal{A}$ during the initialization, so the completeness criterion is satisfied in this case.
- Suppose edge $u v$ is present in graph $G_{t}^{\prime}$. Let $H^{\prime}=(V(H),(E(H) \cup\{u v\}))$ be the graph obtained by adding edge $u v$ to graph $H$, and let $G_{t^{\prime}}^{\prime}=\left(V\left(G_{t}^{\prime}\right),\left(E\left(G_{t}^{\prime}\right) \backslash\{u v\}\right)\right)$ be the graph obtained by deleting edge $u v$ from graph $G_{t}^{\prime}$. Then it is straightforward to verify that (i) the set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$ is exactly the set $O \backslash\{u, v\}$, (ii) $H^{\prime}$ is a residual subgraph for node $t^{\prime}$, and (iii) $G_{t^{\prime}}^{\prime}$ is a subgraph of $G_{t^{\prime}}$ such that the graph $G_{t}^{\prime}, \cup H^{\prime}=G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$. Let $P^{\prime}$ be the partition of $X$ defined by graph $G_{t^{\prime}}^{\prime}$. Then $G_{t^{\prime}}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), P^{\prime}\right)$ such that the union of $G_{t^{\prime}}^{\prime}$ and the residual subgraph $\mathrm{H}^{\prime}$ of $\mathrm{t}^{\prime}$ is a dominating Eulerian subgraph of $G$. That is, $\left(\left(t^{\prime}, X, O \backslash\{u, v\}\right), P^{\prime}\right)$ completes $H^{\prime}$. So by the inductive assumption there exists some partition $Q^{\prime}$ of $X$ in the set $V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$ such that $\left((t, X, O \backslash\{u, v\}), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{\mathrm{G}}^{\prime}$ is a witness for $\left((\mathrm{t}, \mathrm{X}, \mathrm{O} \backslash\{\mathbf{u}, v\}), \mathrm{Q}^{\prime}\right)$ and (ii) $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ is a dominating Eulerian subgraph of G.
Note that $Q^{\prime}$ is the partition of set $X$ defined by the graph $\hat{G}^{\prime}$. Suppose both $u$ and $v$ are in the same block of partition $Q^{\prime}$. Then adding the edge $u v$ to $\hat{G}^{\prime}$ does not change the partition of $X$ defined by $\hat{G}^{\prime}$. It follows that the graph $\hat{G}=\left(V\left(\hat{G}^{\prime}\right), E\left(\hat{G}^{\prime}\right) \cup\{u v\}\right)$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $\left(\left(t, X, O, Q^{\prime}\right)\right.$ and (ii) $\hat{G} \cup H$ is a dominating Eulerian subgraph of G. Thus $\left(\left(t, X, O, Q^{\prime}\right)\right.$ completes the residual subgraph $H$. Now notice that our algorithm adds the partition $\mathrm{Q}^{\prime}$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case.
In the remaining case, vertices $u$ and $v$ are in distinct blocks of partition $\mathrm{Q}^{\prime}$. Let Q be
the partition obtained from $Q^{\prime}$ by merging together the two blocks to which vertices $u$ and $v$ belong, respectively, and leaving the other blocks as they are. Let $\hat{\mathrm{G}}$ be defined as in the previous paragraph. Then the partition of $X$ defined by $\hat{G}$ is $Q$. It follows that $\hat{G}$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $((t, X, O, Q)$ and (ii) $\hat{G} \cup H$ is a dominating Eulerian subgraph of G . Thus $((t, X, O, Q)$ completes the residual subgraph H. Now notice that our algorithm adds the partition Q to the set $\mathcal{A}$. Thus the completeness criterion holds in this case as well.

3. If $\{u, v\} \cap O=\{u\}$ then for each partition $P^{\prime} \in V P\left[t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right]$,
= If vertices $u, v$ are in the same block of $\mathrm{P}^{\prime}$ then the algorithm adds $\mathrm{P}=\mathrm{P}^{\prime}$ to the set $\mathcal{A}$.

- If vertices $u, v$ are in different blocks of $P^{\prime}$ then the algorithm merges these two blocks of $\mathrm{P}^{\prime}$ and adds the resulting partition P -with one fewer block than $\mathrm{P}^{\prime}$ - to the set $\mathcal{A}$.
In either case, by the inductive assumption we have that partition $P^{\prime}$ is valid for the combination ( $\left.t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right)$. Let $G_{t^{\prime}}^{\prime \prime}$ be (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right), P^{\prime}\right)$, and let $G_{t}^{\prime}=\left(V\left(G_{t^{\prime}}^{\prime \prime}\right), E\left(G_{t^{\prime}}^{\prime \prime}\right) \cup\{u v\}\right)$ be the graph obtained from $G_{t^{\prime}}^{\prime \prime}$ by adding the edge $u v$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$. In $G_{t^{\prime}}^{\prime \prime}$ the degree of vertex $u$ is even, and the degree of vertex $v$ is odd. So in $G_{t}^{\prime}$ vertex $u$ has an odd degree, and vertex $v$ has an even degree. It is straightforward to verify that $G_{t}^{\prime}$ is a witness for ( $(\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the addition of partition P to $\mathcal{A}$ preserves the soundness of $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition Q of X such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X}$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $\mathfrak{i} \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ holds, and $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.
d. The set of odd-degree vertices in $G_{t}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$.

Note that by the definition of a residual subgraph, graph H (i) does not contain edge $u v$, and (ii) is a residual subgraph with respect to node $t^{\prime}$ as well. We consider two cases.
= Suppose edge $u v$ is not present in graph $G_{t}^{\prime}$. Then it is straightforward to verify that $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a witness for $\left(\left(\mathbf{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ as well. By the inductive hypothesis there exists some partition $Q$ of $X$ in the set $V P\left[t^{\prime}, X, O\right]$ such that $((t, X, O), Q)$ completes $H$. This same partition Q is present in the set $\mathcal{A}$ as well.

- Suppose edge $u v$ is present in graph $\mathrm{G}_{\mathrm{t}}^{\prime}$. Let $\mathrm{H}^{\prime}=(\mathrm{V}(\mathrm{H}),(\mathrm{E}(\mathrm{H}) \cup\{u v\}))$ be the graph obtained by adding edge $u v$ to graph $H$, and let $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right),\left(\mathrm{E}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \backslash\{u v\}\right)\right)$ be the graph obtained by deleting edge $u v$ from graph $G_{t}^{\prime}$. Then it is straightforward to verify that (i) the set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$ is exactly the set $(O \backslash\{u\}) \cup\{v\}$, (ii) $\mathrm{H}^{\prime}$ is a residual subgraph for node $t^{\prime}$, and (iii) $G_{t^{\prime}}^{\prime}$ is a subgraph of $G_{t^{\prime}}$ such that the graph $G_{t^{\prime}}^{\prime} \cup H^{\prime}=G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$. Let $P^{\prime}$ be the partition of $X$ defined by graph $G_{t^{\prime}}^{\prime}$. Then $G_{t^{\prime}}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right), P^{\prime}\right)$ such that the union of $\mathrm{G}_{\mathrm{t}}^{\prime}$, and the residual subgraph $\mathrm{H}^{\prime}$ of $\mathrm{t}^{\prime}$ is a dominating Eulerian subgraph of $G$. That is, $\left(\left(t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right), \mathrm{P}^{\prime}\right)$ completes $\mathrm{H}^{\prime}$. So by the inductive assumption there exists some partition $Q^{\prime}$ of $X$ in the set $V P\left[t^{\prime}, X,(O \backslash\{u\}) \cup\{v\}\right]$ such that $\left((t, X,(O \backslash\{u\}) \cup\{v\}), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{G}^{\prime}$ is a witness for $\left((t, X,(O \backslash\{u\}) \cup\{v\}), Q^{\prime}\right)$ and (ii) $\hat{G}^{\prime} \cup H^{\prime}$ is a dominating Eulerian subgraph of G.

Note that $Q^{\prime}$ is the partition of set $X$ defined by the graph $\hat{G}^{\prime}$, and that the set of odd-degree vertices in $\hat{G}^{\prime}$ is exactly the set $(\mathbf{O} \backslash\{u\}) \cup\{v\}$. Suppose both $u$ and $v$ are in the same block of partition $Q^{\prime}$. Then adding the edge $u v$ to $\hat{G}^{\prime}(i)$ does not change the partition of X defined by $\hat{\mathrm{G}}^{\prime}$, and (ii) does change the set of odd-degree vertices to O . It follows that the graph $\hat{G}=\left(V\left(\hat{G}^{\prime}\right), E\left(\hat{G}^{\prime}\right) \cup\{u v\}\right)$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $\left(\left(t, X, O, Q^{\prime}\right)\right.$ and (ii) $\hat{G} \cup H$ is a dominating Eulerian subgraph of $G$. Thus ( $\left(t, X, O, Q^{\prime}\right)$ completes the residual subgraph $H$. Now notice that our algorithm adds the partition $Q^{\prime}$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case. In the remaining case, vertices $u$ and $v$ are in distinct blocks of partition $\mathrm{Q}^{\prime}$. Let Q be the partition obtained from $Q^{\prime}$ by merging together the two blocks to which vertices $u$ and $v$ belong, respectively, and leaving the other blocks as they are. Let $\hat{G}$ be defined as in the previous paragraph. Then the partition of $X$ defined by $\hat{G}$ is $Q$. It follows that $\hat{G}$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $((t, X, O, Q)$ and (ii) $\hat{G} \cup H$ is a dominating Eulerian subgraph of $G$. Thus $((t, X, O, Q)$ completes the residual subgraph $H$. Now notice that our algorithm adds the partition $Q$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case as well.
4. The case when $\{u, v\} \cap O=\{v\}$ is symmetrical to the previous case, so we leave out the arguments for this case.
5. If $\{u, v\} \cap O=\emptyset$ then for each partition $P^{\prime} \in V P\left[t^{\prime}, X, O \cup\{u, v\}\right]$,

- If vertices $u, v$ are in the same block of $\mathrm{P}^{\prime}$ then the algorithm adds $\mathrm{P}=\mathrm{P}^{\prime}$ to the set $\mathcal{A}$.
= If vertices $u, v$ are in different blocks of $P^{\prime}$ then the algorithm merges these two blocks of $\mathrm{P}^{\prime}$ and adds the resulting partition P -with one fewer block than $\mathrm{P}^{\prime}$ - to the set $\mathcal{A}$. In either case, by the inductive assumption we have that partition $P^{\prime}$ is valid for the combination ( $t^{\prime}, X, O \cup\{u, v\}$ ). Let $G_{t^{\prime}}^{\prime \prime}$ be (i) a subgraph of $G_{t^{\prime}}$ and (ii) a witness for $\left(\left(t^{\prime}, X, O \cup\{u, v\}\right), \mathrm{P}^{\prime}\right)$, and let $\mathrm{G}_{\mathrm{t}}^{\prime}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right), \mathrm{E}\left(\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime \prime}\right) \cup\{u v\}\right)$ be the graph obtained from $G_{t^{\prime}}^{\prime \prime}$ by adding the edge $u v$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$. Vertices $u, v$ have odd degrees in $G_{t^{\prime}}^{\prime \prime}$, and hence they have even degrees in $G_{t}^{\prime}$. It is straightforward to verify that $G_{t}^{\prime}$ is a witness for $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{P})$. Thus the addition of partition P to $\mathcal{A}$ preserves the soundness of $\mathcal{A}$.
Now we prove completeness. So let H be a residual subgraph with respect to t with $V(H) \cap X_{t}=X$, for which there exists a partition $P=\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that $((t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$ - a witness for $((t, X, O), P)$-such that the following hold:
a. $X_{t} \cap \mathrm{~V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)=\mathrm{X}$.
b. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $i \in\{1,2, \ldots, p\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
c. $v^{\star} \in V\left(G_{t}^{\prime}\right)$ holds, and $V\left(G_{t}^{\prime}\right)$ is a vertex cover of graph $G_{t}$.
d. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
e. The graph $G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$.

Note that by the definition of a residual subgraph, graph H (i) does not contain edge $u v$, and (ii) is a residual subgraph with respect to node $t^{\prime}$ as well. We consider two cases.

- Suppose edge $u v$ is not present in graph $G_{t}^{\prime}$. Then it is straightforward to verify that $\mathrm{G}_{\mathrm{t}}^{\prime}$ is a witness for $\left(\left(\mathbf{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ as well. By the inductive hypothesis there exists some partition Q of X in the set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ such that $((\mathrm{t}, \mathrm{X}, \mathrm{O}), \mathrm{Q})$ completes H . This same partition Q is present in the set $\mathcal{A}$ as well.
- Suppose edge $u v$ is present in graph $G_{t}^{\prime}$. Let $H^{\prime}=(V(H),(E(H) \cup\{u v\}))$ be the graph obtained by adding edge $u v$ to graph $H$, and let $G_{t^{\prime}}^{\prime}=\left(V\left(G_{t}^{\prime}\right),\left(E\left(G_{t}^{\prime}\right) \backslash\{u v\}\right)\right)$ be the
graph obtained by deleting edge $u v$ from graph $G_{t}^{\prime}$. Then it is straightforward to verify that (i) the set of odd-degree vertices in $G_{t^{\prime}}^{\prime}$ is exactly the set $\mathrm{O} \backslash\{u, v\}$, (ii) $H^{\prime}$ is a residual subgraph for node $t^{\prime}$, and (iii) $G_{t^{\prime}}^{\prime}$ is a subgraph of $G_{t^{\prime}}$ such that the graph $G_{t}^{\prime} \cup H^{\prime}=G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$. Let $P^{\prime}$ be the partition of $X$ defined by graph $G_{t^{\prime}}^{\prime}$. Then $G_{t^{\prime}}^{\prime}$ is a witness for ( $\left.\left(t^{\prime}, X, O \backslash\{u, v\}\right), P^{\prime}\right)$ such that the union of $\mathrm{G}_{\mathrm{t}^{\prime}}^{\prime}$, and the residual subgraph $\mathrm{H}^{\prime}$ of $\mathrm{t}^{\prime}$ is a dominating Eulerian subgraph of G . That is, $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O} \backslash\{u, v\}\right), \mathrm{P}^{\prime}\right)$ completes $\mathrm{H}^{\prime}$. So by the inductive assumption there exists some partition $Q^{\prime}$ of $X$ in the set $V P\left[t^{\prime}, X, O \backslash\{u, v\}\right]$ such that $\left((t, X, O \backslash\{u, v\}), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{\mathrm{G}}^{\prime}$ is a witness for $\left((\mathrm{t}, \mathrm{X}, \mathrm{O} \backslash\{\mathbf{u}, v\}), \mathrm{Q}^{\prime}\right)$ and (ii) $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ is a dominating Eulerian subgraph of G.
Note that $\mathrm{Q}^{\prime}$ is the partition of set X defined by the graph $\hat{\mathrm{G}}^{\prime}$, and that the set of odd-degree vertices in $\hat{\mathrm{G}}^{\prime}$ is exactly the set $\mathrm{O} \backslash\{u, v\}$. Suppose both $u$ and $v$ are in the same block of partition $\mathrm{Q}^{\prime}$. Then adding the edge $u v$ to $\hat{\mathrm{G}}^{\prime}$ (i) does not change the partition of X defined by $\hat{\mathrm{G}}^{\prime}$, and (ii) does change the set of odd-degree vertices to O . It follows that the graph $\hat{G}=\left(V\left(\hat{G}^{\prime}\right), E\left(\hat{G}^{\prime}\right) \cup\{u v\}\right)$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $\left(\left(t, X, O, Q^{\prime}\right)\right.$ and (ii) $\hat{\mathrm{G}} \cup H$ is a dominating Eulerian subgraph of $G$. Thus ( $\left(\mathrm{t}, \mathrm{X}, \mathrm{O}, \mathrm{Q}^{\prime}\right)$ completes the residual subgraph $H$. Now notice that our algorithm adds the partition $Q^{\prime}$ to the set $\mathcal{A}$. Thus the completeness criterion holds in this case. In the remaining case, vertices $u$ and $v$ are in distinct blocks of partition $\mathrm{Q}^{\prime}$. Let Q be the partition obtained from $Q^{\prime}$ by merging together the two blocks to which vertices $u$ and $v$ belong, respectively, and leaving the other blocks as they are. Let $\hat{\mathrm{G}}$ be defined as in the previous paragraph. Then the partition of $X$ defined by $\hat{G}$ is $Q$. It follows that $\hat{G}$ is a subgraph of $G_{t}$ such that (i) $\hat{G}$ is a witness for $((t, X, O, Q)$ and (ii) $\hat{G} \cup H$ is a dominating Eulerian subgraph of $G$. Thus $((t, X, O, Q)$ completes the residual subgraph $H$. Now notice that our algorithm adds the partition Q to the set $\mathcal{A}$. Thus the completeness criterion holds in this case as well.
- Lemma 41. Let t be a forget node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $\mathrm{t}^{\prime}$ be the child node of t , and let $v$ be the vertex forgotten at t . Then $v \in X_{t^{\prime}}$ and $X_{t}=X_{t^{\prime}} \backslash\{v\}$, and $v \notin O$ hold. Recall that $\mathrm{P}(v)$ is the block of partition P which contains element $v$ and that $\mathrm{P}-v$ is the partition obtained by eliding $v$ from P . The algorithm adds all partitions in the set $\left\{\mathrm{P}^{\prime}-v ; \mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right],\left|\mathrm{P}^{\prime}(v)\right|>1\right\}$ to $\mathcal{A}$. By the inductive assumption we have that every partition $\mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right]$ is valid for the combination $\left(t^{\prime}, X \cup\{v\}, O\right)$. Note that (i) the graph $G_{t^{\prime}}$ is identical to the graph $G_{t}$, and (ii) for any subgraph $H$ of $G_{t^{\prime}}=G_{t},\left(V(H) \cap X_{t^{\prime}}\right)=X \cup\{v\}$ implies $\left(V(H) \cap X_{t}\right)=X$. It follows that if every connected component of a graph $H$ contains at least two vertices from the set $X \cup\{v\}$ then every connected component of H contains at least one vertex from set $X$. Using these observations it is straightforward to verify that if a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ is a witness for $\left(\left(\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right), \mathrm{P}^{\prime}\right)$ where $v \notin \mathrm{O}$ and $\left|\mathrm{P}^{\prime}(v)\right|>1$ hold, then it is also (i) a subgraph of $\mathrm{G}_{\mathrm{t}}$, and (ii) a witness for $\left((t, X, O), P^{\prime}-v\right)$. Thus for each partition $\mathrm{P}^{\prime} \in V \mathrm{P}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right],\left|\mathrm{P}^{\prime}(v)\right|>1$ the partition $\mathrm{P}^{\prime}-v$ is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Thus all partitions in the set $\left\{\mathrm{P}^{\prime}-v ; \mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right]\right\}$ are valid for $(\mathrm{t}, \mathrm{X}, \mathrm{O})$.

The algorithm also adds all the partitions from $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ to $\mathcal{A}$. By the inductive assumption we have that every partition $\mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is valid for the combination $\left(t^{\prime}, X, O\right)$. It is once again straightforward to verify that if a subgraph $G_{t^{\prime}}^{\prime}$ of $G_{t^{\prime}}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P^{\prime}\right)$ then it is also (i) a subgraph of $G_{t}$, and (ii) a witness for $\left((t, X, O), P^{\prime}\right)$.

Thus each partition $\mathrm{P}^{\prime} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ is valid for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$. Hence all partitions added to the set $\mathcal{A}$ by the algorithm are valid for ( $\mathrm{t}, \mathrm{X}, \mathrm{O}$ ).

We now argue that the set $\mathcal{A}$ satisfies the completeness criterion. So let H be a residual subgraph with respect to $t$ with $\mathrm{V}(\mathrm{H}) \cap X_{\mathrm{t}}=\mathrm{X}$, for which there exists a partition $\mathrm{P}=$ $\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that ( $(t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}-a$ witness for $((t, X, O), P)$-such that the following hold:

1. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
2. $\mathrm{G}_{\mathrm{t}}^{\prime}$ has exactly p connected components $\mathrm{C}_{1}, \mathrm{C}_{2}, \ldots, \mathrm{C}_{\mathrm{p}}$ and for each $\mathfrak{i} \in\{1,2, \ldots, \mathrm{p}\}$, $X^{i} \subseteq V\left(C_{i}\right)$ holds.
3. $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ holds, and $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.
4. The set of odd-degree vertices in $G_{t}^{\prime}$ is exactly the set $O$.
5. The graph $\mathrm{G}_{\mathrm{t}}^{\prime} \cup \mathrm{H}$ is a dominating Eulerian subgraph of $G$.

Suppose graph $\mathrm{G}_{\mathrm{t}}^{\prime}$ does not contain vertex $v$. Then it is easy to verify that H is a residual subgraph with respect to $t^{\prime}$ with $V(H) \cap X_{t^{\prime}}=X$, and that graph $G_{t}^{\prime}$ is a witness for $\left(\left(t^{\prime}, X, O\right), P\right)$ such that the union of graphs $H$ and $G_{t}^{\prime}$ is a dominating Eulerian subgraph of G . That is, $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{P}\right)$ completes H . By inductive assumption there exists a partition $\mathrm{Q} \in \mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right]$ such that $\left(\left(\mathrm{t}^{\prime}, \mathrm{X}, \mathrm{O}\right), \mathrm{Q}\right)$ completes H . Since the algorithm adds this partition Q to $\mathcal{A}$ we get that $\mathcal{A}$ satisfies the completeness criterion in this case.

Now suppose graph $G_{t}^{\prime}$ contains vertex $v$. From the definition of a residual subgraph we know that $v \notin \mathrm{~V}(\mathrm{H})$ holds. Without loss of generality, let it be the case that $v \in \mathrm{C}_{\mathrm{p}}$ holds. Since $X_{t^{\prime}}=X_{t} \cup\{v\}$ we get that $X_{t^{\prime}} \cap V\left(G_{t}^{\prime}\right)=X \cup\{v\}$ holds. Let $H^{\prime}=(V(H) \cup\{v\}, E(H))$ be the graph obtained by adding vertex $v$ (and no extra edges) to graph H . Then it is straightforward to verify that (i) $\mathrm{H}^{\prime}$ is a residual subgraph with respect to $\mathrm{t}^{\prime}$ with $\mathrm{V}\left(\mathrm{H}^{\prime}\right) \cap \mathrm{X}_{\mathrm{t}^{\prime}}=\mathrm{X} \cup\{v\}$, (ii) the graph $G_{t}^{\prime}$ is a witness for the partition $P^{\prime}=\left\{X^{1}, X^{2}, \ldots\left(X^{p} \cup\{v\}\right)\right\}$ of $X \cup\{v\}$ being valid for the combination $\left(t^{\prime}, X \cup\{v\}, O\right)$, and (iii) the graph $G_{t}^{\prime} \cup H^{\prime}$ is a dominating Eulerian subgraph of $G$. That is, $\left(\left(t^{\prime}, X \cup\{v\}, O\right), P^{\prime}\right)$ completes $\mathrm{H}^{\prime}$.

By the inductive assumption there exists some partition $Q^{\prime}$ of $X \cup\{v\}$ in the set $V P\left[t^{\prime}, X \cup\right.$ $\{v\}, O]\}$ such that $\left(\left(t^{\prime}, X \cup\{v\}, O\right), Q^{\prime}\right)$ completes $H^{\prime}$. So there exists a subgraph $\hat{G}^{\prime}$ of $G_{t^{\prime}}$ such that (i) $\hat{\mathrm{G}}^{\prime}$ is a witness for $\left(\left(\mathrm{t}^{\prime}, \mathrm{X} \cup\{v\}, \mathrm{O}\right), \mathrm{Q}^{\prime}\right)$ and (ii) $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ is a dominating Eulerian subgraph of $G$. Note that $X_{\mathrm{t}} \cap \mathrm{V}\left(\hat{\mathrm{G}}^{\prime}\right)=\mathrm{X}$ holds.

Since $v$ had degree zero in graph $\mathrm{H}^{\prime}$ we get that $v$ has a positive even degree in $\hat{\mathrm{G}}^{\prime}$. From the definition of a witness for validity-Definition 30-we get that $\mathrm{Q}^{\prime}$ is the partition of the set $\mathrm{X} \cup\{v\}$ defined by the graph $\hat{\mathrm{G}}^{\prime}$. Let $\mathrm{Q}_{H^{\prime}}$ be the partition of the set $\mathrm{X} \cup\{v\}$ defined by the graph $\mathrm{H}^{\prime}$. Since $\operatorname{deg}_{\mathrm{H}^{\prime}}(v)=0$ holds we get that vertex $v$ appears in a block of size one - namely, $\{v\}$-in $\mathrm{Q}_{H^{\prime}}$. If $\{v\}$ is a block of $\mathrm{Q}^{\prime}$ as well, then $\{v\}$ will also be a block in their join $Q_{H^{\prime}} \sqcup \mathrm{Q}^{\prime}$. But the union of graphs $\mathrm{H}^{\prime}$ and $\hat{\mathrm{G}}^{\prime}$ is connected and so from Lemma 11 we know that $\mathrm{Q}_{\mathrm{H}^{\prime}} \sqcup \mathrm{Q}^{\prime}=\{\{\mathrm{X} \cup\{v\}\}\}$. Thus $\{v\}$ is not a block of $\mathrm{Q}_{\mathrm{H}^{\prime}} \sqcup \mathrm{Q}^{\prime}$, or of $\mathrm{Q}^{\prime}$. So there exists a vertex $v^{\prime} \in X$ such that $v, v^{\prime}$ are in the same block of $\mathrm{Q}^{\prime}$. In particular, this implies that the partition $\mathrm{Q}=\mathrm{Q}^{\prime}-v$, which is the partition of set X defined by graph $\hat{\mathrm{G}}^{\prime}$, has exactly as many blocks as has the partition $\mathrm{Q}^{\prime}$ of $\mathrm{X} \cup\{v\}$.

Putting these together we get that the subgraph $\hat{G}^{\prime}$ of $G_{t}$ is a witness for $((t, X, O), Q=$ $\left.\mathrm{Q}^{\prime}-v\right)$. Now since graph H can be obtained from graph $\mathrm{H}^{\prime}$ by deleting vertex $v$, we get that the graphs $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}^{\prime}$ and $\hat{\mathrm{G}}^{\prime} \cup \mathrm{H}$ are identical. In particular, the latter is a dominating Eulerian subgraph of G. Thus ( $(t, X, O), Q)$ completes the residual graph $H$. Since the algorithm adds partition Q to the set $\mathcal{A}$, we get that $\mathcal{A}$ satisfies the completeness criterion.

- Lemma 42. Let t be a join node of the tree decomposition $\mathcal{T}$ and let $\mathrm{X} \subseteq \mathrm{X}_{\mathrm{t}}, \mathrm{O} \subseteq \mathrm{X}$ be arbitrary subsets of $\mathrm{X}_{\mathrm{t}}, \mathrm{X}$ respectively. The collection $\mathcal{A}$ of partitions computed by the DP for the combination $(\mathrm{t}, \mathrm{X}, \mathrm{O})$ satisfies the correctness criteria.

Proof. Let $t_{1}, t_{2}$ be the children of $t$. Then $X_{t}=X_{t_{1}}=X_{t_{2}}$. Note that $V\left(G_{t}\right)=V\left(G_{t_{1}}\right) \cup$ $V\left(G_{t_{2}}\right)$ and $E\left(G_{t}\right)=E\left(G_{t_{1}}\right) \cup E\left(G_{t_{2}}\right)$ hold, and so graph $G_{t}$ is the union of graphs $G_{t_{1}}$ and $G_{t_{2}}$. Further, since each edge in the graph is introduced at exactly one bag in $\mathcal{T}$ we get that $\mathrm{E}\left(\mathrm{G}_{\mathrm{t}_{1}}\right) \cap \mathrm{E}\left(\mathrm{G}_{\mathrm{t}_{2}}\right)=\emptyset$ holds. Moreover, $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}_{1}}\right) \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}_{2}}\right)=\mathrm{X}_{\mathrm{t}}$ holds as well. The algorithm initializes $\mathcal{A}$ to the empty set. For each way of dividing set O into two disjoint subsets $\mathrm{O}_{1}, \mathrm{O}_{2}$ (one of which could be empty) and for each subset $\hat{O}$ (which could also be empty) of the set $X \backslash O$, the algorithm picks a number of pairs ( $P_{1}, P_{2}$ ) of partitions and adds their joins $P_{1} \sqcup P_{2}$ to the set $\mathcal{A}$. We first show that the partition $P_{1} \sqcup P_{2}$ is valid for the combination $(t, X, O)$, for each choice of pairs ( $\mathrm{P}_{1}, \mathrm{P}_{2}$ ) made by the algorithm.

So let $P_{1} \in V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], P_{2} \in V P\left[t_{2}, X, O_{2} \cup \hat{O}\right]$. By the inductive hypothesis we get that $P_{1}$ is valid for the combination $\left(t_{1}, X, O_{1} \cup \hat{O}\right)$ and $P_{2}$ is valid for the combination $\left(t_{2}, X, O_{2} \cup \hat{O}\right)$. So there exist subgraphs $G_{t_{1}}^{\prime}=\left(V_{t_{1}}^{\prime}, E_{t_{1}}^{\prime}\right)$ of $G_{t_{1}}$ and $G_{t_{2}}^{\prime}=\left(V_{t_{2}}^{\prime}, E_{t_{2}}^{\prime}\right)$ of $G_{t_{2}}$ such that

1. $X_{t} \cap V_{t_{1}}^{\prime}=X=X_{t} \cap V_{t_{2}}^{\prime}$.
2. The vertex set of each connected component of $\mathrm{G}_{\mathrm{t}_{1}}^{\prime}$ and of $\mathrm{G}_{\mathrm{t}_{2}}^{\prime}$ has a non-empty intersection with set $X$. Moreover, $P_{1}$ is the partition of $X$ defined by the subgraph $G_{t_{1}}^{\prime}$ and $P_{2}$ is the partition of $X$ defined by the subgraph $G_{t_{2}}^{\prime}$.
3. Both $v^{\star} \in V_{t_{1}}^{\prime}$ and $v^{\star} \in V_{t_{2}}^{\prime}$ hold. Further, $V_{t_{1}}^{\prime}$ is a vertex cover of graph $G_{t_{1}}$ and $V_{t_{2}}^{\prime}$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}_{2}}$.
4. The set of odd-degree vertices in $G_{t_{1}}^{\prime}$ is exactly the set $\mathrm{O}_{1} \cup \hat{O}$ and the set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}_{2}}^{\prime}$ is exactly the set $\mathrm{O}_{2} \cup \hat{\mathrm{O}}$.
Let $G_{t}^{\prime}=G_{t_{1}}^{\prime} \cup G_{t_{2}}^{\prime}$. Then $G_{t}^{\prime}$ is a subgraph of $G_{t}$, and
5. Since $X_{t} \cap V_{t_{1}}^{\prime}=X=X_{t} \cap V_{t_{2}}^{\prime}$ holds we have that $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$ holds as well.
6. The vertex set of each connected component of $G_{t}^{\prime}$ has a non-empty intersection with set $X$. Moreover, from Lemma 11 we get that $P_{1} \sqcup P_{2}$ is the partition of $X$ defined by the subgraph $G_{t}^{\prime}$.
7. $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ holds, and $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.
8. Since $E\left(G_{t_{1}}\right) \cap E\left(G_{t_{2}}\right)=\emptyset$ holds we get that the degree of any vertex $v$ in graph $G_{t}^{\prime}$ is the sum of its degrees in the two graphs $G_{t_{1}}^{\prime}$ and $G_{t_{2}}^{\prime}$. Since (i) the set of odd-degree vertices in graph $G_{t_{1}}^{\prime}$ is exactly the set $\mathrm{O}_{1} \cup \hat{O}$, (ii) the set of odd-degree vertices in graph $\mathrm{G}_{\mathrm{t}_{2}}^{\prime}$ is exactly the set $\mathrm{O}_{2} \cup \hat{\mathrm{O}}$, and (iii) O is the disjoint union of sets $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$, we get that the set of odd-degree vertices in graph $G_{t}^{\prime}$ is exactly the set $O$.
Thus graph $G_{t}^{\prime}$ is a witness for partition $P_{1} \sqcup P_{2}$ being valid for the combination ( $t, X, O$ ), and so partition $P_{1} \sqcup P_{2} \in \mathcal{A}$ is valid for the combination ( $t, X, O$ ). This proves that collection $\mathcal{A}$ satisfies the soundness criterion.

We now argue that the set $\mathcal{A}$ satisfies the completeness criterion. So let H be a residual subgraph with respect to $t$ with $V(H) \cap X_{t}=X$, for which there exists a partition $P=$ $\left\{X^{1}, X^{2}, \ldots X^{p}\right\}$ of $X$ such that $((t, X, O), P)$ completes $H$. We need to show that the set $\mathcal{A}$ computed by the algorithm contains some partition $Q$ of $X$ such that ( $(t, X, O), Q)$ completes $H$. Observe that there exists a subgraph $G_{t}^{\prime}$ of $G_{t}$-a witness for $((t, X, O), P)$-such that the following hold:

1. $X_{t} \cap V\left(G_{t}^{\prime}\right)=X$.
2. $G_{t}^{\prime}$ has exactly $p$ connected components $C_{1}, C_{2}, \ldots, C_{p}$ and for each $\mathfrak{i} \in\{1,2, \ldots, p\}$, $X^{i} \subseteq \mathrm{~V}\left(\mathrm{C}_{\mathrm{i}}\right)$ holds.
3. $v^{\star} \in \mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ holds, and $\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right)$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.

## XX:44 On Computing the Hamiltonian Index of Graphs

4. The set of odd-degree vertices in $\mathrm{G}_{\mathrm{t}}^{\prime}$ is exactly the set O .
5. The graph $G_{t}^{\prime} \cup H$ is a dominating Eulerian subgraph of $G$.

Let $\mathrm{G}_{1}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}_{1}}\right), \mathrm{E}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{E}\left(\mathrm{G}_{\mathrm{t}_{1}}\right)\right)$ and $\mathrm{G}_{2}=\left(\mathrm{V}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{V}\left(\mathrm{G}_{\mathrm{t}_{2}}\right), \mathrm{E}\left(\mathrm{G}_{\mathrm{t}}^{\prime}\right) \cap \mathrm{E}\left(\mathrm{G}_{\mathrm{t}_{2}}\right)\right)$ be, respectively, the subgraphs of $G_{t}^{\prime}$ defined by the subtrees of $\mathcal{T}$ rooted at nodes $t_{1}$ and $t_{2}$, respectively. Then $G_{t}^{\prime}=G_{1} \cup G_{2}, V\left(G_{1}\right) \cap X_{t_{1}}=V\left(G_{2}\right) \cap X_{t_{2}}=V\left(G_{1}\right) \cap V\left(G_{2}\right)=X$, and $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\emptyset$ all hold. Let $\tilde{O_{1}}, \tilde{O_{2}}$ be the sets of vertices of odd degree in graphs $G_{1}, G_{2}$, respectively. Since graph $\left(H \cup G_{1}\right) \cup G_{2}$ is Eulerian and since $V\left(H \cup G_{1}\right) \cap V\left(G_{2}\right)=X$ holds, we get that (i) $\tilde{O}_{2} \subseteq X$ holds, and (ii) every connected component of graph $G_{2}$ contains at least one vertex from set $X$. By symmetric reasoning we get that (i) $\tilde{\mathrm{O}}_{1} \subseteq X$ holds, and (ii) every connected component of graph $\mathrm{G}_{1}$ contains at least one vertex from set X . Let $\mathrm{O}_{2}=\tilde{\mathrm{O}}_{2} \cap \mathrm{O}$ and $\hat{\mathrm{O}}=\tilde{\mathrm{O}}_{2} \backslash \mathrm{O}$. Then $\tilde{\mathrm{O}}_{2}=\mathrm{O}_{2} \cup \hat{\mathrm{O}}$. Define $\mathrm{O}_{1}=\mathrm{O} \backslash \mathrm{O}_{2}$. Since (i) the set of odd-degree vertices in graph $G_{t}^{\prime}$ is exactly the set $O$, and (ii) $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\emptyset$ holds, we get that the set of odd-degree vertices in graph $\mathrm{G}_{1}$ is $\tilde{O}_{1}=\left(\mathrm{O} \backslash \mathrm{O}_{2}\right) \cup \hat{O}=\mathrm{O}_{1} \cup \hat{O}$.

Let $Q_{2}$ be the partition of set $X$ defined by graph $G_{2}$, and let $R_{1}=H \cup G_{1}$. It is straightforward to verify the following: (i) $R_{1}$ is a residual subgraph with respect to node $t_{2}$ with $V\left(R_{1}\right) \cap X_{t_{2}}=X$; (ii) graph $G_{2}$ is a witness for partition $Q_{2}$ being valid for the combination $\left(t_{2}, X, \tilde{O}_{2}\right)$, and (iii) $G_{2}$ is a certificate for ( $\left.\left(t_{2}, X, \tilde{O}_{2}\right), Q_{2}\right)$ completing the residual graph $R_{1}$. By the inductive assumption there is a partition $P_{2}$ of $X$ in the set $V P\left[t_{2}, X, O_{2} \cup \hat{O}\right]$ such that $\left(\left(t_{2}, X, O_{2} \cup \hat{O}\right), P_{2}\right)$ completes the residual graph $R_{1}$. Let $H_{2}$ be a certificate for $\left(\left(t_{2}, X, O_{2} \cup \hat{O}\right), P_{2}\right)$ completing $R_{1}$. Note that $H_{2}$ is a subgraph of $G_{t_{2}}$, and that $R_{1} \cup H_{2}=\left(H \cup G_{1}\right) \cup H_{2}$ is a dominating Eulerian subgraph of G.

Let $Q_{1}$ be the partition of set $X$ defined by graph $G_{1}$, and let $R_{2}=H \cup H_{2}$. From Lemma 33 we get that the set of odd-degree vertices of the residual subgraph H is exactly the set O, and from Definitions 30 and 31 we get that the set of odd-degree vertices of graph $\mathrm{H}_{2}$ is the set $\mathrm{O}_{2} \cup \hat{O}$. From the definition of a residual subgraph we get that $\mathrm{E}(\mathrm{H}) \cap \mathrm{E}\left(\mathrm{H}_{2}\right)=\emptyset$ holds. It follows that the set of odd-degree vertices of graph $R_{2}$ is $\left(O \backslash O_{2}\right) \cup \hat{O}=O_{1} \cup \hat{O}$, which is exactly the set of odd-degree vertices of graph $\mathrm{G}_{1}$.

It is now straightforward to verify the following: (i) $R_{2}$ is a residual subgraph with respect to node $t_{1}$ with $V\left(R_{2}\right) \cap X_{t_{1}}=X$; (ii) graph $G_{1}$ is a witness for partition $Q_{1}$ being valid for the combination ( $\left.t_{1}, X, O_{1} \cup \hat{O}\right)$, and (iii) $G_{1}$ is a certificate for $\left(\left(t_{1}, X, O_{1} \cup \hat{O}\right), Q_{1}\right)$ completing the residual graph $R_{2}$. By the inductive assumption there is a partition $P_{1}$ of $X$ in the set $V P\left[t_{1}, X, O_{1} \cup \hat{O}\right]$ such that $\left(\left(t_{1}, X, O_{1} \cup \hat{O}\right), P_{1}\right)$ completes the residual graph $R_{2}$. Let $H_{1}$ be a certificate for $\left(\left(t_{1}, X, O_{1} \cup \hat{O}\right), P_{1}\right)$ completing $R_{2}$. Note that $H_{1}$ is a subgraph of $G_{t_{1}}$, and that $R_{2} \cup H_{1}=\left(H \cup H_{2}\right) \cup H_{1}$ is a dominating Eulerian subgraph of $G$.

Let $\hat{H}=H_{1} \cup H_{2}$. Then $\hat{H}$ is a subgraph of $G_{t}$, and

1. Since $X_{t} \cap V\left(H_{1}\right)=X=X_{t} \cap V\left(H_{2}\right)$ holds we have that $X_{t} \cap V(\hat{H})=X$ holds as well.
2. The vertex set of each connected component of $\hat{H}$ has a non-empty intersection with set $X$. Moreover, from Lemma 11 we get that $P_{1} \sqcup P_{2}$ is the partition of $X$ defined by the subgraph $\hat{H}$.
3. $v^{\star} \in \mathrm{V}(\hat{\mathrm{H}})$ holds, and $\mathrm{V}(\hat{\mathrm{H}})$ is a vertex cover of graph $\mathrm{G}_{\mathrm{t}}$.
4. Since $E\left(G_{t_{1}}\right) \cap E\left(G_{t_{2}}\right)=\emptyset$ holds we get that the degree of any vertex $v$ in graph $\hat{H}$ is the sum of its degrees in the two graphs $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$. Since (i) the set of odd-degree vertices in graph $\mathrm{H}_{1}$ is exactly the set $\mathrm{O}_{1} \cup \hat{\mathrm{O}}$, (ii) the set of odd-degree vertices in graph $\mathrm{H}_{2}$ is exactly the set $\mathrm{O}_{2} \cup \hat{\mathrm{O}}$, and (iii) O is the disjoint union of sets $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$, we get that the set of odd-degree vertices in graph $\hat{\mathrm{H}}$ is exactly the set O .
Graph $\hat{H}$ is thus a witness for partition $P_{1} \sqcup P_{2}$ of $X$ being valid for the combination ( $t, X, O$ ), and $H \cup \hat{H}$ is a dominating Eulerian subgraph of G. Thus $\left((t, X, O), P_{1} \sqcup P_{2}\right)$ completes H. Since the algorithm adds partition $\mathrm{P}_{1} \sqcup \mathrm{P}_{2}$ to the set $\mathcal{A}$ we get that $\mathcal{A}$ satisfies the
completeness criterion.

We can now prove

- Theorem 29. There is an algorithm which solves an instance (G, $\mathfrak{T}, \mathrm{tw}$ ) of Dominating Eulerian Subgraph in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}}\right)$ time.

Proof. We first modify $\mathcal{T}$ to make it a "nearly-nice" tree decomposition rooted at r as described at the start of this section. We then execute the dynamic programming steps described above on $\mathcal{T}$. We return yes if the element $\left\{\left\{v^{\star}\right\}\right\}$ is present in the set $\mathrm{VP}[\mathrm{r}, \mathrm{X}=$ $\left.\left\{v^{\star}\right\}, \mathrm{O}=\emptyset\right]$ computed by the DP, and no otherwise.

From Lemma 36 we know that ( $\mathrm{G}, \mathcal{T}, \mathrm{tw}$ ) is a yes instance of Dominating Eulerian Subgraph if and only if the combination $\left(\left(r, X=\left\{v^{\star}\right\}, O=\emptyset\right), P=\left\{\left\{v^{\star}\right\}\right\}\right)$ completes the residual graph $\mathrm{H}=\left(\left\{\nu^{\star}\right\}, \emptyset\right)$. By induction on the structure of the tree decomposition $\mathcal{T}$ and using Observation 37 and Lemmas 38, 39, 40, 41, and 42 we get that the set $\mathrm{VP}[\mathrm{r}, \mathrm{X}=$ $\left.\left\{v^{\star}\right\}, \mathrm{O}=\emptyset\right]$ computed by the algorithm satisfies the correctness criteria. And since $\left\{\left\{\nu^{\star}\right\}\right\}$ is the unique partition of set $\left\{\nu^{\star}\right\}$ we get that the set $\mathrm{VP}\left[\mathrm{r}, \mathrm{X}=\left\{\nu^{\star}\right\}, \mathrm{O}=\emptyset\right]$ computed by the algorithm will contain the partition $\left\{\left\{\nu^{\star}\right\}\right\}$ if and only if $(G, \mathcal{T}, \mathrm{tw})$ is a yes instance of Dominating Eulerian Subgraph.

Note that we compute representative subsets as the last step in the computation at each bag. So we get, while performing computations at an intermediate node $t$, that the number of partitions in any set $\operatorname{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}^{\prime}, \cdot\right]$ for any child node $\mathrm{t}^{\prime}$ of t and subset $\mathrm{X}^{\prime}$ of $\mathrm{X}_{\mathrm{t}^{\prime}}$ is at most $2^{\left(\left|X^{\prime}\right|-1\right)}$ (See Theorem 12|. We use Theorem 10 to perform various operations on one or two partitions - such as adding a block to a partition, merging two blocks of a partition, eliding an element from a partition, or computing the join of two partitions-in polynomial time.

The computation at each leaf node of $\mathcal{T}$ can be done in constant time.
For an introduce vertex node or an introduce edge node or a forget node $t$ and a fixed pair of subsets $X \subseteq X_{t}, O \subseteq X$, the computation of set $\mathcal{A}$ involves-in the worst case - spending polynomial time for each partition $\mathrm{P}^{\prime}$ in some set $\mathrm{VP}\left[\mathrm{t}^{\prime}, \mathrm{X}^{\prime} \subseteq \mathrm{X}, \cdot\right]$. Since the number of partitions in this latter set is at most $2^{\left(\left|X^{\prime}\right|-1\right)} \leqslant 2^{(|X|-1)}$ we get that the set $\mathcal{A}$ can be computed in $\mathcal{O}^{\star}\left(2^{(|X|-1)}\right)$ time, and that the set $\mathcal{B}$ can be computed-see Theorem 12 in $\mathcal{O}^{\star}\left(2^{(|X|-1)} \cdot 2^{(\omega-1) \cdot|X|}\right)=\mathcal{O}^{\star}\left(2^{\omega \cdot|X|}\right)$ time. Since the number of ways of choosing the subset $\mathrm{O} \subseteq \mathrm{X}$ is $2^{\mid \mathrm{X\mid}}$ the entire computation at an introduce vertex, introduce edge, or forget node t can be done in time

$$
\begin{aligned}
\sum_{|X|=0}^{\left|X_{t}\right|}\binom{\left|X_{t}\right|}{|X|} 2^{|X|} \mathcal{O}^{\star}\left(2^{w \cdot|X|}\right) & =\mathcal{O}^{\star}\left(\sum_{|X|=0}^{\mathrm{t} w+1}\binom{\mathrm{t} w+1}{|X|} 2^{(\omega+1)|X|}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2^{(\omega+1)}\right)^{(t w+1)}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2 \cdot 2^{\omega}\right)^{\mathrm{tw}}\right)
\end{aligned}
$$

For a join node $t$ and a fixed subset $X \subseteq X_{t}$ we guess three pairwise disjoint subsets $\hat{O}, O_{1}, O_{2}$ of $X$ in time $4^{|X|}$. For each guess we go over all partitions $P_{1} \in V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], P_{2} \in$ $\mathrm{VP}\left[\mathrm{t}_{2}, \mathrm{X}, \mathrm{O}_{2} \cup \hat{\mathrm{O}}\right]$ and add their join $\mathrm{P}_{1} \sqcup \mathrm{P}_{2}$ to the set $\mathcal{A}$. Since the number of partitions in each of the two sets $V P\left[t_{1}, X, O_{1} \cup \hat{O}\right], V P\left[t_{2}, X, O_{2} \cup \hat{O}\right]$ is at most $2^{(|X|-1)}$, the size of set $\mathcal{A}$ is at most $2^{(2|\mathrm{X}|-2)}$. The entire computation at the join node can be done in time

## XX:46 On Computing the Hamiltonian Index of Graphs

$$
\begin{aligned}
\sum_{|X|=0}^{\left|X_{t}\right|}\binom{\left|X_{t}\right|}{|X|} 4^{|X|}\left(2^{(2|X|-2)}+\mathcal{O}^{\star}\left(2^{(2|X|-2)} \cdot 2^{(\omega-1) \cdot|X|}\right)\right) & =\mathcal{O}^{\star}\left(\sum_{|X|=0}^{t w+1}\binom{t w+1}{|X|} 2^{4|X|-2+\omega|X|-|X|}\right) \\
& =\mathcal{O}^{\star}\left(\sum_{|X|=0}^{t w+1}\binom{\mathrm{t} w+1}{|X|} 2^{(\omega+3)|\mathrm{X}|}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{(t w+1)}\right) \\
& =\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathrm{tw}}\right)
\end{aligned}
$$

The entire DP over $\mathcal{T}$ can thus be done in $\mathcal{O}^{\star}\left(\left(1+2^{(\omega+3)}\right)^{\mathfrak{t} \boldsymbol{w}}\right)$ time.


[^0]:    1 The $\mathcal{O}^{\star}$ () notation hides polynomial factors in input size.
    2 That is: connected, and with all vertices of even degree.

[^1]:    ${ }^{3}$ See Section 2 for definitions.
    ${ }^{4}$ We deem the empty graph-with no vertices-to be connected.

[^2]:    ${ }^{5}$ See the next section for the definition of tree decompositions and treewidth.

[^3]:    6 See Appendix A for an algorithm of our own design which solves Dominating Eulerian Subgraph (and hence EHP and EHC) in the same running time.

