On Compatibility of Two Approaches to Generalization of the Lovász Extension Formula

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Abstract. We present a method of generalization of the Lovász extension formula combining two known approaches - the first of them based on the replacement of the product operator by some suitable binary function F and the second one based on the replacement of the minimum operator by a suitable aggregation function A. We propose generalization by simultaneous replacement of both product and minimum operators and investigate pairs (F, A) yielding an aggregation function for all capacities.

Keywords: Aggregation function · Choquet integral · Capacity · Möbius transform

1 Introduction

Aggregation of several values into a single value proves to be useful in many fields, e.g., multicriteria decision making, image processing, deep learning, fuzzy systems etc. Using the Choquet integral [3] as a mean of aggregation process allows to capture relations between aggregated data through so-called fuzzy measures [9]. This is the reason of the nowadays interest in generalizations of the Choquet integral, for a recent state-of-art see, e.g., [4].

In our paper we focus on generalizations of the Choquet integral expressed by means of the so-called Möbius transform, which is also known as Lovász extension formula, see (2) below. Recently, two different approaches occured - in the first one the Lovász extension formula is modified by replacing of the product operator by some suitable binary function F and the second one is based on the replacement of the minimum operator by a suitable aggregation function A. We study the question, when these two approaches can be used simultaneously and we investigate the functional $I^m_{F,A}$ obtained in this way.

The paper is organized as follows. In the next section, some necessary preliminaries are given. In Sect. 3, we propose the new functional $I^m_{F,A}$ and exemplify the instances, when the obtained functional is an aggregation function for all

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capacities and when it is not. Section 4 contains results concerning the binary case. Finally, some concluding remarks are given.

2 Preliminaries

In this section we recall some definitions and results which will be used in the sequel. We also fix the notation, mostly according to [5], wherein more information concerning the theory of aggregation functions can be found.

Let $n \in \mathbb{N}$ and $N = \{1, \cdots, n\}$.

Definition 1. *A function $A\colon [0,1]^n \to [0,1]$ is an (n-ary) aggregation function if A is monotone and satisfies the boundary conditions $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.*

We denote the class of all n-ary aggregations functions by $\mathcal{A}_{(n)}$.

Definition 2. *An aggregation function $A \in \mathcal{A}_{(n)}$ is*

- *conjunctive, if $A(\mathbf{x}) \leq \min_{i \in N} x_i$ for all $\mathbf{x} \in [0,1]^n$,*
- *disjunctive, if $A(\mathbf{x}) \geq \max_{i \in N} x_i$ for all $\mathbf{x} \in [0,1]^n$.*

Definition 3. *A set function $m\colon 2^N \to [0,1]$ is a capacity if $m(C) \leq m(D)$ whenever $C \subseteq D$ and m satisfies the boundary conditions $m(\emptyset) = 0$, $m(N) = 1$.*

We denote the class of all capacities on 2^N by $\mathcal{M}_{(n)}$.

Definition 4. *The set function $M_m\colon 2^N \to \mathbb{R}$, defined by*

$$M_m(I) = \sum_{K \subseteq I} (-1)^{|I \setminus K|} m(K)$$

for all $I \subseteq N$, is called Möbius transform corresponding to a capacity m.

Möbius transform is invertible by means of the so-called Zeta transform:

$$m(A) = \sum_{B \subseteq A} M_m(B), \tag{1}$$

for every $A \subseteq N$.

Denote $\mathcal{R}_n \subsetneq \mathbb{R}$ the range of the Möbius transform. The bounds of the Möbius transform have recently been studied by Grabisch et al. in [6].

Definition 5. *Let $m\colon 2^N \to [0,1]$ be a capacity and $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$. Then the Choquet integral of $\mathbf{x}$ with respect to m is given by*

$$\mathbf{Ch}_m(\mathbf{x}) = \int_0^1 m(\{i \in N | x_i \geq t\})\, \mathrm{d}t,$$

where the integral on the right-hand side is the Riemann integral.

Proposition 1. *Let $m: 2^N \to [0,1]$ and $\mathbf{x} \in [0,1]^n$. Then the discrete Choquet integral can be expressed as:*

$$\mathbf{Ch}_m(\mathbf{x}) = \sum_{\emptyset \neq B \subseteq N} \left(M_m(B) \cdot \min_{i \in B} x_i \right). \tag{2}$$

Formula (2) is also known as the *Lovász extension* formula [8].

Now we recall two approaches to generalization of the formula (2). The first one is due to Kolesárová et al. [7] and is based on replacing the minimum operator in (2) by some other aggregation function in the following way:

Let $m \in \mathcal{M}_{(n)}$ be a capacity, $A \in \mathcal{A}_{(n)}$ be an aggregation function. Define $F_{m,A}\colon [0,1]^n \to \mathbb{R}$ by

$$F_{m,A}(x_1, \ldots, x_n) = \sum_{B \subseteq N} M_m(B) A(\mathbf{x}_B), \tag{3}$$

where $(\mathbf{x}_B)_i = x_i$ whenever $i \in B$ and $(\mathbf{x}_B)_i = 1$ otherwise. The authors focused on characterization of aggregation functions A yielding, for all capacities $m \in \mathcal{M}_{(n)}$, an aggregation function $F_{m,A}$ extending the capacity m, i.e., on such A that $F_{m,A} \in \mathcal{A}_{(n)}$ and $F_{m,A}(\mathbf{1}_B) = m(B)$ for all $B \subseteq N$ (here $\mathbf{1}_B$ stands for the indicator of the set B).

Remark 1. There was shown in [7] that (among others) all copulas are suitable to be taken in rôle of A in (3). For instance, taking $A = \Pi$, where $\Pi(\mathbf{x}) = \prod_{i=1}^n x_i$ is the product copula, we obtain the well-known Owen multilinear extension (see [10]).

The second approach occured recently in [2] and is based on replacing the product of $M_m(A)$ and minimum operator in the formula (2) by some function $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ in the following way:

Let $m \in \mathcal{M}_{(n)}$, $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be a function bounded on $[0,1]^2$. Define the function $\mathfrak{I}_{\mathfrak{m}}^F\colon [0,1]^n \to \mathbb{R}$ by

$$\mathfrak{I}_m^F(\mathbf{x}) = \sum_{\emptyset \neq B \subseteq N} F(M_m(B), \min_{i \in B}\{x_i\}). \tag{4}$$

The authors focused on functions F yielding an aggregation function $\mathfrak{I}_m^F$ for all capacities $m \in \mathcal{M}_{(n)}$.

Remark 2. It was shown in [2] that all functions F yielding for all $m \in \mathcal{M}_{(n)}$ aggregation functions $\mathfrak{I}_m^F$ with a given diagonal section $\delta \in \mathcal{A}_{(1)}$ are exactly those of the form

$$F(u, v) = u\, h(v) + \frac{\delta(v) - h(v)}{2^n - 1}, \tag{5}$$

where $h\colon [0,1] \to \mathbb{R}$ is a function satisfying

$$-\frac{\delta(y) - \delta(x)}{2^n - 2} \leq h(y) - h(x) \leq \delta(y) - \delta(x),$$

for all $(x, y) \in [0,1]^2$, such that $x < y$.

However, there is no full characterization of all functions F yielding an aggregation function $\mathfrak{I}_m^F$ for every $m \in \mathcal{M}_{(n)}$ in [2].

3 Double Generalization of the Lovász Extension Formula

Let $F\colon \mathbb{R} \times [0,1] \to [0,1]$ be a function bounded on $[0,1]^2$, A be an aggregation function $A \in \mathcal{A}_{(n)}$, m be a capacity $m \in \mathcal{M}_{(n)}$. We define the function $\mathcal{I}_{F,A}^m\colon [0,1]^n \to \mathbb{R}$ as

$$\mathcal{I}_{F,A}^m(\mathbf{x}) = \sum_{\emptyset \neq B \subseteq N} F(M_m(B), A(\mathbf{x}_B)), \tag{6}$$

where $(\mathbf{x}_B)_i = x_i$ whenever $i \in B$ and $(\mathbf{x}_B)_i = 1$ otherwise.

Lemma 1. *Let $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be a function bounded on $[0,1]^2$ and $c \in \mathbb{R}$. Let $F_c\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be a function defined by*

$$F_c(x, y) = F(x, y) + c(x - \frac{1}{2^n - 1}).$$

Then, that for any $m \in \mathcal{M}_{(n)}$, it holds $\mathcal{I}_{F,A}^m(\mathbf{x}) = \mathcal{I}_{F_c,A}^m(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

Proof. Since $\sum_{\emptyset \neq B \subseteq N} c \left(M_m(B) - \frac{1}{2^n - 1} \right) = 0$, the result follows.

Consequently, one can consider $F(0,0) = 0$ with no loss of generality (compare with Proposition 3.1 in [2]).

Let us define

$$\mathcal{F}_0 = \{F\colon \mathbb{R} \times [0,1] \to \mathbb{R} \mid F(0,0) = 0 \text{ and } F \text{ is bounded on } [0,1]^2\}$$

Definition 6. *A function $F \in \mathcal{F}_0$ is I-*compatible *with an aggregation function $A \in \mathcal{A}_{(n)}$ iff $\mathcal{I}_{F,A}^m \in \mathcal{A}_{(n)}$ for all $m \in \mathcal{M}_{(n)}$.*

Note that, according to Remark 1, the product operator $\Pi(u, v) = uv$ is I-compatible with every copula. Next, according to Remark 2, all binary functions of the form (5) are I-compatible with $A = \min$.

Example 1. Let $F(u, v) = \frac{v}{2^n - 1}$, $A \in \mathcal{A}_{(n)}$ be a conjunctive aggregation function. We have

$$\mathcal{I}_{F,A}^m(\mathbf{x}) = \frac{1}{2^n - 1} \sum_{\emptyset \neq B \subseteq N} A(\mathbf{x}_B).$$

Clearly, it is a monotone function and $\mathcal{I}_{F,A}^m(\mathbf{1}) = 1$. Moreover, conjunctivity of A gives $\mathcal{I}_{F,A}^m(\mathbf{0}) = 0$. Thus, $\mathcal{I}_{F,A}^m$ is an aggregation function for all capacities $m \in \mathcal{M}_{(n)}$ and therefore F is I-compatible with every conjunctive aggregation function $A \in \mathcal{A}_{(n)}$.

Example 2. Let $f\colon [0,1] \to [0,1]$ be a nondecreasing function such that $f(0) = 0$ and $f(1) = 1$, i.e., $f \in \mathcal{A}_{(1)}$. Let $F(u,v) = (2-2^n)u + f(v)$. Then F is I-compatible with every disjunctive aggregation function $A \in \mathcal{A}_{(n)}$. Indeed, disjunctivity of A implies $A(\mathbf{x}_B) = 1$ for all $x \in [0,1]^n$, $\emptyset \neq B \subsetneq N$. Then, using (1), we obtain

$$\begin{aligned}\mathcal{I}^m_{F,A}(\mathbf{x}) &= (2-2^n) \sum_{\emptyset \neq B \subseteq N} M_m(B) + \sum_{\emptyset \neq B \subseteq N} f(A(\mathbf{x}_B)) \\ &= 2 - 2^n + f(A(\mathbf{x})) + \sum_{\emptyset \neq B \subsetneq N} f(A(\mathbf{x}_B)) \\ &= 2 - 2^n + f(A(\mathbf{x})) + 2^n - 2 = f(A(\mathbf{x})),\end{aligned}$$

which is an aggregation function for all $m \in \mathcal{M}_{(n)}$.

On the other hand, for $n > 1$, F is not I-compatible with the minimal aggregation function A_* defined as $A_*(\mathbf{x}) = 1$ if $\mathbf{x} = \mathbf{1}$ and $A_*(\mathbf{x}) = 0$ otherwise, since in this case $\mathcal{I}^m_{F,A_*}(\mathbf{x}) = 2 - 2^n$ for all $\mathbf{x} \neq \mathbf{1}$. Note that for $n = 1$ we obtain $\mathcal{I}^m_{F,A_*} = A_*$.

For a measure $m \in \mathcal{M}_{(2)}$ let us denote $m(\{1\}) = a$ and $m(\{2\}) = a$.

Example 3. Let $n = 2$. Let $F(u,v) = u\, v^{u+1}$, $A(x,y) = \max\{x+y-1, 0\}$. Then

$$\mathcal{I}^m_{F,A}(x,y) = \begin{cases} ax^{a+1} + by^{b+1} + (1-a-b)(x+y-1)^{2-a-b} & \text{if } x+y \geq 1 \\ ax^{a+1} + by^{b+1} & \text{otherwise} \end{cases},$$

which is an aggregation function for all $m \in \mathcal{M}_{(2)}$, thus F is I-compatible with A.

However, taking a disjunctive aggregation function in rôle of A, we obtain

$$\mathcal{I}^m_{F,A}(x,y) = a + b + (1-a-b)A(x,y)^{2-a-b},$$

which is not an aggregation function for all capacities up to the minimal one ($a = b = 0$). Hence, F is not I-compatible with any disjunctive aggregation function.

4 Binary Case

Let $n = 2$. Then the function $\mathcal{I}^m_{F,A}$ defined by (6) can be expressed as

$$\mathcal{I}^m_{F,A}(x,y) = F(a, A(x,1)) + F(b, A(1,y)) + F(1-a-b, A(x,y)). \tag{7}$$

Proposition 2. *Let $F \in \mathcal{F}_0$, $A \in \mathcal{A}_{(2)}$. Then F is I-compatible with A iff the following conditions are satisfied*

(i) There exist constants $k, \kappa \in \mathbb{R}$ such that for any $u \in \mathcal{R}_2 = [-1,1]$ it holds
$F(u, A(0,1)) = F(u, A(1,0)) = k(u - \frac{1}{2})$
$F(u,0) = ku,$
$F(u,1) = \kappa u + \frac{1-\kappa}{3}.$

(ii) For all $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$ it holds

$$F(a, A(x',1)) - F(a, A(x,1)) + F(1-a-b, A(x',y)) - F(1-a-b, A(x,y)) \geq 0$$

and

$$F(b, A(1,y')) - F(b, A(1,y)) + F(1-a-b, A(x,y')) - F(1-a-b, A(x,y)) \geq 0,$$

for any $a, b \in [0,1]$.

Proof. It can easily be checked that conditions (i) ensure boundary conditions $\mathcal{I}^m_{F,A}(0,0) = 0$ and $\mathcal{I}^m_{F,A}(1,1) = 1$. To show necessity, let us consider the following equation:

$$\mathcal{I}^m_{F,A}(0,0) = F(a, A(0,1)) + F(b, A(1,0)) + F(1-a-b, A(0,0)) = 0,$$

for all $a, b \in [0,1]$.

Denoting $F(u, A(0,1)) = f(u)$, $F(u, A(1,0)) = h(u)$ and $F(u, 0) = g(u)$, for $u \in [-1,1]$, the previous equation takes form

$$f(a) + h(b) + g(1-a-b) = 0, \tag{8}$$

for all $a, b \in [0,1]$. Following techniques used for solving Pexider's equation (see [1]), we can put $a = 0$ and $b = 0$ respectively, obtaining

$$\begin{aligned} f(0) + h(b) + g(1-b) &= 0, \\ f(a) + h(0) + g(1-a) &= 0. \end{aligned}$$

Thus, for any $t \in [0,1]$, we have

$$\begin{aligned} f(0) + h(t) + g(1-t) &= 0, \\ f(t) + h(0) + g(1-t) &= 0. \end{aligned}$$

Consequently

$$h(t) = f(t) + f(0) - h(0), \tag{9}$$
$$g(t) = -f(1-t) - h(0). \tag{10}$$

Therefore, formula (8) turns into

$$f(a+b) = f(a) + f(b) + f(0) - 2h(0),$$

for all $a, b \in [0,1]$. Now, denoting $\varphi(t) = f(t) + f(0) - 2h(0)$, we get

$$\varphi(a+b) = \varphi(a) + \varphi(b), \tag{11}$$

which is the Cauchy equation. Taking $a = b = 0$, we get $\varphi(0) = 0$. Therefore, putting $a = t, b = -t$, we get $\varphi(t) = -\varphi(-t)$, i.e., φ is an odd function. Since we suppose F to be bounded on $[0,1]^2$, according to Aczél [1], all solutions of the

Eq. (11) on the interval $[-1,1]$ can be expressed as $\varphi(t)=kt$, for some $k\in\mathbb{R}$. Therefore,

$$f(t)=kt-f(0)+2h(0),$$

for all $t\in[-1,1]$, which for $t=0$ gives $f(0)=h(0)$. Denoting $f(0)=c$, by (9) and (10) we obtain

$$\begin{aligned}f(t)&=kt+c,\\ h(t)&=kt+c,\\ g(t)&=-f(1-t)-h(0)=kt-k-2c.\end{aligned}$$

Since by assumption $g(0)=F(0,0)=0$, we have $c=-\frac{k}{2}$ and consequently,

$$f(t)=h(t)=k(t-\frac{1}{2}), \tag{12}$$

$$g(t)=kt, \tag{13}$$

for all $t\in[-1,1]$ as asserted.

The second boundary condition for $\mathcal{I}^m_{F,A}$ gives

$$\mathcal{I}^m_{F,A}(1,1)=F(a,A(1,1))+F(b,A(1,1))+F(1-a-b,A(1,1))=1,$$

for all $a,b\in[0,1]$. As A is an aggregation function, it holds $A(1,1)=1$. Denoting $F(u,1)=\psi(u)$ we obtain

$$\psi(a)+\psi(b)+\psi(1-a-b)=1,$$

for all $a,b\in[0,1]$. Similarly as above, this equation can be transformed into the Cauchy equation (see also [2]) having all solutions of the form $\psi(t)=\kappa t+\frac{1-\kappa}{3}$, for $\kappa\in\mathbb{R}$ and $t\in[-1,1]$.

The conditions (ii) are equivalent to monotonicity of $\mathcal{I}^m_{F,A}$, which completes the proof.

Considering aggregation functions satisfying $A(0,1)=A(1,0)=0$ (e.g., all conjunctive aggregation functions are involved in this subclass), the conditions in Proposition 2 ensuring the boundary conditions of $I^m_{F,A}$ can be simplified in the following way.

Corollary 1. *Let $F\in\mathcal{F}_0$, $A\in\mathcal{A}_{(2)}$ be an aggregation function with $A(0,1)=A(1,0)=0$. Then the following holds:*

(i) $I^m_{F,A}(0,0)=0$ *iff* $F(u,0)=0$ *for any* $u\in\mathcal{R}_2$,
(ii) $I^m_{F,A}(1,1)=1$ *iff there exist a constant* $\kappa\in\mathbb{R}$ *such that* $F(u,1)=\kappa u+\frac{1-\kappa}{3}$ *for any* $u\in\mathcal{R}_2$*. Moreover, if* F *is* I*-compatible with* A*, then* $\kappa\in[-\frac{1}{2},1]$.

Proof. We have $F(u, A(0,1)) = F(u, A(1,0)) = F(u, 0)$ for all $u \in \mathcal{R}_n$. The conditions (i) in Proposition 2 yield $k(u - \frac{1}{2}) = ku$, and consequently $k = 0$, thus $F(u, 0) = 0$ for all $u \in \mathcal{R}_n$ as asserted.

Supposing that F is I-compatible with A and considering nondecreasingness of $\mathcal{I}^m_{F,A}$ in the first variable, we obtain

$$\begin{aligned} 0 &\leq \mathcal{I}^m_{F,A}(0,0) - \mathcal{I}^m_{F,A}(1,0) \\ &= F(a, A(0,1)) - F(a, A(1,1)) + F(1-a-b, A(0,0)) - F(1-a-b, A(1,0)) \\ &= F(a,0) - F(a,1) + F(1-a-b,0) - F(1-a-b,0) \\ &= F(a,0) - F(a,1), \end{aligned}$$

for all $a \in [0,1]$.

Hence,

$$0 = F(u,0) \leq F(u,1) = \kappa u + \frac{1-\kappa}{3}$$

for all $u \in [0,1]$ and consequently $-\frac{1}{2} \leq \kappa \leq 1$, which completes the proof.

Considering aggregation functions satisfying $A(0,1) = A(1,0) = 1$ (e.g., all disjunctive aggregation functions are involved in this subclass), the conditions in Proposition 2 ensuring the boundary conditions of $I^m_{F,A}$ can be simplified in the following way.

Corollary 2. *Let $F \in \mathcal{F}_0$, $A \in \mathcal{A}_{(2)}$ be an aggregation function with $A(0,1) = A(1,0) = 1$. Then the following holds:*

(i) $I^m_{F,A}(0,0) = 0$ *iff* $F(u,0) = -2u$ *for any* $u \in \mathcal{R}_2$,
(ii) $I^m_{F,A}(1,1) = 1$ *iff* $F(u,1) = -2u+1$ *for any* $u \in \mathcal{R}_2$,

Proof. Since $A(0,1) = A(1,0) = 1$, the Eq. (8) takes form

$$f(a) + f(b) + g(1-a-b) = 0,$$

for all $a, b \in [0,1]$. Taking $b = 1-a$ and considering $g(0) = 0$ we obtain

$$f(a) = -f(1-a),$$

for all $a \in [0,1]$, and thus $f(\frac{1}{2}) = 0$. Proposition 2(i) yields

$$F\left(\frac{1}{2}, 1\right) = \frac{\kappa}{2} + \frac{1-\kappa}{3} = 0,$$

thus $\kappa = -2$, and consequently formulae (12),(13) imply the assertion.

5 Conclusion

We have introduced a new functional $I^m_{F,A}$ generalizing the Lovász extension formula (or the Choquet integral expressed in terms of Möbius transform) using simultaneously two known approaches. We have investigated when the obtained functional is an aggregation function for all capacities and exemplified positive and negative instances. In case of the binary functional we have found a characterization of all pairs (F, A) which are I-compatible, i.e., yielding an aggregation function $I^m_{F,A}$ for all capacities m. In our future reasearch we will focus on the characterization of all I-compatible pairs (F, A) in general n-ary case. Another interesting unsolved problem is the problem of giving back capacity, i.e., characterization of pairs (F, A) satisfying $I^m_{F,A}(\mathbf{1}_E) = m(E)$ for all $E \subseteq N$.

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