Foundations of Finitely Supported Structures

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A Set Theoretical Viewpoint

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"The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite."

David Hilbert

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Preface

Finitely supported structures have historical roots in the permutation models of Zermelo-Fraenkel set theory with atoms elaborated by Fraenkel and Mostowski in the 1930s in order to prove the independence of the axiom of choice from the other axioms of a set theory with atoms. They are also related to the recent developments of Fraenkel and Mostowski axiomatic set theory and of nominal sets; from the beginning of this century, finitely supported structures appeared in computer science to describe new ways of presenting the syntax of formal systems involving variable-binding operations. Inductively defined finitely supported sets involving the name-abstraction together with Cartesian product and disjoint union can encode syntax modulo renaming of bound variables. In this way, the standard theory of algebraic data types can be extended to include signatures involving binding operators. In particular, there is an associated notion of structural recursion for defining syntax-manipulating functions and a notion of proof by structural induction. Various generalizations of finitely supported sets were used in order to study automata, languages or Turing machines that operate over infinite alphabets; for this a relaxed notion of finiteness called 'orbit finiteness' was defined and means 'having a finite number of orbits under a certain group action'. Finitely supported sets were studied from both a set theoretical perspective (by M.J. Gabbay who introduced the so called axiomatic Fraenkel-Mostowski set theory which is actually Zermelo-Fraenkel set theory with atoms together with a new finite support axiom requiring the existence of a finite support for every hierarchical set theoretical construction) and a categorical perspective (by A. Pitts who defined nominal sets as classical Zermelo-Fraenkel sets equipped with a canonical group action of the group of permutations of a fixed ZF set of basic elements called 'the set of atoms' by analogy with the Fraenkel-Mostowski approach, satisfying additionally a finite support requirement; nominal sets represent an alternative Zermelo-Fraenkel approach to the non-standard axiomatic Fraenkel-Mostowski set theory).

In this book we also equip classical sets with permutation actions. The world of finitely supported sets contains the family of classical (non-atomic) Zermelo-Fraenkel sets (having the property that all of their elements are empty supported), and the family of atomic sets (which contain at least one 'basic element/atom' some-where in their structure) having finite supports as elements in the powerset of a set equipped with a permutation action. The main goal of this book is to present a set theoretical approach for studying the foundations of finitely supported sets and of related topics. In this sense we analyze the consistency of various forms of choice, as well as the consistency of results regarding cardinality, maximality and infinity, in the framework of finitely supported sets. We also introduce finitely supported algebraic structures as finitely supported sets that are equipped with finitely supported algebraic laws or with finitely supported relations. We present detailed examples of finitely supported partially ordered sets and finitely supported lattices, and we provide new properties of them. Some properties (especially fixed point properties, properties regarding cardinalities order, or results of comparing various forms of infinity) are specific to the theory of finitely supported sets, leading from the finite support requirement. A complete listing of the properties of basic elements (atoms) in the framework of finitely supported sets is also carried out. The notion of infinity is described within finitely supported sets, and several definitions of infinity are compared internally in this new framework. Finally, we present the concepts of freshness and abstractions from a slightly different perspective than in the theory of nominal sets.

The translation of a result from a non-atomic framework into an atomic framework could be quite complicated. For example, in Zermelo-Fraenkel framework it is known that both Kurepa maximal antichain principle and multiple choice principle imply the axiom of choice. Such a result is not preserved in Zermelo-Fraenkel set theory with atoms as proved by Jech. The translation of a result from a non-atomic framework into an atomic framework of sets with finite supports is even much more complicated. We analyze if a classical Zermelo-Fraenkel result (obtained for non-atomic sets) can be adequately reformulated by replacing 'set' with 'finitely supported set (under the canonical permutation action)' in order to remain valid also for atomic sets with finite support. We investigate what results in the classical non-atomic set theory are preserved in the theory of finitely supported atomic sets. We also analyze if there are specific properties of finitely supported sets that do not have a related Zermelo-Fraenkel (non-atomic) correspondent. In this way, infinite structures hierarchically constructed from the related set of basic elements can be characterized in a finitary manner by analyzing their finite supports. A meta-theoretical principle that works within the world of finitely supported sets states that for any finite set S of atoms, anything that is definable (in a higher-order logic) from S-supported structures using S-supported constructions is S-supported. However, the formal application of this method actually consists in a hierarchical step-by-step construction of the support of a certain structure by employing the supports of the substructures of a related structure, and has limitations related to results requiring choice principle or hidden choice.

The book represents a set theoretical development for the (set theoretical) foundations of the theory of finitely supported sets and structures (either originally presented as Fraenkel-Mostowski sets, or later defined as nominal sets). We collect various results on this topic and present them in a uniform manner. More than half of the results presented in this book are original, especially all the results regarding choice principles and their equivalences, results regarding cardinalities (Trichotomy, Cantor-Schröder-Bernstein theorem and its dual, cardinals arithmetic, cardinals ordering, Dedekind infinity, Tarski infinity, Mostowski infinity), results regarding the relationship between various forms of infinity, specific fixed point properties for finitely supported ordered structures, constructions of finitely supported algebraic structures with their specific properties, important properties of atoms (and also of functions on atoms and of higher-order constructions on atoms), and properties of connecting atomic and non-atomic sets. Therefore, this is a pure theoretical book accessible to a broad audience. We do not discuss here computer science applications of finitely supported sets (which are treated in [44] by using nominal sets).

To conclude, we focus on set theoretical foundations and go back to the original Fraenkel and Mostowski approach. Nominal sets are called in this book 'invariant sets' motivated by Tarski's approach on logicality (logical notions are, according to Tarski, those notions which are left invariant under the effect of the one-to-one transformations of the universe of discourse onto itself). We discuss foundations of the finitely supported sets, meaning that we analyze the consistency of various Zermelo-Fraenkel results within the framework of the invariant sets where atomic structures are involved, and also present specific properties of atomic structures. There is no major difference regarding 'Finitely Supported Mathematics' (which is a generic name for the theory of finitely supported algebraic structures) and the 'nominal approach' related to basic definitions, except that the nominal approach (whose value we certainly recognize) is related to computer science applications, while we work on foundations of mathematics and experimental sciences (by studying the validity, the consistency and the inconsistency of various results within the framework of atomic finitely supported sets). Our goal is not to re-brand the nominal framework, but to provide a full collection of set theoretical results regarding the foundations of finitely supported structures.

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