NP-completeness Results for Graph Burning on Geometric Graphs

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Abstract. Graph burning runs on discrete time steps. The aim is to burn all the vertices in a given graph in the least number of time steps. This number is known to be the burning number of the graph. The spread of social influence, an alarm, or a social contagion can be modeled using graph burning. The less the burning number, the faster the spread. Optimal burning of general graphs is NP-Hard. There is a 3-approximation algorithm to burn general graphs where as better approximation factors are there for many sub classes. Here we study burning of grids; provide a lower bound for burning arbitrary grids and a 2-approximation algorithm for burning square grids. On the other hand, burning path forests, spider graphs, and trees with maximum degree three is already known to be NP-Complete. In this article we show burning problem to be NP-Complete on connected interval graphs, permutation graphs and several other geometric graph classes as corollaries.

1 Introduction

The spread of social influence in order to analyze a social network is an important topic of study [4,15,16]. Kramer et al. [17] have highlighted that the underlying network plays an essential role in the spread of an emotional contagion; they have nullified the necessity of in-person interaction and non-verbal cues. With the aim to model such problems, *Graph Burning* was introduced in [7]. Graph burning is also inspired by other contact processes like *firefighting* [11], *graph cleaning* [2], and *graph bootstrap percolation* [3]. Burning a graph can be used to model the spread of a meme, gossip, or a social contagion, influence or emotion. It can also be used to model the viral infections: the exposure to infections and proliferation of virus.

Graph burning runs on discrete time-steps (or rounds) as follows: in each time-step t, first (a) an arbitrary vertex is burnt from "outside" (it is selected as a $fire\ source$), and then, (b) the fire spreads to the vertices that are one hop neighbors of the already burnt vertices (burnt by round t-1); this process continues till all the vertices of the given graph are burned. Observe that some fire source selected at round t does not spread fire to its one hop neighbors in the same round. The sequence of fire sources, selected one in each round until a graph is completely burnt, is called a $burning\ sequence$ of that graph. The minimum

time steps (equivalently, number of fire sources) required to burn a graph G is called the *burning length* or the *burning number* of G, and is denoted by b(G). The less the value of b(G), the faster it is to spread the fire, and therefore burn all the vertices in G. The graph burning problem is to find an optimal burning sequence for a given graph G. At places, we use burning problem to refer the same.

The underlying decision problem for graph burning is as follows: the given input is an arbitrary graph G and an integer k, the task is to determine if Gcan be burned in k or less rounds. Bessy et al. [5] showed that optimal graph burning is an NP-Complete problem. They also showed that burning spider graphs, trees with maximum degree three and path forests is NP-Complete. In this article, we study the graph burning problem on interval graphs and grids. Interval graphs are formed from a set of closed intervals on the real line such that each interval corresponds to a vertex and the vertices corresponding to two such intervals are connected only if they overlap on the real line. Grids are formed by a set of equidistant horizontal and vertical lines intersecting at right angles such that each intersection point corresponds to a vertex and all the (induced) line segments joining those vertices are considered as edges. We prove NP-Completeness results for interval graphs. Our construction and proof technique are similar to [5]. We also provide matching bounds for burning grids. Our Contribution: We provide a lower bound for the burning number of grids of arbitrary size and a 2-approximation algorithm for burning square grids (Section 4). We prove burning connected interval graphs (Section 5) and general permutation graphs (Section 6) to be NP-Complete. We also report hardness results on some more graph classes (Section 7.1) as corollaries of our constructions and proofs.

2 Preliminary definitions and symbols

We mention below some of the notations used in this article. Let G be a graph. We denote the set of vertices in G by V(G). The distance between two vertices imply the number of edges contained in the shortest path between those two vertices in G. The radical center of a graph means the vertex from which the shortest distance to the furthest vertex is minimum. We define \bigcup_{S} to be the left sequential union. This operation can add a single element to a sequence, or merge two sequences. As an example, let P = (a, b) be a sequence, then after executing the statement $P = P \cup_{S} (c)$, P becomes (c, a, b). Similarly \bigcup_{S} is defined as the right sequential union. Let Q_1 , Q_2 be two paths. By joining these two paths in order Q_1 , Q_2 , we mean adding an edge between the last vertex of Q_1 and the first vertex of the Q_2 . Let Q_1 be a set of natural numbers. We denote the sum of all numbers in Q_1 as Q_2 . The largest element in Q_1 is denoted by Q_2 .

Let W be a non-empty set of vertices such that $W \subset V(G)$. The set of vertices that are at most at a distance i from W in G, including W is denoted by $G.N_i[W]$. The set W may be a set containing a single element. Let $S = (x_1, x_2, ..., x_k)$ be a

burning sequence of G of size k such that, x_i is chosen as the fire source in round i. The burning cluster (or simply cluster, when it is clear from the context) of a fire source x_i is the set $G.N_{k-i}[x_i]$. Precisely, it is the set of vertices, to whom the fire source x_i is able to spread fire to in the remaining (k-i) rounds. Observe that, x_i would be able to burn all of it's (k-i) hop neighbors. Now it is easy to see that, if S is able to burn G completely, then eq. (1) must hold true [5].

$$G.N_{k-1}[x_1] \cup G.N_{k-2}[x_2] \cup \dots \cup G.N_0[x_k] = V(G)$$
(1)

For the NP-Completeness proof, we reduce distinct 3-partition problem to our problem. The input of the distinct 3-partition problem is a set of distinct natural numbers, $X = \{a_1, a_2, ..., a_{3n}\}$, such that $\sum_{i=1}^{3n} a_i = nB$ where $\frac{B}{4} < a_i < \frac{B}{2}$. The task is to determine if X can be partitioned into n sets, each containing 3 elements such that sum of each set equals B. Note that, B can only be a natural number as it is a sum of 3 natural numbers. It is well known that the distinct 3-partition problem is NP-Complete in the strong sense (see [12,5]).

3 Related Works

The burning problem was introduced by Bonato et al. (2016) [7]. This work showed that the burning number of a path or cycle of length p is $\lceil \sqrt{p} \rceil$ along with some other properties and results. Bessy et al. [5] showed that burning a general graph is NP-Complete: they showed that burning spider graphs, trees, and path-forests are NP-Complete. A 3-approximation algorithm for burning general graphs was described in [5]. Bonato et al. [8] proposed a 2-approximation algorithm for burning trees. A 1.5-approximation algorithm for burning path-forests was described in [9]. A 2-approximation algorithm for burning graphs that are bounded by a diameter of constant length was described in [13]. There are works providing upper bounds on the burning number of some classes of graphs. Authors in [10] as well as [9] showed that burning number of spider graphs of order n is at most \sqrt{n} . Bessy et al. [6] provided a bound on the burning number of a connected graph of order n, and a special class of trees. Simon et al. [1] presented systems that utilize burning in the spread of an alarm through a network. Simon et al. [18] provided heuristics to minimize the time steps in burning a graph. Kamali et al. [13] provides upper bound on burning number for the graphs with bounded path length and also for the graphs with minimum degree δ . Along with this, authors in [13] (also [14]) discussed bounds on the burning number of interval graphs and showed almost tight bounds. Although, they have not provided any algorithm to find an optimal burning sequence. Despite the known bounds on the burning number and the fact that most of other properties of interval graphs can be computed in polynomial time, we show that burning connected interval graphs turns out to be NP-Complete. Also we study graph burning on grids which is a graph of constant minimum degree.

4 Burning grids

In this section we study graph burning problem on grids by providing a lower bound for grids of arbitrary size and an 2-approximation algorithm for square grids. According to [13], the upper bound on burning number of graphs with constant minimum degree is $O(\sqrt{n})$. Here we provide a better upper bound and a matching lower bound for this specific class of graphs.

First we analyze at most how many nodes can be burnt by an arbitrary fire source inside the grid. We show an example in Figure 1 (a). Let the maximum number of vertices that can be burned by a single fire source in k rounds be denoted by f_k . We compute f_k using the recurrence relations as follows.

$$f_1 = 1$$

$$f_k = 4(k-1) + f_{k-1}$$

At k^{th} time step after a fire source is placed, the number of vertices which can be burned is $1+4+8+12+\ldots+4(k-1)=1+4(1+2+3+\ldots+(k-1))=1+4\times\frac{k(k-1)}{2}=2k(k-1)+1.$

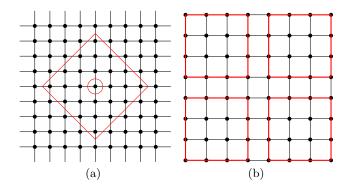


Fig. 1. (a) On a grid, a fire source (circled) is able to burn 1 vertex in 1 round, 5 vertices in 2 rounds, 13 vertices in 3 rounds, 25 vertices in 4 rounds, and so on. (b) An 8×8 grid divided into four 4×4 subgrids.

The Lower Bound: We prove the following lemma on the lower bound of burning number on any arbitrary grid of size $l \times b$.

Lemma 1. To burn a grid of size $l \times b$ or more, we need a burning sequence containing at least $(l \times b)^{\frac{1}{3}}$ fire sources.

Proof. Consider a grid of size $l \times b$, where l and b are any positive integers. As discussed above, $f_k = 2k(k-1)+1$. So, if i be the burning number of the grid, then the total number of vertices that are burned by this burning sequence will be $1+5+13+25+...+2i(i-1)+1=(2(0+2+6+12+20+...+(i^2-i))+i)=2((1^2-1)+(2^2-2)+(3^2-3)+...+(i^2-i))+i=2((1^2+2^2+3^2+...+i^2)-(1+2+3+...+i))+i=2\left(\frac{i(i+1)(2i+1)}{6}-\frac{i(i+1)}{2}\right)+i=\frac{i(i+1)(2i+1)}{3}-i(i+1)+i=\frac{i(i+1)(2i-2)}{3}+i=\frac{i(2i^2-2+3)}{3}=\frac{i(2i^2+1)}{3}.$

A burning sequence of size i shall be able to burn at most $\frac{2i^3+i}{3}$ vertices on the grid. So to burn the grid in i rounds, we must have,

$$\frac{2i^3 + i}{3} \ge l \times b \tag{2}$$

Since $\frac{2i^3+i}{3} \le i^3 \quad \forall i \ge 1$ and the burning sequence burns all the vertices, we have $i^3 \ge \frac{2i^3+i}{3} \ge l \times b$. This implies $i \ge (l \times b)^{\frac{1}{3}}$.

We state the following corollary.

Corollary 1. To burn a square grid of size n, the burning number needs to be at least $n^{\frac{1}{3}}$.

Now we describe the following algorithm to burn an arbitrary grid G of size $l \times b$. We further show that this is a 2-approximation algorithm for burning any $l \times l$ square grid with $l \geq 403$.

The Algorithm: Divide G into subgrids (see Figure 1 (b) for example) of dimensions $l^{\frac{2}{3}} \times b^{\frac{2}{3}}$. Represent the resultant subgrids by $g_1, g_2, ..., g_k$ where k is the count of subgrids obtained. Let S be the sequence of fire sources, initially empty. For $1 \leq i \leq k$, put the radical center x_i of subgrid g_i as the i-th fire source in S. If G is not completely burnt by those k fire sources, then in each step $i \geq k+1$, continue putting unburnt vertices from G in S until G is completely burnt.

Theorem 1. Our algorithm is able to burn a grid G within an approximation factor of 2 if G is a $l \times l$ square grid with $l \ge 403$.

Proof. The algorithm divides the grid in to at most $(l^{\frac{1}{3}}+1)\times(l^{\frac{1}{3}}+1)$ subgrids. In each round $i,\ x_i$ is set to be the i-th fire source. As the fire source is placed at the radical center of a subgrid, and the radius of the subgrid is $l^{\frac{2}{3}}$, so it takes $l^{\frac{2}{3}}$ rounds to burn the corresponding subgrid. As the last fire source may take up to $l^{\frac{2}{3}}$ rounds to burn the respective subgrid, our algorithm takes a total of at most $\lceil (l^{\frac{1}{3}}+1)\times(l^{\frac{1}{3}}+1)+l^{\frac{2}{3}} \rceil$ rounds i.e., $\lceil 2l^{\frac{2}{3}}+2l^{\frac{1}{3}}+1 \rceil$ rounds to burn G completely. Next we see what can not be burnt using half of the rounds that our algorithm takes in worst case.

It is impossible to burn any $l \times l$ square grid with $l \geq 403$ in less than $\frac{\lceil 2l^{\frac{2}{3}}+2l^{\frac{1}{3}}+1 \rceil}{2}$ rounds as $i=\frac{\lceil 2l^{\frac{2}{3}}+2l^{\frac{1}{3}}+1 \rceil}{2}$ value does not satisfy Equation 2 for b=l where $l \geq 403$. Remember, to burn G completely in i rounds, Equation 2 must be satisfied. Hence the proof.

5 Burning interval graphs

We show that burning connected interval graphs is NP-Complete by giving a reduction from the distinct 3-partition problem. We construct interval graph from any given instance of the distinct 3-partition problem. We do so by replacing each spider structure by a "comb structure" in the construction of the NP-Completeness proof for burning trees in [5], which we elaborate later in this

section. But before going to that, we have the following discussion that tries to relate burning an interval graph to burning a path.

Bonato et al. (2016) [7] proved that a path or a cycle of n vertices can be burned in $\lceil \sqrt{n} \rceil$ steps. Note the following observation from the above fact.

Observation 1 The burning clusters of each of the n fire sources of any optimal burning sequence of a path of n^2 vertices are pairwise disjoint.

We provide an example of burning a path of size nine as shown in the Figure 2. The vertex v_3 is chosen as fire source x_1 in time t=1 and x_1 is burned at this step. In time t=2, v_7 is chosen as the next fire source x_2 and it is burned in this step. Along with this, the one hop neighbors v_2 , v_4 of the already burnt (by step 1) vertex v_3 also are burned in this step. In time t=3, v_9 is selected as the third fire source and subsequently is burned in this step. Also the one hop neighbors v_1 , v_5 , v_6 and v_8 are burned in this step by the spread of fire from the already burnt vertices (by step 2).

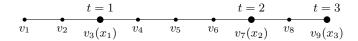


Fig. 2. Path of length nine is burnt in three steps.

We would like to recall another result from [13,14] on the bounds on burning number of interval graphs as the following observation.

Observation 2 Let L be a maximum length path among the all pair shortest paths of an interval graph G. Then $b(L) \le b(G) \le b(L) + 1$.

Also note that, finding such L is easy to do in polynomial time. We can simply compute all pair shortest path and choose the maximum length path among all. Then we can see from the proof of Observation 2 that burning an interval graph in (b(L)+1) rounds, i.e., at most in (b(G)+1) rounds is trivial. We study whether finding a burning sequence of length b(G) is possible in polynomial time, especially if b(G) = b(L) for an interval graph G. We show that determining whether b(G) = b(L) is NP-Complete.

General idea: First we provide a general idea behind our approach. We prove the Np-Completeness of burning interval graphs by giving a reduction from the distinct 3-partition problem. We construct interval graph from any given instance of the distinct 3-partition problem. We show that burning this interval graph is possible optimally in polynomial time if and only if one can solve the distinct 3-partition problem. While describing the idea, we refer to few notations here which are defined in Section 5.1.

We start with any input X of the distinct 3-partition problem. First we construct another set X' from X such that all the elements of X' are odd. The reason behind moving to X' is, we aim to use the fact that the sizes of

the burning clusters of the fire sources on a path are all odds if the length of the path is a perfect square. First we construct a path P_I of length $(2m+1)^2$ (where $m = \max(X)$) by combining few subpaths of shorter lengths. Note that $b(P_I) = 2m+1$. Then we add few vertices and corresponding edges to some of the subpaths T_j of P_I in such a way that it remains an interval graph. We call it IG(X) (Section 5.1). The optimal burning number b(IG(X)) takes the value 2m+1 whenever X'(and eventually X) can be partitioned according to the distinct 3-partition problem (Lemma 3). So we keep the burning number of the path P_I and the interval graph IG(X) same.

Additional vertices and edges are added to the sub paths T_j to form structures T_j^c (refer Figure 3) in such a way that to burn IG(X) optimally, one must have to burn each T_j^c only with one fire source (Lemma 2). Not only that, one must have to put that fire source on T_j^c in a particular round depending on the length of the subpath T_j (Lemma 5). With the help of these results and another couple of results, we finally show that, to burn this interval graph optimally in b(IG(X)) steps, one needs to solve the distinct 3-partition problem on the input X. This makes our problem an NP-Complete problem (Section 5.3 Theorem 2).

5.1 Interval graph construction

Let n be a natural number. Let $X = \{a_1, a_2, \cdots, a_{3n}\}$ be an input to a distinct 3-partition problem. So, $n = \frac{|X|}{3}$ and $B = \frac{s(X)}{n}$. Let $m = \max(X)$, and k = m - 3n. Let F_m be the set of first m natural numbers, $F_m = \{1, 2, 3, ..., m\}$. Also let F'_m be the set of first m odd numbers, $F'_m = \{2 \ f_i - 1 : f_i \in F_m\} = \{1, 3, 5, ..., 2m - 1\}$. Let $X' = \{2 \ a_i - 1 : a_i \in X\}$, $B' = \frac{s(X')}{n}$. Observe that $s(X') = \sum_{i=1}^{3n} 2 \ a_i - 1 = 2nB - 3n$, so B' = 2B - 3. It is easy to observe that any solution of X gives a solution of X' and vice versa. Let $Y = F'_m \setminus X'$.

Let there be n paths $Q_1, Q_2..., Q_n$, each of order B'. Consider k paths $Q'_1, Q'_2, ..., Q'_k$ such that each Q'_j ($\forall \ 1 \leq j \leq k$) is of order of j^{th} largest number in Y, where k = |Y|. Clearly the total number of vertices in $Q_1, Q_2..., Q_n, Q'_1, Q'_2, ..., Q'_k$ is m^2 , i.e., equals $s(F'_m)$. Consider another m+1 paths $T_1, T_2, ..., T_{m+1}$ such that each T_j ($\forall \ 1 \leq j \leq m+1$) is of order of 2(2m+1-j)+1. Total number of vertices in $T_1, T_2, ..., T_{m+1}$ is $\sum_{j=1}^{m+1} \left(2(2m+1-j)+1\right) = (3m^2+4m+1)$. We join these paths in the following order to form a larger path:

 $Q_1, T_1, Q_2, T_2, ..., Q_n, T_n, Q'_1, T_{n+1}, Q'_2, T_{n+2}, ..., Q'_k, T_{n+k}, T_{n+k+1}, ..., T_{m+1}.$ We denote this path as P_I . The total number of vertices in P_I is $m^2 + 3m^2 + 4m + 1 = (2m+1)^2$. Hence $b(P_I) = (2m+1)$.

Now we add few more vertices to P_I in such a way that it remains an interval graph and the optimal burning number of the graph remains same as $b(P_I)$. We add a distinct vertex connected to each vertex from 2nd to 2nd-last vertices of T_j , $\forall \ 1 \leq j \leq m+1$ (Figure 3 illustrates an example T_j along with the added vertices and edges (vertically upwards w.r.t. T_j). This forms a kind of comb structure; we call it T_j^c). Let this graph be called IG(X). Now we calculate total number of vertices in IG(X). Number of vertices added to each T_j is (2(2m+1-j)+1)-2=(4m+1-j). Hence total number of vertices added to P_I is $\sum_{j=1}^{m+1} (4m+1-j)=(3m^2+2m-1)$. So, total number of vertices in

IG(X) is $(2m+1)^2 + (3m^2 + 2m - 1) = (7m^2 + 6m)$. One such example of IG(X) is shown in Figure 4 corresponding to the numerical example given in Section 5.2.

Observe that P_I is a diameter of IG(X) and there is no cycle in IG(X). Also, all the vertices which are not in P_I are connected to some vertex of P_I by an edge. Hence IG(X) is a valid interval graph.

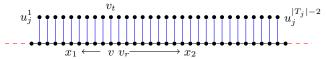


Fig. 3. Structure of a T_j with 33 vertices, along with the extra vertices connected to it. The dashed line represents the fact that other subpaths may be connected to a T_j on either or both ends.

Let u_j^1 be the vertex connected to the 2nd vertex of each T_j and $u_j^{|T_j|-2}$ be the vertex connected to its 2nd-last vertex of T_j , where $|T_j|$ stands for the number of vertices in the subpath T_j . Let $A^{T_j} = \{u_j^1, u_j^2, ..., u_j^{|T_j|-2}\}$ be the set of all $|T_j| - 2$ additional vertices corresponding to T_j . Now we mention an important observation regarding burning T_j^c .

5.2 Example construction

In this section, we show the construction of IG(X) from a particular input set X. Let $X=\{10,11,12,14,15,16\}$. Then $n=2,\ m=16,\ B=39,$ and k=10. Also $F_m=\{1,2,...,16\}$ and $F'_m=\{1,3,...,31\}$. Further, $X'=\{19,21,23,27,29,31\}$, B'=75=2B-3 and $Y=\{1,3,5,7,9,11,13,15,17,25\}$. Observe that Q_1 and Q_2 are paths of size 75, and each $Q'_1,Q'_2,...,Q'_k$ are paths of order of 25, 17, 15, 13, 11, 9, 7, 5, 3, 1 respectively. $T_1,T_2,T_3,...T_{m+1}$ are of order of 65, 63, 61..., 33 respectively. We add a vertex connected to each vertex from 2nd to 2nd-last vertices of $T_j(1 \le j \le m+1)$. Observe that this is a valid interval graph. The constructed example IG(X) is shown in Figure 4.

Observation 3 If T_j^c is burnt by putting $m \geq 2$ fire sources on T_j , then the burning clusters of at least two of these fire sources overlap (i.e., contain common vertices) of T_j .

Proof. Let that some T_j^c be completely burnt by two or more fire sources and yet there is no overlap between the burning clusters of any of those fire sources. Since all the fire sources are on T_j , which is a sub path of P_I , we say two fire sources on T_j are adjacent if there is a path in T_j between those two fire sources such that the path does not contain any other fire sources. For any two adjacent fire sources let us assume that there is no vertex which lies in the burning clusters of both the fire sources. Let v be a vertex on the path joining those two adjacent

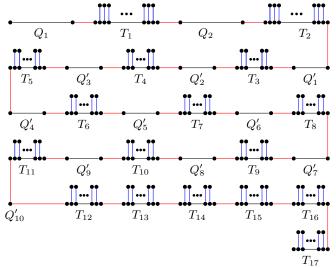


Fig. 4. Construction of an example IG(X).

fire sources x_1 and x_2 , such that the vertices in the left side of v including it (vertices towards x_1 as shown in Figure 3 using the left arrow) are burnt by x_1 and the vertices in the right of v (excluding v) are burnt by x_2 .

Let the vertex that is just right to v is v_r . By pigeonhole principle, we have that at least one of v or v_r having a neighbor v_t in T_j^c which is not in T_j . Without the loss of generality, let that v is having such a neighbor. Since the burning cluster of x_1 extends till v and not to its one hop neighbor v_r ($\in T_j$), so it does not burn the other one hop neighbor v_t ($\notin T_j$) too. It is easy to see that the second fire source can not burn v_t . This is contradiction to our assumption that T_j^c is burnt completely without overlapping clusters.

The following observation is immediate.

Observation 4 If a single fire source is able to burn T_j in t rounds, then T_j^c would also be burnt by it in the same number of rounds.

Lemma 2. If at least one T_j is burnt using more than one fire sources, then P_I can not be burnt optimally, i.e., in $b(P_I)(=2m+1)$ steps.

Proof. Since P_I is a simple path of length $(2m+1)^2$, according to Observation 1, each fire source in an optimal burning sequence must burn disjoint set of vertices of P_I . Let $x_1, x_2, ..., x_{2m+1}$ be an optimal burning sequence of P_I such that some T_j is burnt using more than one fire sources, then according to Observation 3, at least two fire sources burn at least one common vertex of P_I and hence $x_1, x_2, ..., x_{2m+1}$ can not be an optimal burning sequence for P_I .

5.3 NP-Completeness

Lemma 3. If X' has a solution for the distinct 3-partition problem, then burning number of IG(X) is 2m + 1.

Proof. If X' has a solution for the distinct 3-partition problem, there would be n sets of three numbers each, sum of which is B'. Recall that length of each Q_i is B'. Hence, $Q_1, ..., Q_n$ can be partitioned into further subpaths $Q_1'', ..., Q_{3n}''$. Let us call the partitions of Q_i as $Q_{3(i-1)+1}''$, $Q_{3(i-1)+2}''$, and Q_{3i}'' . Since X' is a set of odd numbers, length of each of these 3n subpaths are odd.

Let $P' = \{Q''_1, ..., Q''_{3n}, Q'_1, ..., Q'_k, T_1, ..., T_{m+1}\}$. Let r_i be the $((2m+1)-i+1)^{th} = (2m-i+2)^{th}$ vertex on the i^{th} largest subpath in P'. Then, the burning sequence $S' = (r_1, r_2, ..., r_{2m+1})$ can burn P_I and subsequently IG(X). This implies that $b(IG(X)) \leq 2m+1$. Since IG(X) has a subpath of length $(2m+1)^2$ in form of P_I , we have $b(IG(X)) \geq 2m+1$. Hence, b(IG(X)) = 2m+1.

Lemma 4. Each fire source y_i of any optimal burning sequence $(y_1, y_2, ..., y_{2m+1})$ of IG(X) must be on P_I .

Proof. We prove it by contradiction. If for any i, y_i is on P_I , then subgraph induced by $G.N_{2m+1-i}[y_i] \cap P_I$ has length at most 2(2m+1-i)+1. Let we put a fire source y_i on some vertex of A^{T_j} for some j, which is not on P_I , and still burn IG(X) in 2m+1 steps. Then subgraph induced by $G.N_{2m+1-i}[y_i] \cap P_I$ is a path of length less than 2(2m+1-i)+1. This along with Equation (1) implies that $|\bigcup_{i=1}^{2m+1} G.N_{2m+1-i}[y_i] \cap P_I| < (2m+1)^2$. So, even P_I is not burnt. This is a contradiction to our assumption that IG(X) is burnt in 2m+1 steps. Therefore each y_i must be a put on some vertex in P_I .

Let $S' = (y_1, y_2, ..., y_{2m+1})$ be any optimal burning sequence. Let r_i be the $(2m-i+2)^{th}$ vertex on the i^{th} largest sub path in P' as described in the proof of Lemma 3. Observe that T_j 's are the largest m+1 sub paths in P'.

Lemma 5. We must have $y_i = r_i, \forall 1 \le i \le m+1$.

Proof. We are going to prove this lemma using the strong induction hypothesis. We have that each $u_j^k \in A^{T_j}$ for some j must receive fire from some y_i in P_I , as all fire sources must be on P_I (Lemma 4). For i=1, the only vertex connected to both u_1^1 and $u_1^{|T_1|-2}$ and within a distance 2m+1-i=2m, is r_1 . Now we must have $y_1=r_1$, else, if we put y_1 somewhere else, then neither y_1 nor any other fire source can burn T_1^c alone. Also, we can not use multiple fire sources to burn T_1^c as an optimal burning of IG(X) does not allow that (Lemma 2). So, we must have that $y_1=r_1$. Now to establish strong induction, let that we need to have y_k on r_k for $1 \le k \le m$. Since r_k is already used to burn T_k^c , the only fire source that can burn T_{k+1}^c alone, is r_{k+1} . Recall that T_{k+1} has the largest length among the remaining subpaths after T_1, T_2, \cdots, T_k are burnt. And we can not use multiple fire sources to burn T_{k+1}^c (Lemma 2). Also the only vertex connected to both u_{k+1}^1 and $u_{k+1}^{|T_{k+1}|-2}$ within distance 2m+1-(k+1) is r_{k+1} . So, we must have that $y_{k+1}=r_{k+1}$. This completes the proof.

Let $P'' = IG(X) \setminus (T_1^c \cup T_2^c \cup ... \cup T_{m+1}^c)$. Now we present the following lemma on burning this remaining subgraph P''. That is P'' is a path forest consists of the subpaths $Q_1, Q_2, ..., Q_n, Q_1', Q_2', ..., Q_k'$. Now we present the following lemma on burning P''.

Lemma 6. There is a partition of P'', induced by the fire sources y_i $(m+1 \le i \le 2m+1)$ of the optimal burning sequence S', into paths of orders in F'_m .

Proof. From Lemma 5, we have that $\forall 1 \leq i \leq m+1$, all the vertices in T_i^c , would be burnt by y_i . Therefore, we have to burn the vertices in $Q_1, ..., Q_n, Q_1', ..., Q_k'$ by the fire sources $\{y_{m+2}, y_{m+3}, \cdots, y_{2m+1}\}$ (the remaining m sources of fire). Since P'' is a disjoint union of paths, so we have that $\forall i$ such that $m+2 \leq i \leq 2m+1$, the subgraph induced by the vertices in $G.N_{2m+1-i}[y_i]$ is a path of length at most 2(2m+1-i)+1. Moreover, we have that the path forest P'' is of order $\sum_{i=1}^{m}(2i-1)=m^2$. This implies that for each i with $m+2 \leq i \leq 2m+1$, the subgraph induced by the vertices in $G.N_{2m+1-i}[y_i]$ is a path of order equal to 2(2m+1-i)+1, otherwise we cannot burn all the vertices of P'' by these m fire sources which is a contradiction to the fact that S' is a optimal burning sequence. Therefore there must be a partition of P'', induced by the burning sequence $y_{m+2}, y_{m+3}, ..., y_{2m+1}$, into subpaths of length as per each element in $F'_m = \{1, 3, 5, ..., 2m-1\}$.

Theorem 2. Optimal burning of an interval graph is NP-Complete.

Proof. one part is already proved in Lemma 3. Here we show the other part. Let say we have a 2m+1 round optimal solution of the burning problem. Each T_j^c must get burned by exactly one fire source as per Lemma 5. From Lemma 6, we claim that the remaining path forest must be burned by the rest of available fire sources corresponding to the set F_m' .

Now, if each of the Q_i' is burned by a single fire source, then they must be burned by the fire sources corresponding to the integers belonging to set Y. Hence the remaining fire sources burning Q_i 's are burnt by fire sources corresponding to the integers belonging to the set X'. As the size of each Q_i is B', which is always odd (B' = 2B - 3), and also B' > 2m - 1 (from the definition of distinct 3-partition problem, m < B/2), so no Q_i can be burnt by a single fire source. Again it can not be burnt by two fire sources as sum of any two numbers in X' are even. Also no Q_i can be burnt by 4 or more fire sources as then by pigeon whole principle there would be at least one Q_i which needs to be burned by at most 2 fire sources, which is not possible. Hence each Q_i must be burnt by exactly three fire sources.

Else, if Q_i' 's are not burned by single fire sources, we apply the following process subject to each subpath Q_j' for $1 \leq j \leq k$. Let that some subpath Q_j' is burned using multiple fire sources. Since the given solution is optimal so these burning clusters are non overlapping. Not only that, the sum of the cluster sizes of these fire sources is exactly same as order of Q_j' . Now some fire source with cluster size equal to order of Q_j' must be present on some other subpath. We can interchange that fire source (whose cluster size is Q_j') by these fire sources

(which are presently burning Q'_j). This way we can make each subpath Q'_j to be burnt by a single fire source whose cluster size is equal to Q'_j . This takes O(m) time. After this we again arrived to the case discussed above and we can see that each Q_i are burnt by exactly three fire sources corresponding to the integers in X'. Therefore we have a solution of the distinct 3-partition problem whose input set is X'. This, in turn, gives us the solution of the distinct 3-partition problem on the input set X.

Therefore, we have reduced the burning problem of IG(X) from the distinct 3-partition problem in pseudo-polynomial time. Since, the distinct 3-partition problem is NP-Complete in the strong sense, burning IG(X) is also NP-Complete in the strong sense.

6 Burning permutation graphs

First we define permutation graphs. A permutation graph is constructed from an original sequence of objects O = (1, 2, 3, ..., k) which are numbers here and its permutation $P = (p_1, p_2, p_3, ..., p_k)$ such that there is an edge between two vertices corresponding to number i and j respectively, if i < j and j occurs before i in P.

We show in the following part of this section that burning permutation graphs is NP-Complete as well. The idea is similar to that of the interval graph. We start with an input X of a distinct 3-partition problem and reduce it to input X'of another distinct 3-partition problem. From that we construct a set of numbers O and finally a permutation P such that the permutation graph of O w.r.t. Pbecomes a path forest P(X). This $P = P_1 \cup_{s/} P_2 \cup_{s/} \cdots \cup_{s/} P_{n+k}$ is a sequential union of permutations P_i , such that the graph corresponds to O and each P_i forms a path. The path forest corresponding to P is exactly similar to P'' that we used in the Lemma 6. More specifically, order wise path corresponding to each P_j for all $1 \leq j \leq n$ is same as the respective subpath Q_i for all $1 \leq i \leq n$ and path corresponding to each P_j for all $n+1 \le j \le n+k$ is same as the respective subpath Q_i for all $1 \le i \le k$ in P''. Then similar argument works here, i.e., if X'has a solution for the distinct 3-partition problem (and so is X), then we need to find that to burn the path forest corresponding to P, i.e., the permutation graph P(X) optimally. So the main focus of this section is on the construction of such permutation graphs from given inputs of distinct 3-partition problem.

6.1 Generation of arbitrary path forest from a permutation of numbers

Let X be an arbitrary multiset of l = |X| positive integers. Let O be the original sequence of numbers 1 to s(X). Now, we are going to construct |X| permutations $P_1, P_2, ..., P_l$ in a specific manner such that these will produce path forest of l disjoint simple paths. The permutation P is simply the sequential union of the above permutations, i.e., $P = P_1 \cup_{s/P_2} \cup_{s/P_1} \cdots \cup_{s/P_l} P_l$.

Each $P_j = \{p_j^1, p_i^2, ..., p_i^{t_j}\}$ is a permutation of the consecutive numbers x_j to y_j belonging to O where $t_j = y_j - x_j + 1$. The sets of such consecutive numbers those corresponds to the l permutations, are pairwise disjoint. This makes the union of all paths corresponding to the permutations, a path forest.

We provide a formula to compute x_j , y_j as follows. Let $y_0 = 0$. $\forall 1 \leq j \leq l$, $x_j = y_{j-1} + 1$ and $y_j = y_{j-1} + X[j]$, where X[j] is the j^{th} element of X. This also implies that the total number of vertices in the permutation graph shall be $y_l = s(X)$.

We provide general formula to find p_i^h for each j and for all $h \in (1, t_j)$ such that P_j corresponds to a path.

- For all $1 \le j \le l$ s.t. t_j is even with $|t_j| \ge 5$:
 For all odd h, $1 \le h \le (t_j 3), p_j^h = 2 + (x_j + h 1)$. For the remaining odd value of $h = t_j - 1, p_j^{t_j - 1} = y_j$.
 - For all even $h, 4 \le i \le t_j, p_i^h = h 2$ and for the remaining even value of $h = 2, p_i^2 = x_j$
- For all $1 \le j \le l$ s.t. t_j is odd with $|t_j| \ge 5$:
 For all odd h, $1 \le h \le t_j 2$, $p_j^h = 2 + (x_j + h 1)$. For the remaining odd value of $h = t_j$, $p_i^{t_j} = y_j - 1$.
 - For all even $h, 4 \leq h \leq t_j 1, p_j^h = (x_j + h 1) 2$ and for the remaining even value of h = 2, $p_i^2 = x_i$
- $-|t_j| \le 4$: If $t_j = 1$, then $P_j = (x_j)$.
 - If $t_j = 2$, then $P_j = (y_j, x_j)$.

 - If $t_j = 3$, then $P_j = (y_j, x_j, x_j + 1)$. If $t_j = 4$, then $P_j = (x_j + 1, y_j, x_j, x_j + 2)$

We also provide an example of such a construction that returns a path. Let O = (1, 2, 3, 4, 5, 6, 7, 8). Then consider P = (3, 1, 5, 2, 7, 4, 8, 6) be the subject permutation of O. The permutation graph G formed from this pair (O, P) is shown in Figure 5. In the figure, v_i are the vertices corresponding to the object $i \in O$.



Fig. 5. Representation of permutation graph corresponding to (O, P), where O =(1, 2, 3, 4, 5, 6, 7, 8) and P = (3, 1, 5, 2, 7, 4, 8, 6).

Now, $P = P_1 \cup_{s/} ... \cup_{s/} P_l = (p_1^1, ..., p_1^{t_1}, p_2^1, ..., p_2^{t_2}, ..., p_1^{t_1}, ..., p_l^{t_l})$ is the subject permutation of O.

We call P(X) to be the permutation graph corresponding to the original sequence O, and its subject permutation P. $\forall 1 \leq j \leq l$ let Q_j be the subgraph in P(X) induced by the permutation $P_j = (p_j^1, p_j^2, ..., p_j^{t_j})$ of the original sequence $(x_j, ..., y_j)$. Observe that $P(X) = Q_1 \cup Q_2 \cup ... \cup Q_{n+k}$ is a path forest where the paths $Q_1, Q_2, ..., Q_{n+k}$ are disjoint from each other.

Although it immediately follows from [5] that burning permutation graphs is NP-Complete since we can construct any path forest from permutation graphs, we are giving a reduction of the distinct 3-partition problem in the following part of this section to show NP-Completeness.

6.2 Permutation graph general construction for NP-Completeness

Let X be an input set to a distinct 3-partition problem; let $n = \frac{|X|}{3}$, $m = \max(X)$, $B = \frac{s(X)}{n}$, and k = m - 3n. Let F_m be the set of first m numbers, $F_m = \{1, 2, 3, ..., m\}$, and F'_m be the set of first m odd numbers, $F'_m = \{2 \ f_i - 1 : f_i \in F_m\} = \{1, 3, 5, ..., 2m - 1\}$. Let $X' = \{2 \ a_i - 1 : a_i \in X\}$, $B' = \frac{s(X')}{n}$. Observe that $s(X') = \sum_{i=1}^{3n} 2 \ a_i - 1 = 2nB - 3n$, so B' = 2B - 3. Let $Y = F'_m \setminus X'$. Let O be the original sequence of numbers 1 to $s(F'_m)$, $O = (1, 2, 3, ..., m^2)$. Now, we are going to construct n + k permutations $P_1, P_2, ..., P_{n+k}$ in a specific manner such that these will produce path forest of (n+k) disjoint simple paths, where k = |Y|. The permutation P is simply the sequential union of the above permutations, i.e., $P = P_1 \cup_{s/} P_2 \cup_{s/} \cdots \cup_{s/} P_{n+k}$.

Each $P_j = \{p_j^1, p_j^2, ..., p_j^{t_j}\}$ is a permutation of the consecutive numbers x_j to y_j belonging to O where $t_j = y_j - x_j + 1$. The sets of such consecutive numbers those corresponds to the (n+k) permutations, are pairwise disjoint. This makes the union of all paths corresponding to the permutations, a path forest. Below we first provide a formula to calculate x_j , y_j . Since each of P_1, P_2, \cdots, P_n is of order B' (i.e., $t_j = B' \ \forall 1 \leq j \leq n$), hence the P_j is a permutation of the numbers $(j-1) \times B' + 1$ to $j \times B'$. We define this formally in the following way. Let $y_0 = 0$. Then $\forall 1 \leq j \leq n, x_j = y_{j-1} + 1$, and $y_j = j \times B'$. For the remaining permutations $P_{n+1}, P_{n+2}, \cdots, P_{n+k}, \ \forall n+1 \leq j \leq n+k, \ x_j = y_{j-1} + 1$ and $y_j = y_{j-1} + L_i^Y$, where i = j - n and L_i^Y is the ith largest element of Y. Note that, i varies from 1 to k. Also observe the following,

$$y_{n+k} = y_n + s(F'_m \setminus X') = nB' + s(F'_m \setminus X')$$

= $s(X') + s(F'_m \setminus X') = s(F'_m) = m^2$

Hence, total number of elements in $\bigcup_{j=1}^{n+k} P_j$ is m^2 .

Now, $P = P_1 \cup_{s/} ... \cup_{s/} P_{n+k} = (p_1^1, ..., p_1^{t_1}, p_2^1, ..., p_2^{t_2}, ..., p_{n+k}^1, ..., p_{n+k}^{t_{n+k}}) = (p_1, p_2, ..., p_{m^2})$ is the subject permutation of O.

We call P(X) to be the permutation graph corresponding to the original sequence O, and its subject permutation P. $\forall \ 1 \le j \le n+k$ let Q_j be the subgraph in P(X) induced by the permutation $P_j = (p_j^1, p_j^2, ..., p_j^{t_j})$ of the original sequence $(x_j, ..., y_j)$. Observe that $P(X) = Q_1 \cup Q_2 \cup ... \cup Q_{n+k}$ is a path forest where the paths $Q_1, Q_2, ..., Q_{n+k}$ are disjoint from each other. Now we prove the NP-Completeness result which mostly follows from the earlier proof on the interval graphs.

6.3 Example Construction

Let $X = \{10, 11, 12, 14, 15, 16\} \implies n = 2, m = 16, B = 39, \text{ and } k = 10.$ $F_m = \{1, 2, ..., 16\}, \text{ and } F'_m = \{1, 3, ..., 31\}.$ $X' = \{19, 21, 23, 27, 29, 31\}, B' = 75 = 2B - 3.$ $Y = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 25\}.$

We finally form paths Q_1 and Q_2 each of order of 75. Also, we form paths $Q_3, Q_4, ..., Q_{12}$ of order of 25, 17, 15, 13, 11, 9, 7, 5, 3, 1 respectively. P(X) is a path forest of the paths $Q_1, ..., Q_{12}$, which are disjoint from each other. Burning number of P(X) in this case is m = 16.

6.4 NP-Completeness

The logic is similar to that of the interval graph. The burning number of P(X) is m if X' has a solution of the distinct 3-partition problem. As then length B' of each of $Q_1, Q_2, ..., Q_n$ can be written as a sum of three odd numbers from X'. And length of each of the remaining $Q_{n+1}, Q_{n+2}, ..., Q_{n+k}$ are also odds as each of the lengths corresponds to an element in Y. So, to burn P(X) in m steps, one needs to solve the distinct 3-partition problem on X' (equivalently on X).

Similarly if we have an optimal burning sequence of P(X), we get a solution of the distinct 3-partition problem for X as discussed the last paragraph of the proof of Theorem 2. This leads us to the following theorem.

Theorem 3. Burning of general permutation graphs is NP-Complete.

7 Discussion

7.1 Some more Hardness Results

In this section we report hardness results on few more graph classes that mostly follow from our result on the interval graph. A disk graph is formed from an arrangement of disks on a Euclidean plane such that there is a vertex in the disk graph corresponding to each disk, and if there is an overlap between a pair of disks, then there shall be an edge between their corresponding vertices in the disk graph. Since any interval graph is valid to be a disc graph, we have the following.

Corollary 2. Optimal burning of disc graphs is NP-Complete even if the underlying disc representation is given.

In a *unit distance graph*, the edges can be drawn in a euclidean plane such that each edge is of unit length. In *matchstick graph*, the edges can be drawn in a euclidean plane such that each edge is of unit length and they do not intersect each other. The graph class that we have constructed is valid to be a unit distance graph and a matchstick graph. So we have Corollary 3 as follows.

Corollary 3. Optimal burning of unit distance graphs and matchetick graphs is NP-Complete.

7.2 Conclusion and future work

In this article we show a lower bound for the burning number of grids of arbitrary size and give a two approximation algorithm for burning square grids. We also show that the graph burning problem is NP-Complete on interval graphs and permutation graphs along with several corollaries that show NP-Completeness of burning other geometric graphs as well.

It remains an open question whether burning grids is an NP-Complete problem. Another related direction is to try and improve the 3-approximation algorithm provided in [5] for burning general graphs.

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