# Lazy Queue Layouts of Posets 

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#### Abstract

We investigate the queue number of posets in terms of their width, that is, the maximum number of pairwise incomparable elements. A long-standing conjecture of Heath and Pemmaraju asserts that every poset of width $w$ has queue number at most $w$. The conjecture has been confirmed for posets of width $w=2$ via so-called lazy linear extension. We extend and thoroughly analyze lazy linear extensions for posets of width $w>2$. Our analysis implies an upper bound of $(w-1)^{2}+1$ on the queue number of width- $w$ posets, which is tight for the strategy and yields an improvement over the previously best-known bound. Further, we provide an example of a poset that requires at least $w+1$ queues in every linear extension, thereby disproving the conjecture for posets of width $w>2$.


Keywords: Queue layouts, Posets, Linear Extensions

## 1 Introduction

A queue layout of a graph consists of a total order $\prec$ of its vertices and a partition of its edges into queues such that no two edges in a single queue nest, that is, there are no edges $(u, v)$ and $(x, y)$ in a queue with $u \prec x \prec y \prec v$. If the input graph is directed, then the total order has to be compatible with its edge directions, i.e., it has to be a topological ordering of it $[12,13]$. The minimum number of queues needed in a queue layout of a graph is commonly referred to as its queue number.

There is a rich literature exploring bounds on the queue number of different classes of graphs $[1,10,14,16,17,18]$. A remarkable work by Dujmović et al. [7] proves that the queue number of (undirected) planar graphs is constant, thus improving upon previous (poly-)logarithmic bounds $[3,5,6]$ and resolving an old conjecture by Heath, Leighton and Rosenberg [10]. For a survey, we refer to [8].

In this paper, we investigate bounds on the queue number of posets. Recall that a poset $\langle P,<\rangle$ is a finite set of elements $P$ equipped with a partial order $<$; refer to Section 2 for formal definitions. The queue number of $\langle P,<\rangle$ is the queue number of the acyclic digraph $G(P,<)$ associated with the poset that contains
all non-transitive relations among the elements of $P$. This digraph is known as the cover graph and can be visualized using a Hasse diagram; see Fig. 1.

The study of the queue number of posets was initiated in 1997 by Heath and Pemmaraju [11], who provided bounds on the queue number of a poset expressed in terms of its width, that is, the maximum number of pairwise incomparable elements with respect to $<$. In particular, they observed that the queue number of a poset of width $w$ cannot exceed $w^{2}$ and posed the following conjecture.

Conjecture 1 (Heath and Pemmaraju [11]) Every poset of width w has queue number at most $w$.

Heath and Pemmaraju [11] made a step towards settling the conjecture by providing a linear upper bound of $4 w-1$ on the queue number of planar posets of width $w$. This bound was recently improved to $3 w-2$ by Knauer, Micek, and Ueckerdt [15], who also gave a planar poset whose queue number is exactly $w$, thus establishing a lower bound. Furthermore, they investigated (non-planar) posets of width 2, and proved that their queue number is at most 2. Therefore, Conjecture 1 holds when $w=2$.*
Our Contribution. We present improvements upon the aforementioned results, thus continuing the study of the queue number of posets expressed in terms of their width, which is one of the open problems by Dujmović et al. [7].
(i) For a fixed total order of a graph, the queue number is the size of a maximum rainbow, that is, a set of pairwise nested edges [10]. Thus to determine the queue number of a poset $\langle P,<\rangle$ one has to compute a linear extension (that is, a total order complying with $<$ ), which minimizes the size of a maximum rainbow. In Theorem 5 in Appendix B.1, we present a poset and a linear extension of it which yields a rainbow of size $w^{2}$. Knauer et al. [15] studied a special type of linear extensions, called lazy, for posets of width-2 to show that their queue number is at most 2. Thus, it is tempting to generalize and analyze lazy linear extensions for posets of width $w>2$. We provide such an analysis and show that the maximum size of a rainbow in a lazy linear extension of a width- $w$ poset is at most $w^{2}-w$ (Theorem 1 in Section 3). Furthermore, we show that the bound is worst-case optimal for lazy linear extensions (Theorem 6 in Appendix B.2).
(ii) The above bound already provides an improvement over the existing upper bound on the queue number of posets. However, a carefully chosen lazy linear extension, which we call most recently used (MRU), further improves the bound to $(w-1)^{2}+1$ (Theorem 2 in Section 4). Therefore, the queue number of a width- $w$ poset is at most $(w-1)^{2}+1$. Again we show this bound to be worst-case optimal for MRU extensions (Theorem 7 in Appendix B.3).
(iii) We demonstrate a non-planar poset of width 3 whose queue number is 4 (Theorem 3). We generalize this example to posets of width $w>3$ (Theorem 4), thus disproving Conjecture 1. These two proofs are mostly deferred to Appendix D.

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Fig. 1: (a) The Hasse diagram of a width-4 poset; gray elements are pairwise incomparable; the chains of a certain decomposition are shown by vertical lines. (b) A 2-queue layout with a 2-rainbow formed by edges $\left(v_{2}, v_{5}\right)$ and $\left(v_{6}, v_{8}\right)$.

## 2 Preliminaries

A partial order over a finite set of elements $P$ is a binary relation $<$ that is irreflexive and transitive. A set $P$ together with a partial order, $<$, is a partially ordered set (or simply a poset) and is denoted by $\langle P,<\rangle$. Two elements $x$ and $y$ with $x<y$ or $y<x$ are called comparable; otherwise $x$ and $y$ are incomparable. A subset of pairwise comparable (incomparable) elements of a poset is called a chain (antichain, respectively). The width of a poset is defined as the cardinality of a largest antichain. For two elements $x$ and $y$ of $P$ with $x<y$, we say that $x$ is covered by $y$ if there is no element $z \in P$ such that $x<z<y$. A poset $\langle P,<\rangle$ is naturally associated with an acyclic digraph $G(P,<)$, called the cover graph, whose vertex-set $V$ consists of the elements of $P$, and there exists an edge from $u$ to $v$ if $u$ is covered by $v$; see Fig. 1a. By definition, $G(P,<)$ has no transitive edges.

A linear extension $L$ of a poset $\langle P,<\rangle$ is a total order of $P$, which complies with $<$, that is, for every two elements $x$ and $y$ in $P$ with $x<y, x$ precedes $y$ in $L$. Given a linear extension $L$ of a poset, we write $x \prec y$ to denote that $x$ precedes $y$ in $L$; if in addition $x$ and $y$ may coincide, we write $x \preceq y$. We use $\left[x_{1}, x_{2}, \ldots, x_{k}\right]$ to denote $x_{i} \prec x_{i+1}$ for all $1 \leq i<k$; such a subsequence of $L$ is also called a pattern. Let $F=\left\{\left(x_{i}, y_{i}\right) ; i=1,2, \ldots, k\right\}$ be a set of $k \geq 2$ independent (that is, having no common endpoints) edges of $G(P,<)$. It follows that $x_{i} \prec y_{i}$ for all $1 \leq i \leq k$. If $\left[x_{1}, \ldots, x_{k}, y_{k}, \ldots, y_{1}\right]$ holds in $L$, then the edges of $F$ form a $k$-rainbow (see Fig. 1b). Edge $\left(x_{i}, y_{i}\right)$ nests edge $\left(x_{j}, y_{j}\right)$, if $1 \leq i<j \leq k$.

A queue layout of an acyclic digraph $G$ consists of a total order of its vertices that is compatible with the edge directions of $G$ and of a partition of its edges into queues, such that no two edges in a queue are nested. The queue number of $G$ is the minimum number of queues required by its queue layouts. The queue number of a poset $\langle P,<\rangle$ is the queue number of its cover graph $G(P,<)$. Equivalently, the queue number of $\langle P,<\rangle$ is at most $k$ if and only if it admits a linear extension $L$ such that no $(k+1)$-rainbow is formed by some of the edges of $G(P,<)$ [14]. If certain edges form a rainbow in $L$, we say that $L$ contains the rainbow.

The elements of a poset $\langle P,<\rangle$ of width $w$ can be partitioned into $w$ chains [4]. Note that such a partition is not necessarily unique. In the following, we fix this partition, and treat it as a function $\mathcal{C}: P \rightarrow\{1, \ldots, w\}$ such that if $\mathcal{C}(u)=\mathcal{C}(v)$
and $u \neq v$, then either $u<v$ or $v<u$. We use $\mathcal{R}, \mathcal{B}$, and $\mathcal{G}$ to denote specific chains from a chain decomposition. A set of edges of the cover graph $G(P,<)$ of the poset that form a rainbow in a linear extension is called an incoming $\mathcal{R}$-rainbow $T_{\mathcal{R}}$ of size $s$ if it consists of $s$ edges $\left(u_{1}, r_{1}\right), \ldots,\left(u_{s}, r_{s}\right)$ such that $r_{i} \in \mathcal{R}$ for all $1 \leq i \leq s$ and $\mathcal{C}\left(u_{i}\right) \neq \mathcal{C}\left(u_{j}\right)$ for all $1 \leq i, j \leq s$ with $i \neq j$. If $s=w, T_{\mathcal{R}}$ is called complete and is denoted by $T_{\mathcal{R}}^{*}$. An edge $e$ of $T_{\mathcal{R}}$ with both endpoints in $\mathcal{R}$ is called an $\mathcal{R}$-self edge. For example, $T_{\mathcal{R}}^{*} \backslash\{e\}$ is an incoming $\mathcal{R}$-rainbow of size $w-1$ without the $\mathcal{R}$-self edge $e$. Similar notation is used for chains $\mathcal{B}$ and $\mathcal{G}$.

## 3 Lazy Linear Extensions

First let us recall two properties of linear extensions, whose proofs immediately follow from the fact that a cover graph of a poset contains no transitive edges.

Proposition 1 A linear extension of a poset $\langle P,<\rangle$ does not contain pattern $\left[r_{1} \ldots r_{2} \ldots r_{3}\right]$, where $\mathcal{C}\left(r_{1}\right)=\mathcal{C}\left(r_{2}\right)=\mathcal{C}\left(r_{3}\right)$ and $\left(r_{1}, r_{3}\right)$ is an edge of $G(P,<)$.

Proposition 2 A linear extension of a poset $\langle P,<\rangle$ does not contain pattern $\left[r_{1} \ldots r_{2} \ldots b_{2} \ldots b_{1}\right]$, where $\mathcal{C}\left(r_{1}\right)=\mathcal{C}\left(r_{2}\right), \mathcal{C}\left(b_{1}\right)=\mathcal{C}\left(b_{2}\right)$, and $\left(r_{1}, b_{1}\right)$ and $\left(r_{2}, b_{2}\right)$ are edges of $G(P,<)$.

Proposition 2 implies that for any linear extension of a poset, the maximum size of a rainbow is at most $w^{2}[11]$. Theorem 5 in Appendix B. 1 shows that for every $w \geq 2$, there exists a width- $w$ poset and a linear extension of it containing a $w^{2}$-rainbow. Hence, a linear extension has be to chosen carefully, if one seeks for a bound on the queue number of posets that is strictly less than $w^{2}$.

In this section, we present and analyze such an extension, which we call lazy. Assume that a poset is given with a decomposition into $w$ chains. Intuitively, a lazy linear extension is constructed incrementally starting from a minimal element of the poset. In every iteration, the next element is chosen from the same chain, if possible. Formally, for $i=1, \ldots, n-1$, assume that we have computed a lazy linear extension $L$ for $i$ vertices of $G(P,<)$ and let $v_{i}$ be last vertex in $L$ (if any). To determine the next vertex $v_{i+1}$ of $L$, we compute the following set consisting of all source-vertices of the subgraph of $G(P,<)$ induced by $V \backslash L$ :

$$
\begin{equation*}
S=\{v \in V \backslash L: \nexists(u, v) \in E \text { with } u \in V \backslash L\} \tag{1}
\end{equation*}
$$

If there is a vertex $u$ in $S$ with $\mathcal{C}(u)=\mathcal{C}\left(v_{i}\right)$, we set $v_{i+1}=u$; otherwise $v_{i+1}$ is freely chosen from $S$; see Algorithm 1 in Appendix A. For the example of Fig. 1a, observe that $v_{1} \prec v_{4} \prec v_{2} \prec v_{3} \prec v_{6} \prec v_{7} \prec v_{5} \prec v_{8}$ is a lazy linear extension.

Lemma 1. If a lazy linear extension $L$ of poset $\langle P,<\rangle$ contains the pattern $\left[r_{1} \ldots b \ldots r_{2}\right]$, where $\mathcal{C}\left(r_{1}\right)=\mathcal{C}\left(r_{2}\right) \neq \mathcal{C}(b)$, then there exists some $x \in P$ with $\mathcal{C}(x) \neq \mathcal{C}\left(r_{1}\right)$ between $r_{1}$ and $r_{2}$ in $L$, such that $x<r_{2}$.

Proof. Since the pattern is $\left[r_{1} \ldots b \ldots r_{2}\right], G(P,<)$ contains an edge from a vertex $x$ with $\mathcal{C}(x) \neq \mathcal{C}\left(r_{1}\right)$ to a vertex $y \in \mathcal{C}\left(r_{1}\right)$ that is between $r_{1}$ and $r_{2}$ in $L$ (notice that $x$ may or may not coincide with $b$ ). Since the edge belongs to $G(P,<)$, it follows that $x<y \leq r_{2}$.

Lemma 2. A lazy linear extension of poset $\langle P,<\rangle$ does not contain pattern

where $\left(u_{1}, r_{1}\right), \ldots,\left(u_{w-1}, r_{w-1}\right)$ form an incoming $\mathcal{C}(r)$-rainbow of size $w-1$, such that $\mathcal{C}(r) \neq \mathcal{C}\left(u_{i}\right)$ for all $1 \leq i \leq w-1$ and $\mathcal{C}(r) \neq \mathcal{C}(b)$.

Proof. Assume to the contrary that there is a lazy linear extension $L$ containing the pattern. Since $\left[r \ldots b \ldots r_{w-1}\right]$ holds in $L$, by Lemma 1 , there is $x$ with $\mathcal{C}(x) \neq$ $\mathcal{C}\left(r_{w-1}\right)$ between $r$ and $r_{w-1}$ in $L$ such that $x<r_{w-1}$. Since $\mathcal{C}(x) \neq \mathcal{C}\left(r_{w-1}\right)$, there is $1 \leq j \leq w-1$ such that $\mathcal{C}(x)=\mathcal{C}\left(u_{j}\right)$, which implies $u_{j}<x$. Thus:


Since $u_{j}<x<r_{w-1} \leq r_{j}$, there is a path from $u_{j}$ to $r_{j}$ in $G(P,<)$. Thus, edge ( $u_{j}, r_{j}$ ) is transitive; a contradiction.

Theorem 1. The maximum size of a rainbow formed by the edges of $G(P,<)$ in a lazy linear extension of a poset $\langle P,<\rangle$ of width $w$ is at most $w^{2}-w$.

Proof. Assume to the contrary that there is a lazy linear extension $L$ that contains a $\left(w^{2}-w+1\right)$-rainbow $T$. By Proposition 2 and the pigeonhole principle, $T$ contains at least one complete incoming rainbow of size $w$; denote it by $T_{\mathcal{R}}^{*}$ and the corresponding chain by $\mathcal{R}$. By Proposition 1 , the $\mathcal{R}$-self edge of $T_{\mathcal{R}}^{*}$ is innermost in $T_{\mathcal{R}}^{*}$. Thus, if $\left(u_{1}, r_{1}\right), \ldots,\left(u_{w}, r_{w}\right)$ are the edges of $T_{\mathcal{R}}^{*}$ and $u_{w} \in \mathcal{R}$, then without loss of generality, we may assume that the following holds in $L$.


We next show that $\left(u_{w}, r_{w}\right)$ is the innermost and $\left(u_{w-1}, r_{w-1}\right)$ is the second innermost edge in $T$. Assume to the contrary that there exists an edge $(x, y)$ in $T$ that does not belong to $T_{\mathcal{R}}^{*}$ (that is, $\left.\mathcal{C}(y) \neq \mathcal{R}\right)$ and which is nested by $\left(u_{w-1}, r_{w-1}\right)$. Regardless of whether $(x, y)$ nests $\left(u_{w}, r_{w}\right)$ or not, we deduce the following.


Together with $u_{w} \in \mathcal{R}$ and $y \notin \mathcal{R}$, we apply Lemma 2 , which yields a contradiction. Since $\left(u_{w}, r_{w}\right)$ and $\left(u_{w-1}, r_{w-1}\right)$ are the two innermost edges of $T$, it follows that $T$ does not contain another complete incoming rainbow of size $w$.

Hence, each of the remaining $w-1$ incoming rainbows has size exactly $w-1$. Consider vertex $u_{w-1}$ and let without loss of generality $\mathcal{C}\left(u_{w-1}\right)=\mathcal{B}$. By Proposition $1, \mathcal{B} \neq \mathcal{R}$. We claim that the incoming $\mathcal{B}$-rainbow $T_{\mathcal{B}}$ does not contain the $\mathcal{B}$-self edge. Assuming the contrary, this $\mathcal{B}$-self edge nests $\left(u_{w-1}, r_{w-1}\right)$ because $\left(u_{w}, r_{w}\right)$ and $\left(u_{w-1}, r_{w-1}\right)$ are the two innermost edges of $T$. Since $\mathcal{C}\left(u_{w-1}\right)=\mathcal{B}$, we obtain a contradiction by Proposition 1 . Thus, $T_{\mathcal{B}}$ is a $\mathcal{B}$-rainbow of size $w-1$ containing no $\mathcal{B}$-self edge. All edges of $T_{\mathcal{B}}$ nest $\left(u_{w-1}, r_{w-1}\right)$, which yields the forbidden pattern of Lemma 2 formed by vertices of $T_{\mathcal{B}}, u_{w-1} \in \mathcal{B}$, and $r_{w-1} \in \mathcal{R}$; a contradiction.

Theorem 6 in Appendix B. 2 shows that our analysis is tight, i.e., there are posets of width $w$ and corresponding lazy linear extensions containing $\left(w^{2}-w\right)$-rainbows.

## 4 MRU Extensions

We now define a special type of lazy linear extensions for a width- $w$ poset $\langle P,<\rangle$, which we call most recently used, or simply $M R U$. For $i=1, \ldots, n-1$, assume that we have computed a linear extension $L$ for $i$ vertices of $G(P,<)$, which are denoted by $v_{1}, \ldots, v_{i}$. To determine the next vertex of $L$, we compute set $S$ of Eq. (1). Among all vertices in $S$, we select one from the most recently used chain (if any). Formally, we select a vertex $u \in S$ such that $\mathcal{C}(u)=\mathcal{C}\left(v_{j}\right)$ for the largest $1 \leq j \leq i$. If such vertex does not exist, we choose $v_{i+1}$ arbitrarily from $S$; see Algorithm 2 in Appendix A. For the example of Fig. 1a, observe that $v_{1} \prec v_{4} \prec v_{2} \prec v_{3} \prec v_{6} \prec v_{5} \prec v_{7} \prec v_{8}$ is an MRU extension.

For a linear extension $L$ of poset $\langle P,<\rangle$, and two elements $x$ and $y$ in $P$, let $\mathcal{C}[x, y]$ be the subset of chains whose elements appear between $x$ and $y$ (inclusively) in $L$, that is, $\mathcal{C}[x, y]=\{\mathcal{C}(z): x \preceq z \preceq y\}$.

Lemma 3. Let $L$ be an MRU extension of a width-w poset $\langle P,<\rangle$ containing pattern $\left[r_{1} \ldots r_{2} \ldots b\right]$, such that $\mathcal{C}\left(r_{1}\right)=\mathcal{C}\left(r_{2}\right) \neq \mathcal{C}(b)$ and there is no element in $L$ between $r_{1}$ and $r_{2}$ from chain $\mathcal{C}\left(r_{1}\right)$. If $\mathcal{C}\left[r_{1}, r_{2}\right]=\mathcal{C}\left[r_{1}, b\right]$, then $r_{2}<b$.

Proof. Assume to the contrary that there is some $b$ for which $r_{2}<b$ does not hold. Without loss of generality, let $b$ be the first (after $r_{2}$ ) of those elements in $L$. Since $\mathcal{C}\left[r_{1}, r_{2}\right]=\mathcal{C}\left[r_{1}, b\right]$, there are elements between $r_{1}$ and $r_{2}$ in $L$ from chain $\mathcal{C}(b)$. Let $b_{1}$ be the last such element in $L$. Hence, $r_{1} \prec b_{1} \prec r_{2} \prec b$. Consider the incremental construction of $L$. Since there is no element between $r_{1}$ and $r_{2}$ in $L$ from chain $\mathcal{C}\left(r_{1}\right)$, the chain of $b$ was "more recent" than the one of $r_{2}$, when $r_{2}$ was chosen as the next element. Thus, there is an edge $(x, b)$ in $G(P,<)$ with $r_{2} \prec x$ in $L$. Since $b$ is the first element that is not comparable to $r_{2}$, then $r_{2}<x$ holds. Hence, $r_{2}<b$; a contradiction to our assumption that $r_{2}<b$ does not hold.

Corollary 1. Let $L$ be an MRU extension of a width-w poset $\langle P,<\rangle$ containing pattern $\left[r_{1} \ldots r_{2}\right]$, such that $\mathcal{C}\left(r_{1}\right)=\mathcal{C}\left(r_{2}\right)$ and there is no element in $L$ between $r_{1}$ and $r_{2}$ from chain $\mathcal{C}\left(r_{1}\right)$. If $\left|\mathcal{C}\left[r_{1}, r_{2}\right]\right|=w$, then $r_{2}$ is comparable to all subsequent elements in $L$.

Next we describe a forbidden pattern which is central in our proofs.
Lemma 4. An MRU extension $L$ of a width-w poset $\langle P,<\rangle$ does not contain the following pattern, even if $u_{k}=b_{1}$


- $\mathcal{C}\left(u_{i}\right) \neq \mathcal{C}\left(u_{j}\right)$ for $1 \leq i, j \leq w$ with $i \neq j$,
- $\left(u_{1}, r_{1}\right), \ldots,\left(u_{k}, r_{k}\right)$ form an incoming $\mathcal{R}$-rainbow of size $k$ for some $1 \leq k \leq w$,
- between $b_{1}$ and $b_{2}$ in $L$, there is an element from $\mathcal{R}$ but no elements from $\mathcal{B}=\mathcal{C}\left(b_{1}\right)=\mathcal{C}\left(b_{2}\right)$.

Proof. Since there are no elements between $b_{1}$ and $b_{2}$ in $L$ from $\mathcal{B}$ and since $\mathcal{C}\left(u_{i}\right) \neq \mathcal{C}\left(u_{j}\right)$ for $1 \leq i, j \leq w$ with $i \neq j$, one of $u_{1}, \ldots, u_{k}$ belongs to $\mathcal{B}$. Let $u_{i}$ be this element with $1 \leq i \leq k$, that is, $\mathcal{C}\left(u_{i}\right)=\mathcal{B}$. Since $\left(u_{1}, r_{1}\right), \ldots,\left(u_{k}, r_{k}\right)$ form an incoming $\mathcal{R}$-rainbow, $\left(u_{i}, r_{i}\right)$ is an edge of $G(P,<)$. Notice that $\left[u_{i} \ldots b_{1} \ldots b_{2} \ldots r_{i}\right]$ holds in $L$ and that $u_{i}=b_{1}$ may hold if $i=k$.

Our proof is by induction on $|\mathcal{C}|-\left|\mathcal{C}\left[b_{1}, b_{2}\right]\right|$, which ranges between 0 and $w-2$. In the base case $|\mathcal{C}|-\left|\mathcal{C}\left[b_{1}, b_{2}\right]\right|=0$, that is, $\left|\mathcal{C}\left[b_{1}, b_{2}\right]\right|=w$. By Corollary $1, b_{2}$ is comparable to all subsequent elements in $L$. In particular, $b_{2}<r_{i}$, which implies that $\left(u_{i}, r_{i}\right)$ is transitive in $G(P,<)$, since $u_{i} \leq b_{1}<b_{2}<r_{i}$; a contradiction.

Assume $|\mathcal{C}|-\left|\mathcal{C}\left[b_{1}, b_{2}\right]\right|>0$. Let $r_{0}$ be the first vertex from $\mathcal{R}$ after $b_{2}$ in $L$, that is, $r_{0} \preceq r_{k}$. If there are no elements between $b_{2}$ and $r_{0}$ from $\mathcal{C} \backslash \mathcal{C}\left[b_{1}, b_{2}\right]$ (that is, $\mathcal{C}\left[b_{1}, b_{2}\right]=\mathcal{C}\left[b_{2}, r_{0}\right]$ ), then by Lemma 3 it follows that $b_{2}<r_{0}$, which implies $u_{i} \leq b_{1}<b_{2}<r_{0} \leq r_{i}$. Thus, edge $\left(u_{i}, r_{i}\right)$ is transitive in $G(P,<)$; a contradiction. Therefore, we may assume that there are elements between $b_{2}$ and $r_{0}$ in $L$ from $\mathcal{C} \backslash \mathcal{C}\left[b_{1}, b_{2}\right]$. Let $g_{1}$ be the first such element; denote $\mathcal{C}\left(g_{1}\right)=\mathcal{G}$. Since between $b_{1}$ and $b_{2}$ in $L$ there is an element from $\mathcal{R}$ (that is, $\mathcal{R} \in \mathcal{C}\left[b_{1}, b_{2}\right]$ ), $\mathcal{G} \neq \mathcal{R}$ holds. Similarly, $\mathcal{G} \neq \mathcal{B}$. Let $\left(u_{\ell}, r_{\ell}\right)$ be the edge of the incoming $\mathcal{R}$-rainbow with $\mathcal{C}\left(u_{\ell}\right)=\mathcal{G}$; notice that such an edge exists as $\mathcal{G} \in \mathcal{C} \backslash \mathcal{C}\left[b_{1}, b_{2}\right]$. Since $r_{0}$ is the first element from $\mathcal{R}$ after $b_{2}$ in $L, r_{0} \preceq r_{\ell}$. Thus, [ $u_{\ell} \ldots b_{1} \ldots b_{2} \ldots g_{1} \ldots r_{0} \ldots r_{\ell}$ ] holds in $L$ such that $\mathcal{C}\left(u_{\ell}\right)=\mathcal{G} \notin\{\mathcal{R}, \mathcal{B}\}$. Let $g_{2}$ be the last element between $u_{\ell}$ and $b_{1}$ from $\mathcal{G}$, that is, $u_{\ell} \preceq g_{2} \prec b_{1}$ in $L$. Now, consider the pattern:

which satisfies the conditions of the lemma, since between $g_{2}$ and $g_{1}$ in $L$ there is an element of $\mathcal{R}$ (namely, the one between $b_{1}$ and $b_{2}$ in $L$ ) and no elements of $\mathcal{G}$ (by the choice of $g_{1}$ and $g_{2}$ ). Further, $|\mathcal{C}|-\left|\mathcal{C}\left[g_{2}, g_{1}\right]\right|<|\mathcal{C}|-\left|\mathcal{C}\left[b_{1}, b_{2}\right]\right|$, since $\{\mathcal{G}\}=\mathcal{C}\left[g_{2}, g_{1}\right] \backslash \mathcal{C}\left[b_{1}, b_{2}\right]$. By the inductive hypothesis, the aforementioned pattern is not contained in $L$. Thus, also the initial one is not contained.

In the next five lemmas we study configurations that cannot appear in a rainbow formed by the edges of $G(P,<)$ in an MRU extension.

Lemma 5. Let $\mathcal{R}$ and $\mathcal{B}$ be different chains of a width-w poset. Then a rainbow in an MRU extension of the poset does not contain all edges from

$$
T_{\mathcal{R}}^{*} \cup\left\{\left(b_{1}, b_{2}\right)\right\},
$$

where $b_{1}, b_{2} \in \mathcal{B}$ and $T_{\mathcal{R}}^{*}$ is a complete incoming $\mathcal{R}$-rainbow.
Proof. Assume to the contrary that a rainbow $T$ contains an incoming $\mathcal{R}$-rainbow formed by edges $\left(u_{1}, r_{1}\right), \ldots,\left(u_{w}, r_{w}\right)$ and an edge $\left(b_{1}, b_{2}\right)$ with $b_{1}, b_{2} \in \mathcal{B}$. As in the proof of Theorem 1, we can show that $\left(u_{w-1}, r_{w-1}\right)$ and $\left(u_{w}, r_{w}\right)$ are the two innermost edges of $T$, and $\mathcal{C}\left(u_{w}\right)=\mathcal{R}$. Assume without loss of generality that $u_{k} \prec b_{1} \prec u_{k+1}$ in $L$ for some $1 \leq k \leq w-1$, which implies that $r_{k+1} \prec b_{2} \prec r_{k}$. Thus, the following holds in $L$.


By Proposition 1, there are no elements from $\mathcal{B}$ between $b_{1}$ and $b_{2}$. Hence, the conditions of Lemma 4 hold for the pattern; a contradiction.

Lemma 6. Let $\mathcal{R}$ and $\mathcal{B}$ be different chains of a width-w poset. Then a rainbow in an MRU extension of the poset does not contain all edges from

$$
T_{\mathcal{R}}^{*} \backslash\left\{\left(r_{1}, r_{2}\right)\right\} \quad \cup \quad T_{\mathcal{B}}^{*} \backslash\left\{\left(b_{1}, b_{2}\right)\right\}
$$

where $r_{1}, r_{2} \in \mathcal{R}, b_{1}, b_{2} \in \mathcal{B}$, and $T_{\mathcal{R}}^{*}, T_{\mathcal{B}}^{*}$ are complete incoming $\mathcal{R}$-rainbow and $\mathcal{B}$-rainbow, respectively.

Proof. Let $T_{\mathcal{R}}$ be an incoming $\mathcal{R}$-rainbow of size $w-1$ without the $\mathcal{R}$-self edge; define $T_{\mathcal{B}}$ symmetrically. Assume to the contrary that a rainbow $T$ in an MRU extension $L$ contains both $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$. Let $\left(u_{w-1}, r_{w-1}\right)$ and $\left(v_{w-1}, b_{w-1}\right)$ be the innermost edges of $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$ in $T$, respectively. Without loss of generality, assume that $\left(v_{w-1}, b_{w-1}\right)$ nests $\left(u_{w-1}, r_{w-1}\right)$. This implies the following in $L$ :


By Lemma 2 applied to $T_{\mathcal{B}}$, there are no elements from $\mathcal{B}$ between $v_{w-1}$ and $r_{w-1}$ in $L$. Consider edge $\left(u_{i}, r_{i}\right)$ of $T_{\mathcal{R}}$ such that $u_{i} \in \mathcal{B}$. Element $u_{i}$ ensures that there are some elements preceding $v_{w-1}$ in $L$ that belong to $\mathcal{B}$. Let $b_{\ell}$ be the last such element in $L$, that is, $b_{\ell} \preceq v_{w-1}$. Symmetrically, let $b_{r}$ be the first element from $\mathcal{B}$ following $r_{w-1}$ in $L$, that is, $r_{w-1} \prec b_{r} \preceq b_{w-1}$, and we have:


By the choice of $b_{\ell}$ and $b_{r}$, we further know that between $b_{\ell}$ and $b_{r}$ there are no elements from $\mathcal{B}$, but there is an element from $\mathcal{R}$, namely $r_{w-1}$. Let $\left(u_{1}, r_{1}\right), \ldots,\left(u_{k}, r_{k}\right)$ be the edges of $T_{\mathcal{R}}$ that nest both $b_{\ell}$ and $b_{r}$ in $L$. Assuming that $u_{w}=r_{w-1}$, we conclude that the following holds in $L$ :


Since between $b_{\ell}$ and $b_{r}$ there are no elements from $\mathcal{B}$, but there is an element from $\mathcal{R}$, we have the forbidden pattern of Lemma 4; a contradiction.

Lemma 7. Let $\mathcal{R}, \mathcal{B}, \mathcal{G}$ be pairwise different chains of a width-w poset. Then a rainbow in an MRU extension of the poset does not contain all edges from

$$
T_{\mathcal{R}}^{*} \backslash\left\{\left(g_{1}, r\right)\right\} \cup T_{\mathcal{B}}^{*} \backslash\left\{\left(g_{2}, b\right)\right\},
$$

where $g_{1}, g_{2} \in \mathcal{G}, r \in \mathcal{R}, b \in \mathcal{B}$, and $T_{\mathcal{R}}^{*}, T_{\mathcal{B}}^{*}$ are complete incoming $\mathcal{R}$-rainbow and $\mathcal{B}$-rainbow, respectively.

Proof. Assume to the contrary that a rainbow $T$ contains both $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$ as in the statement of the lemma. Let $\left(u_{1}, r_{1}\right), \ldots,\left(u_{w-1}, r_{w-1}\right)$ be the edges of $T_{\mathcal{R}}$ and $\left(v_{1}, b_{1}\right), \ldots,\left(v_{w-1}, b_{w-1}\right)$ be the edges of $T_{\mathcal{B}}$, where $\left(u_{w-1}, r_{w-1}\right)$ and $\left(v_{w-1}, b_{w-1}\right)$ are the $\mathcal{R}$ - and $\mathcal{B}$-self edges, respectively. By Proposition 1, $\left(u_{w-1}, r_{w-1}\right)$ and $\left(v_{w-1}, b_{w-1}\right)$ are innermost edges in $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$. Without loss of generality, assume that $\left(v_{w-1}, b_{w-1}\right)$ nests $\left(u_{w-1}, r_{w-1}\right)$, and that $v_{w-1}$ appears between vertices $u_{k}$ and $u_{k+1}$ of $T_{\mathcal{R}}$, which implies that $r_{k+1} \prec b_{w-1} \prec r_{k}$. Hence, the following holds in $L$ :


By Proposition 1, there is no vertex of $\mathcal{B}$ between $v_{w-1}$ and $b_{w-1}$ in $L$. If there is a vertex from $\mathcal{G}$ between $v_{w-1}$ and $b_{w-1}$ in $L$, then we have the forbidden pattern of Lemma 4 , since $\mathcal{C}\left(u_{i}\right) \neq \mathcal{G}$ for all $1 \leq i \leq w-1$.


Otherwise, by Lemma 1 , there is some $x \notin \mathcal{B}$ between $v_{w-1}$ and $b_{w-1}$ in $L$, such that $x<b_{w-1}$. As mentioned above, $x \notin \mathcal{G}$ either. Thus, the incoming $\mathcal{B}$-rainbow contains edge $\left(v_{i}, b_{i}\right)$, which nests $\left(v_{w-1}, b_{w-1}\right)$, such that $\mathcal{C}\left(v_{i}\right)=\mathcal{C}(x)$. Since $v_{i}<x<b_{w-1}<b_{i}$, the edge ( $v_{i}, b_{i}$ ) is transitive; a contradiction.

Lemma 8. Let $\mathcal{R}, \mathcal{B}, \mathcal{G}$ be pairwise different chains of a width-w poset. Then a rainbow in an MRU extension of the poset does not contain all edges from

$$
T_{\mathcal{B}}^{*} \backslash\left\{\left(b_{1}, b_{2}\right)\right\} \cup T_{\mathcal{R}}^{*} \backslash\left\{\left(m_{r}, r\right)\right\} \cup T_{\mathcal{G}}^{*} \backslash\left\{\left(m_{g}, g\right)\right\},
$$

where $b_{1}, b_{2} \in \mathcal{B}, m_{r} \in V \backslash \mathcal{R}, r \in \mathcal{R}, m_{g} \in V \backslash \mathcal{G}, g \in \mathcal{G}$, and $T_{\mathcal{B}}^{*}, T_{\mathcal{R}}^{*}, T_{\mathcal{G}}^{*}$ are complete incoming $\mathcal{B}$-rainbow, $\mathcal{R}$-rainbow $\mathcal{G}$-rainbow, respectively.

Proof. Assume to the contrary that a rainbow $T$ contains three incoming rainbows, $T_{\mathcal{B}}, T_{\mathcal{R}}$, and $T_{\mathcal{G}}$, as in the statement of the lemma. Without loss of generality, assume that the $\mathcal{G}$-self edge ( $g_{1}, g_{2}$ ) is nested by the $\mathcal{R}$-self edge, $\left(r_{1}, r_{2}\right)$; that
is, $r_{1} \prec g_{1} \prec g_{2} \prec r_{2}$. Denote the edges of $T_{\mathcal{B}}$ by $\left(u_{i}, b_{u_{i}}\right)$ for $1 \leq i \leq w-1$, and assume that the following holds in $L$ for some $k \leq w-1$.


Suppose there exists a vertex $x \in \mathcal{B}$ such that $r_{1} \prec x \prec r_{2}$; then $r_{1}$ and $r_{2}$ together with $x$ and edges of $T_{\mathcal{B}}$ form the forbidden pattern of Lemma 4. Thus, there are no vertices from $\mathcal{B}$ between $r_{1}$ and $r_{2}$ in $L$, and ( $u_{k}, b_{u_{k}}$ ) is the innermost edge of $T_{\mathcal{B}}$ in $T$. Therefore, we can find two consecutive vertices in chain $\mathcal{B}, b^{\prime}$ and $b^{\prime \prime}$, such that $b^{\prime} \prec r_{1} \prec r_{2} \prec b^{\prime \prime} \preceq b_{u_{k}}$. Here $b^{\prime}$ exists because by Lemma 7 at least one of the two edges, $(b, r),(b, g)$, is in $T$ as part of $T_{\mathcal{R}}, T_{\mathcal{G}}$, respectively. Further, by Lemma 2, the interval between $u_{k}$ and $b_{u_{k}}$ does not contain pattern $\left[u_{k} \ldots b \ldots x \ldots b_{u_{k}}\right]$, where $b \in \mathcal{B}, x \notin \mathcal{B}$. Thus, $b^{\prime} \prec u_{k}$ and the interval of $L$ between $b^{\prime \prime}$ and $b_{u_{k}}$ contains vertices only from $\mathcal{B}\left(b^{\prime \prime}=b_{u_{k}}\right.$ is possible).


Now if there exists a vertex from $\mathcal{C}\left(m_{r}\right)$ between $b^{\prime}$ and $b^{\prime \prime}$, then $\left[b^{\prime} \ldots r_{1} \ldots b^{\prime \prime}\right]$ together with the edges of $T_{\mathcal{R}}$ form the forbidden pattern of Lemma 4. Thus, there are no vertices from $\mathcal{C}\left(m_{r}\right)$ between $b^{\prime}$ and $b^{\prime \prime}$.

Finally, consider vertices $r_{1}$ and $r_{2}$ that are consecutive in $\mathcal{R}$. By Lemma 1 and the fact that $r_{1} \prec g_{1} \prec r_{2}$, there is $x \notin \mathcal{C}\left(m_{r}\right)$ between $r_{1}$ and $r_{2}$ such that $x<r_{2}$. Since $x \notin \mathcal{C}\left(m_{r}\right)$, rainbow $T_{\mathcal{R}}$ contains edge $\left(y, r_{y}\right)$ for some $r_{y} \in \mathcal{R}$ such that $\mathcal{C}(y)=\mathcal{C}(x)$. Edge $\left(y, r_{y}\right)$ is transitive, as $y<x<r_{2}<r_{y}$; a contradiction.

Lemma 9. Let $\mathcal{R}, \mathcal{B}, \mathcal{G}$ be pairwise different chains of a width-w poset. Then a rainbow in an MRU extension of the poset does not contain all edges from

$$
T_{\mathcal{B}}^{*} \backslash\left\{\left(m_{b}, b\right)\right\} \quad \cup \quad T_{\mathcal{R}}^{*} \backslash\left\{\left(m_{r}, r\right)\right\} \quad \cup \quad T_{\mathcal{G}}^{*} \backslash\left\{\left(m_{g}, g\right)\right\}
$$

where $m_{b} \in V \backslash \mathcal{B}, b \in \mathcal{B}, m_{r} \in V \backslash \mathcal{R}, r \in \mathcal{R}, m_{g} \in V \backslash \mathcal{G}, g \in \mathcal{G}$, and $T_{\mathcal{B}}^{*}, T_{\mathcal{R}}^{*}, T_{\mathcal{G}}^{*}$ are complete incoming $\mathcal{B}$-rainbow, $\mathcal{R}$-rainbow $\mathcal{G}$-rainbow, respectively.

Proof. Assume to the contrary that a rainbow $T$ contains three incoming rainbows $T_{\mathcal{B}}, T_{\mathcal{R}}$, and $T_{\mathcal{G}}$, as in the statement of the lemma for some MRU extension $L$ of the poset. By Lemma $7, \mathcal{C}\left(m_{b}\right), \mathcal{C}\left(m_{r}\right)$, and $\mathcal{C}\left(m_{g}\right)$ are pairwise distinct chains.

Without loss of generality, assume that the $\mathcal{R}$-self edge, $\left(r_{1}, r_{2}\right)$, nests the $\mathcal{B}$-self edge, $\left(b_{1}, b_{2}\right)$, which in turn nests the $\mathcal{G}$-self edge, $\left(g_{1}, g_{2}\right)$. Namely, $r_{1} \prec$ $b_{1} \prec g_{1} \prec g_{2} \prec b_{2} \prec r_{2}$. Denote the edges of $T_{\mathcal{B}}$ by $\left(u_{i}, b_{u_{i}}\right)$ for $1 \leq i \leq w-1$, and assume that

holds in $L$ for some $k \leq w-1$. If there is a vertex from $\mathcal{C}\left(m_{b}\right)$ between $r_{1}$ and $r_{2}$ in $L$, then the forbidden pattern of Lemma 4 is formed by $\left[r_{1} \ldots b_{1} \ldots r_{2}\right]$ and edges
of $T_{\mathcal{B}}$. Otherwise by Lemma 1 , there is some $x \notin \mathcal{C}\left(m_{b}\right)$ between $b_{1}$ and $b_{2}$ such that $x<b_{2}$. Since $\left|T_{\mathcal{B}}\right|=w-1, T_{\mathcal{B}}$ contains edge $\left(y, b_{y}\right)$ for some $b_{y} \in \mathcal{B}$ such that $\mathcal{C}(y)=\mathcal{C}(x)$. Since $y<x<b_{2}<b_{y},\left(y, b_{y}\right)$ is transitive; a contradiction.

Now we state the main result of the section.
Theorem 2. The maximum size of a rainbow formed by the edges of $G(P,<)$ in an MRU extension of a poset $\langle P,<\rangle$ of width $w$ is at most $(w-1)^{2}+1$.

Proof. When $w=2$, the theorem holds for any lazy linear extension by Theorem 1 and thus for MRU. Hence, we focus on the case $w \geq 3$. Assume to the contrary that an MRU extension contains a rainbow $T$ of size $(w-1)^{2}+1$. Let $T_{\mathcal{B}}, T_{\mathcal{R}}$, $T_{\mathcal{G}}$ be the largest incoming rainbows in $T$ corresponding to chains $\mathcal{B}, \mathcal{R}$, and $\mathcal{G}$, respectively. Assume without loss of generality that $\left|T_{\mathcal{B}}\right| \geq\left|T_{\mathcal{R}}\right| \geq\left|T_{\mathcal{G}}\right|$. By the pigeonhole principle, we have $\left|T_{\mathcal{B}}\right| \geq\left|T_{\mathcal{R}}\right| \geq w-1$. We claim that $\left|T_{\mathcal{B}}\right|=w-1$. Indeed, if $\left|T_{\mathcal{B}}\right|=w$, then by Lemma $5, T_{\mathcal{R}}$ does not contain the $\mathcal{R}$-self edge. Thus, $T$ contains $T_{\mathcal{B}}^{*}$ and $T_{\mathcal{R}}^{*} \backslash\left\{\left(r_{1}, r_{2}\right)\right\}$ with $r_{1}, r_{2} \in \mathcal{R}$; a contradiction by Lemma 6 .

Thus, $\left|T_{\mathcal{B}}\right|=\left|T_{\mathcal{R}}\right|=\left|T_{\mathcal{G}}\right|=w-1$ follows, and we distinguish cases based on the number of self edges in $T_{\mathcal{B}}, T_{\mathcal{R}}$, and $T_{\mathcal{G}}$. If each of them contain its self edge, then we have the forbidden configuration of Lemma 9. If two of $T_{\mathcal{B}}, T_{\mathcal{R}}$, and $T_{\mathcal{G}}$ contain a self edge, then we have the forbidden configuration of Lemma 8. Finally, if at most one of $T_{\mathcal{B}}, T_{\mathcal{R}}$, and $T_{\mathcal{G}}$ contains a self edge, say $T_{\mathcal{B}}$, then $T_{\mathcal{R}}$ and $T_{\mathcal{G}}$ form the forbidden configuration of Lemma 6. This concludes the proof.

Theorem 7 in Appendix B. 3 shows that our analysis is tight, i.e., there are posets of width $w$ and corresponding MRU extensions containing $\left((w-1)^{2}+1\right)$-rainbows.

## 5 A Counterexample to Conjecture 1

Here we sketch our approach to disprove Conjecture 1. We describe a poset in terms of its cover graph $G(p, q)$; see Fig. 2. For $p \geq q-3$, graph $G(p, q)$ consists of $2 p+q$ vertices $a_{1}, \ldots, a_{p}, b_{1}, \ldots, b_{q}$, and $c_{1}, \ldots, c_{p}$ that form three chains of lengths $p, q$, and $p$, respectively. For all $1 \leq i \leq p$ and for all $1 \leq j \leq q$, the edges $\left(a_{i}, a_{i+1}\right),\left(b_{j}, b_{j+1}\right)$ and $\left(c_{i}, c_{i+1}\right)$ form the intra-chain edges of $G(p, q)$. Graph $G(p, q)$ also contains the following inter-chain edges: (i) $\left(a_{i}, c_{i+3}\right)$ and ( $c_{i}, a_{i+3}$ ) for all $1 \leq i+3 \leq p$, and (ii) $\left(a_{i}, b_{i}\right)$ and $\left(c_{i}, b_{i}\right)$ for all $1 \leq i \leq q$. We denote by $\widetilde{G}(p, q)$ the graph obtained by adding $\left(b_{1}, a_{p}\right)$ and $\left(b_{1}, c_{p}\right)$ to $G(p, q)$.

Theorem 3. $\widetilde{G}(31,22)$ requires 4 queues in every linear extension.
Sketch. We provide lower bounds on the queue number for simple subgraphs of $\widetilde{G}(p, q)$ (Lemmas 10 and 11) and then for more complicated ones (Lemmas 12 and 13) for appropriate values of $p$ and $q$. We distinguish two cases depending on the length of edge $\left(b_{1}, c_{p}\right)$ in a linear extension $L$ of $\widetilde{G}(p, q)$. Either the edge is "short" (that is, $b_{1}$ is close to $c_{p}$ in $L$ ) or "long". In the first case, the existence of a 4 -rainbow is derived from the properties of the subgraphs. In the latter case, edge $\left(b_{1}, c_{p}\right)$ nests a large subgraph of $\widetilde{G}(p, q)$, which needs 3 queues.


Fig. 2: Illustration of graph $\widetilde{G}(p, q)$ with $p=16$ and $q=11$.

To prove that Conjecture 1 does not hold for $w>3$, we employ an auxiliary lemma implicitly used in [15]; see Lemma 14 in Appendix D for details.

Theorem 4. For every $w \geq 3$, there is a width-w poset with queue number $w+1$.

## 6 Conclusions

In this paper, we explored the relationship between the queue number and the width of posets. We disproved Conjecture 1 and we focused on two natural types of linear extensions, lazy and MRU. That led to an improvement of the upper bound on the queue number of posets. A natural future direction is reduce the gap between the lower bound, $w+1$, and the upper bound, $(w-1)^{2}+1$, on the queue number of posets of width $w>2$. In particular, we do not know whether the queue number of width-3 posets is four or five, and whether a subquadratic upper bound is possible. It is also intriguing to ask whether Conjecture 1 holds for planar width- $w$ posets whose best-known upper bound is currently $3 w-2$ [15].

Another related open problem is on the stack number of directed acyclic graphs (DAGs). The stack number is defined analogously to the queue number except that no two edges in a single stack cross. Heath et al. $[12,13]$ asked whether the stack number of upward planar DAGs is bounded by a constant. While the question has been settled for some subclasses of planar digraphs [9], the general problem remains unsolved. This is in contrast with the stack number of undirected planar graphs, which has been shown recently to be exactly four [2].

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## Appendix

## A Pseudocode for the Algorithms

In this section, we provide pseudocode for computing a lazy linear extension (Algorithm 1) and an MRU extension (Algorithm 2) of a poset of width $w$.

```
Algorithm 1: Lazy Linear Extension
    Input : The cover graph \(G=(V, E)\) of a width- \(w\) poset \(\langle P,<\rangle\) with \(n\) elements
                and a chain partition \(\mathcal{C}\)
    Output: A linear extension \(L: v_{1} \prec v_{2} \prec \cdots \prec v_{n}\) of \(G\).
    for \(i=1\) to \(n\) do
        \(v_{i} \leftarrow \emptyset ;\)
        // find vertices from \(V \backslash L\) having no incoming edges from \(V \backslash L\)
        \(S \leftarrow\{v \in V \backslash L: \nexists(u, v) \in E\) with \(u \in V \backslash L\} ;\)
        foreach \(u \in S\) do /* iterating over candidates */
            if \(\mathcal{C}(u)=\mathcal{C}\left(v_{i-1}\right)\) then
                \(v_{i} \leftarrow u ;\)
        if \(v_{i}=\emptyset\) then \(v_{i} \leftarrow \operatorname{arbitrary}(S)\);
        \(L \leftarrow L \oplus\left\{v_{i}\right\} ;\)
    return \(L\);
```

```
Algorithm 2: MRU Extension
    Input : The cover graph \(G=(V, E)\) of a width- \(w\) poset \(\langle P,<\rangle\) with \(n\) elements
        and a chain partition \(\mathcal{C}\)
    Output: A linear extension \(L: v_{1} \prec v_{2} \prec \cdots \prec v_{n}\) of \(G\).
    for \(i=1\) to \(n\) do
        \(v_{i} \leftarrow \emptyset ;\)
        // find vertices from \(V \backslash L\) having no incoming edges from \(V \backslash L\)
        \(S \leftarrow\{v \in V \backslash L: \nexists(u, v) \in E\) with \(u \in V \backslash L\} ;\)
        for \(j=i-1\) down to 1 do \(\quad / *\) iterating over reversed \(L * /\)
        foreach \(u \in S\) do /* iterating over candidates */
            if \(\mathcal{C}(u)=\mathcal{C}\left(v_{j}\right)\) then \(/ *\) check corresponding element of \(L\) */
                \(v_{i} \leftarrow u ;\)
                break;
            if \(v_{i}=\emptyset\) then \(v_{i} \leftarrow \operatorname{arbitrary}(S)\);
            \(L \leftarrow L \oplus\left\{v_{i}\right\} ;\)
    return \(L\);
```


## B Lower Bounds

## B. 1 A Lower Bound for General Linear Extensions

In the following, we prove that a linear extension of a poset of width $w$ may result in a rainbow of size $w^{2}$ for the edges of its cover graph, which suggests that the bound by Heath and Pemmaraju [11] is worst-case optimal. Notice that the same claim is made by Knauer et al. [15]. However, the poset that they claim to require $w^{2}$ queues (in some linear extension of it) is defined on $2 w$ elements. As a result, its cover graph cannot have more than $w$ independent edges. Thus, also the largest rainbow that can be formed by any linear extension is of size at most $w$, that is, $w$ is an upper bound on the queue number of this poset.

Theorem 5. For every even $w \geq 2$, there is a width-w poset and a linear extension of it which results in a rainbow of size $w^{2}$ for the edges of its cover graph.

Proof. For even $w \geq 2$, we construct a poset $\left\langle P_{w},<\right\rangle$ of width $w$ and we demonstrate a linear extension of it, which results in a queue layout of $G\left(P_{w},<\right)$ with $w^{2}$ queues. We describe $\left\langle P_{w},<\right\rangle$ in terms of its cover graph $G\left(P_{w},<\right)$, which contains $w$ chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w}$ of length $2 w$ that form paths in $G\left(P_{w},<\right)$. We denote the $j$-th vertex of the $i$-th chain $\mathcal{C}_{i}$ by $v_{i, j}$, where $1 \leq i \leq w$ and $1 \leq j \leq 2 w$. Since each chain is a path in $G\left(P_{w},<\right),\left(v_{i, j}, v_{i, j+1}\right)$ is an edge in $G\left(P_{w},<\right)$ for every $1 \leq i \leq w$ and $1 \leq j \leq 2 w-1$. The first and the last $w$ vertices of each such path partition the vertex-set of $G\left(P_{w},<\right)$ into two sets $S$ and $T$, respectively, that is, $S=\cup_{i=1}^{w}\left\{v_{i, 1}, \ldots, v_{i, w}\right\}$ and $T=\cup_{i=1}^{w}\left\{v_{i, w+1}, \ldots, v_{i, 2 w}\right\}$. Observe that each chain has exactly one edge, called middle-edge, connecting a vertex in $S$ to a vertex in $T$. We describe the inter-chain edges of $G\left(P_{w},<\right)$ in an iterative way. Assume that we have introduced the inter-chain edges that form the connections between the first $i-1$ chains and let $\mathcal{C}_{i}$ be the next chain to consider. First, we introduce the outgoing inter-chain edges from the vertices of $\mathcal{C}_{i}$ as follows. For $k=1, \ldots, i-1$, we connect the $k$-th vertex $v_{i, k}$ of chain $\mathcal{C}_{i}$ to the $(2 w-i+1)$ th vertex $v_{i-k, 2 w-i+1}$ of chain $\mathcal{C}_{i-k}$, that is, we introduce ( $v_{i, k}, v_{i-k, 2 w-i+1}$ ) in $G\left(P_{w},<\right)$. We next introduce the incoming inter-chain edges to vertices of $\mathcal{C}_{i}$ as follows. For $k=1, \ldots, i-1$, we connect the $(w-i+k)$-th vertex of $k$-th chain $\mathcal{C}_{k}$ to the $(2 w-k+1)$-th vertex of chain $\mathcal{C}_{i}$, that is, we introduce edge $\left(v_{k, w-i+k}, v_{i, 2 w-k+1}\right)$ in $G\left(P_{w},<\right)$. This completes the construction of $G\left(P_{w},<\right)$ and thus of poset $\left\langle P_{w},<\right\rangle$.

By construction, the inter-chain edges of $G\left(P_{w},<\right)$ connect only vertices from $S$ to vertices in $T$, and from each chain there is only one (outgoing) interchain edge to every other chain. This implies that an inter-chain edge cannot be transitive in $G\left(P_{w},<\right)$. On the other hand, an intra-chain edge $(u, v)$ also cannot be transitive because its source $u$ needs an outgoing inter-chain edge (which classifies $u$ in $S$ ) and its target $v$ an incoming inter-chain edge (which classifies $v$ in $T$ ). This implies that $(u, v)$ is a middle edge. In this case, however, our construction ensures that there are inter-chain edges attached to neither $u$ nor $v$. Thus, $G\left(P_{w},<\right)$ is transitively reduced. Since $G\left(P_{w},<\right)$ is by construction


Fig. 3: Illustration for the proof of Theorem 5: The cover graph $G\left(P_{w},<\right)$ of a poset $\left\langle P_{w},<\right\rangle$ with $w=4$ and a linear extension (indicated with gray numbers) of it which yields a rainbow of size 16 .
acyclic, we conclude that $\left\langle P_{w},<\right\rangle$ is a poset. Since any two vertices in the same chain are comparable, the width of $\left\langle P_{w},<\right\rangle$ equals to the number of sources (or sinks) of chains, which is $w$.

To complete the proof, we next describe a linear extension of $G\left(P_{w},<\right)$ which necessarily yields a $w^{2}$-rainbow. For $i=1, \ldots, w$ and for $j=1, \ldots, w-1$, the $j$-th vertex $v_{i, j}$ of chain $\mathcal{C}_{i}$ is the $((i-1)(w-1)+j)$-th vertex in the extension. For $i=1, \ldots, w$, the $w$-th vertex $v_{i, w}$ of chain $\mathcal{C}_{i}$ is the $(w(w-1)+w-(i-1))$-th vertex in the extension. For $i=1, \ldots, w$ and for $j=1, \ldots, w$, the $(w+j)$-th vertex $v_{i, w+j}$ of chain $\mathcal{C}_{i}$ is the $\left(w^{2}+j w+(i-1)\right)$-th vertex in the extension. In this linear extension, all inter-chain edges (which are in total $w(w-1)$ ) and all middle edges (which are in total $w$ ) form a rainbow of size $w^{2}$.

## B. 2 A Lower Bound for Lazy Linear Extension

Theorem 6. For every $w \geq 2$, there exists a width-w poset, which has a lazy linear extension resulting in a rainbow of size $w^{2}-w$ for the edges of its cover graph.

Proof. For $w \geq 2$, we construct a poset $\left\langle P_{w},<_{w}\right\rangle$ of width $w$ and we demonstrate a lazy linear extension $L_{w}$ of it, which results in a queue layout of $G\left(P_{w},<_{w}\right)$ with $w^{2}-w$ queues. We describe $\left\langle P_{w},<_{w}\right\rangle$ in terms of its cover graph $G\left(P_{w},<_{w}\right)$. We define $G\left(P_{w},<_{w}\right)$ recursively based on the graph $G\left(P_{w-1},<_{w-1}\right)$ of width $w-1$, for which we assume that it admits a lazy linear extension $L_{w-1}$, such that the edges of $G\left(P_{w-1},<_{w-1}\right)$ form a rainbow of size exactly $(w-1)^{2}-(w-1)$ in $L_{w-1}$. Since $G\left(P_{w-1},<_{w-1}\right)$ has width $w-1$, its vertex-set can be partitioned into $w-1$ chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$ [4]. As an invariant property in the recursive definition of $G\left(P_{w},<_{w}\right)$, we assume that the first and the last vertices in $L_{w-1}$ belong to two different chains of the partition, say w.l.o.g. to $\mathcal{C}_{1}$ and $\mathcal{C}_{w-1}$, respectively.

In the base case $w=2$, cover graph $G\left(P_{2},<_{2}\right)$ consists of five vertices $v_{1}, \ldots, v_{5}$ and four edges $\left(v_{1}, v_{2}\right),\left(v_{1}, v_{5}\right),\left(v_{3}, v_{4}\right)$ and $\left(v_{4}, v_{5}\right)$. It is not difficult


Fig. 4: Illustration for Theorem 6; $q$ denotes the number of vertices of $G\left(P_{w-1},<_{w-1}\right)$, that is, $q=3(w-1)^{2}-(w-1)-5$.
to see that $G\left(P_{2},<_{2}\right)$ has width 2 and for the chain partition $\mathcal{C}_{1}=\left\{v_{1}, v_{2}\right\}$, $\mathcal{C}_{2}=\left\{v_{3}, v_{4}, v_{5}\right\}$ the linear extension $v_{1} \prec \ldots \prec v_{5}$ is a lazy linear extension of it, which satisfies the invariant property and results in a 2-rainbow formed by $\left(v_{1}, v_{5}\right)$ and $\left(v_{3}, v_{4}\right)$.

Graph $G\left(P_{w},<_{w}\right)$ is obtained by augmenting $G\left(P_{w-1},<_{w-1}\right)$ with $6 w-4$ vertices. Hence, $G\left(P_{w},<_{w}\right)$ contains $3 w^{2}-w-5$ vertices in total. We further enrich the chain partition $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$ of $G\left(P_{w-1},<_{w-1}\right)$ by one additional chain $\mathcal{C}_{w}$ in $G\left(P_{w},<_{w}\right)$; see Fig. 4. In particular, chain $\mathcal{C}_{w}$ contains $2(w-1)$ vertices $v_{w, 1}, \ldots, v_{w, 2 w-2}$ vertices that form a path in this order in $G\left(P_{w},<_{w}\right)$. Chain $\mathcal{C}_{1}$ of $G\left(P_{w-1},<_{w-1}\right)$ in enriched with five additional vertices $v_{1,1}, v_{1,2}, \bar{v}_{1,2}, v_{1,3}$ and $v_{1,4}$ in $G\left(P_{w},<_{w}\right)$, such that $v_{1,1}$ is connected to $v_{1,2}, v_{1,2}$ is connected $\bar{v}_{1,2}$ and $\bar{v}_{1,2}$ is connected to the first vertex of chain $\mathcal{C}_{i}$ in $L_{w-1}$ for all $1 \leq i \leq w-1$, the last vertex of $\mathcal{C}_{1}$ in $L_{w-1}$ is connected to $v_{1,3}$, and $v_{1,3}$ is connected to $v_{1,4}$. For $i=2, \ldots, w-2$, chain $\mathcal{C}_{i}$ of $G\left(P_{w-1},<_{w-1}\right)$ is enriched with four vertices $v_{i, 1}, v_{i, 2}, v_{i, 3}$ and $v_{i, 4}$ in $G\left(P_{w},<_{w}\right)$, such that $v_{i, 1}$ is connected to $v_{i, 2}, v_{i, 2}$ is connected to the first vertex of chain $\mathcal{C}_{i}$ in $L_{w-1}$, the last vertex of chain $\mathcal{C}_{i}$ in $L_{w-1}$ is connected to $v_{i, 3}$, and vertex $v_{i, 3}$ is connected to $v_{i, 4}$. Finally, chain $\mathcal{C}_{w-1}$ is enriched with five vertices $v_{w-1,1}, v_{w-1,2}, \bar{v}_{w-1,3}, v_{w-1,3}$ and $v_{w-1,4}$, such that vertex $v_{w-1,1}$ is connected to $v_{w-1,2}, v_{w-1,2}$ is connected to the first vertex of $\mathcal{C}_{w-1}$ in $L_{w-1}$, the last vertex of $\mathcal{C}_{w-1}$ in $L_{w-1}$ is connected to $\bar{v}_{w-1,3}$, $\bar{v}_{w-1,3}$ is connected to $v_{i, 3}$ for all $1 \leq i \leq w-1$ and $v_{i, 3}$ is connected to $v_{w-1,4}$ for all $1 \leq i \leq w$. We complete the construction of $G\left(P_{w},<_{w}\right)$ by adding the following edges (colored orange in Fig. 4): (i) $\left(v_{i, 1}, v_{w, w+i-1}\right)$ for all $1 \leq i \leq w-1$, (ii) $\left(v_{w, i}, v_{w-i, 4}\right)$ for all $1 \leq i \leq w-1$.

The construction ensures that $G\left(P_{w},<_{w}\right)$ contains no transitive edges and that its width is $w$, since all the newly added vertices either are comparable to vertices of $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$ or belong to the newly introduced chain $\mathcal{C}_{w}$. Hence, $P_{w}$
is a well-defined width- $w$ poset. Now, consider the following linear extension $L_{w}$ of $G\left(P_{w},<_{w}\right)$ :

$$
\begin{array}{r}
{\left[v_{w, 1}, \ldots, v_{w, w-1}, v_{w-1,1}, v_{w-1,2}, \ldots v_{1,1}, v_{1,2}, \bar{v}_{1,2}, L_{w-1}, \bar{v}_{w-1,3}, v_{w-1,3}\right.} \\
v_{w, w}, \ldots, v_{w, 2 w-2}, v_{w-2,3}, v_{w-2,4}, \ldots, v_{1,3}, v_{1,4}, v_{w-1,4}
\end{array}
$$

It can be easily checked that $L_{w}$ is a lazy linear extension of $G\left(P_{w},<_{w}\right)$, under our invariant property that the first and the last vertices of $L_{w-1}$ belong to two different chains in $\left\{\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}\right\}$, which we assume to be $\mathcal{C}_{1}$ and $\mathcal{C}_{w-1}$, respectively. Note that since the first vertex of $L_{w}$ belongs to $\mathcal{C}_{w}$ while its last vertex to $\mathcal{C}_{w-1}$, the invariant property is maintained in the course of the recursion. We complete the proof by observing that the $w-1$ edges stemming from the first $w-1$ vertices of $\mathcal{C}_{w}$ towards the last vertices of the chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$ and the $w-1$ edges stemming from the first $w-1$ vertices of $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$ towards the last $w-1$ vertices of chain $\mathcal{C}_{w}$ form a rainbow of size $2 w-2$ in $L_{w}$ (see the orange edges in Fig. 4), which nests the rainbow of size $(w-1)^{2}-(w-1)$ of $L_{w-1}$. Thus, we have identified a rainbow of total size $w^{2}-w$ in $L_{w}$, as desired.

## B. 3 A Lower Bound for MRU Extension

Theorem 7. For every $w \geq 2$, there exists a width-w poset, which has an MRU extension resulting in a rainbow of size $(w-1)^{2}+1$ for the edges of its cover graph.

Proof. As in the proof of Theorem 6, we describe poset $\left\langle P_{w},<_{w}\right\rangle$ in terms of its cover graph $G\left(P_{w},<_{w}\right)$. Similar to the proof of Theorem $6 G\left(P_{w},<_{w}\right)$ is defined recursively based on graph $G\left(P_{w-1},<_{w-1}\right)$ which is of width $w-1$ and thus its vertex-set admits a partition into $w-1$ chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$. As an invariant property in the recursive definition of $G\left(P_{w},<_{w}\right)$ we now assume that $G\left(P_{w-1},<_{w-1}\right)$ admits an MRU extension $L_{w-1}$ resulting in a rainbow of size $(w-1)^{2}+1$ for the edges of $G\left(P_{w-1},<_{w-1}\right)$, in which for every $1 \leq i<w$ the first vertex of $\mathcal{C}_{i}$ appears before the first vertex of $\mathcal{C}_{i+1}$ in $L_{w-1}$, while the last vertex of $\mathcal{C}_{i}$ appears after the last vertex of $\mathcal{C}_{i+1}$ in $L_{w-1}$. Note that this property is stronger than the corresponding one we imposed for $G\left(P_{w},<_{w}\right)$. The base graph $G\left(P_{2},<_{2}\right)$ is exactly the same as the one in the proof of Theorem 6 , and it is not difficult to see that $v_{1} \prec \ldots \prec v_{5}$ is an MRU extension of it satisfying also the stronger invariant property.

The first step in the construction of graph $G\left(P_{w},<_{w}\right)$ based on $G\left(P_{w-1},<_{w-1}\right)$ is exactly the same as in the proof of Theorem 6 but without the edge $\left(v_{1,1}, v_{w, w}\right)$, which is now replaced by $\left(\bar{v}_{1,2}, v_{w, w}\right)$; see Fig. 5. In a second step, we introduce a vertex $v_{w, 0}$ being the first vertex in the path formed by the vertices of chain $\mathcal{C}_{w}$. This vertex is also connected to $v_{i, 2}$ for all $1 \leq i \leq w-1$. Finally, we add the following edges to $G\left(P_{w},<_{w}\right)$, namely, for all $1 \leq i<j \leq w-1$, we connect $v_{i, 3}$ to $v_{j, 4}$. Note that $G\left(P_{w},<_{w}\right)$ is acyclic and transitively reduced as desired, while its width is $w$. We construct an appropriate linear extension $L_{w}$ of it as follows:


Fig. 5: Illustration for Theorem 7; q denotes the number of vertices of $G\left(P_{w-1},<_{w-1}\right)$, that is, $q=3(w-1)^{2}-7$.

$$
\begin{array}{r}
{\left[v_{1,1}, \ldots, v_{w-1,1}, v_{w, 0}, v_{w, 1}, \ldots, v_{w, w-1}, v_{w-1,2}, v_{w-2,2}, \ldots v_{1,2}, \bar{v}_{1,2}, L_{w-1}\right.} \\
\left.\bar{v}_{w-1,3}, v_{w-1,3}, v_{w-2,3}, \ldots, v_{1,3}, v_{1,4}, \ldots v_{w-1,4}, v_{w, w+1}, v_{w, 2 w-2}\right]
\end{array}
$$

It can be easily checked that $L_{w}$ is an MRU extension of $G\left(P_{w},<_{w}\right)$, under strong invariant property. In particular, at vertex $\bar{v}_{1,2}$ of the aforementioned extension chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w}$ are in this order from the most recent to the least recent one. By the invariant property, at vertex $\bar{v}_{w-1,3}$ chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w}$ are in the reverse order, that is, from the least recent to the most recent one. Since for the first vertices of every chain in $L_{w}$ it holds $v_{1,1} \prec \ldots \prec v_{w-1,1} \prec v_{w, 0}$, while for the corresponding last vertices it holds $v_{1,4} \prec \cdots \prec v_{w-1,4} \prec v_{w, 2 w-2}$, the strong invariant property is maintained in $L_{w}$.

We complete the proof by observing that the $w-1$ edges stemming from the first $w-1$ vertices of $\mathcal{C}_{w}$ towards the last vertices of the chains $\mathcal{C}_{1}, \ldots, \mathcal{C}_{w-1}$ and the $w-2$ edges stemming from the first $w-2$ vertices of $\mathcal{C}_{2}, \ldots, \mathcal{C}_{w-1}$ towards the last $w-2$ vertices of chain $\mathcal{C}_{w}$ form a rainbow of size $2 w-3$ in $L_{w}$ (refer to the orange edges in Fig. 5), which nests the rainbow of size $(w-2)^{2}+1$ of $L_{w-1}$. Hence, we identified a rainbow of total size $(w-1)^{2}+1$ in $L_{w}$, as desired.

## C A Note on the Upper Bound of Knauer et al. [15]

Here we discuss a problem in the approach of Knauer et al. [15] to derive the upper bound of $w^{2}-2\lfloor w / 2\rfloor$ on the queue number of posets of width $w$. Knauer et al. used a simple form of the lazy linear extension that we discuss in Section 3 to prove that the queue number of a poset of width 2 is at most 2 . Using the result, they derived the bound of $w^{2}-2\lfloor w / 2\rfloor$ on the queue number of a poset


Fig. 6: Illustration a poset of width 4 together with a chain partition $\mathcal{C}_{1}, \ldots, \mathcal{C}_{4}$.
$\langle P,<\rangle$ of width $w$ by pairing up chains of the chain partition of $\langle P,<\rangle$. The pairing yields $\lfloor w / 2\rfloor$ pairs, each of which induces a poset of width 2 , and thus, admits a lazy linear extension with the maximum rainbow of size 2 .

The critical step is to combine the linear extensions of the pairs to a linear extension of the original poset by "respecting all these partial linear extensions", as stated in [15]. The step is problematic even for $w=4$. To see this, consider the poset illustrated in Fig. 6 through its cover graph. This poset has width 4 and $\mathcal{C}_{1}, \ldots, \mathcal{C}_{4}$ is a chain partition. It is not difficult to see that the poset induced by $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ admits the following lazy linear extension:

$$
L_{1}: v_{2} \prec v_{6} \prec v_{1} \prec v_{5} .
$$

The poset induced by $\mathcal{C}_{3}$ and $\mathcal{C}_{4}$ admits the following lazy linear extension:

$$
L_{2}: v_{3} \prec v_{4} \prec v_{8} \prec v_{7} .
$$

According to [15], the two linear extensions, $L_{1}$ and $L_{2}$, are combined into a linear extension $L$ of the original poset. In particular, the following holds in $L$ :
$-v_{1} \prec v_{8}$, due to edge ( $v_{1}, v_{8}$ ),

- $v_{8} \prec v_{7}$, since this holds in $L_{2}$,
- $v_{7} \prec v_{6}$, due to edge ( $v_{7}, v_{6}$ ).

By transitivity, it follows that $v_{1} \prec v_{6}$ in $L$. However, $v_{6} \prec v_{1}$ in $L_{1}$, a contradiction.

We conclude that a crucial argument is missing in [15]. It is not clear how to avoid such a problem for an approach in which two linear extensions are combined into a single one. It is tempting to argue about specific lazy linear extensions (such as MRU), but unfortunately those are identical for width-2 posets.

## D Details on the Counterexample to Conjecture 1

In this section, we give the details of the proofs of Theorems 3 and 4. Recall the definitions of cover graphs $G(p, q)$ and $\widetilde{G}(p, q)$ from Section 5 . It is easy to verify that both $G(p, q)$ and $\widetilde{G}(p, q)$ are transitively reduced, acyclic and of width 3. For $i=1, \ldots, q-3$, we denote by $T_{a}(i)$ the subgraph of $G(p, q)$ induced by
the vertices $a_{i}, \ldots, a_{i+6}$ and the vertex $c_{i+3}$. Accordingly, $T_{c}(i)$ is the subgraph of $G(p, q)$ induced by the vertices $c_{i}, \ldots, c_{i+6}$ and the vertex $a_{i+3}$; see Fig. 7b. We further denote by $X_{a}(i)$ the subgraph of $G(p, q)$ induced by the vertices $a_{i+1}, \ldots, a_{i+4}, c_{i}, \ldots, c_{i+5}$ and symmetrically by $X_{c}(i)$ the subgraph of $G(p, q)$ induced by the vertices $a_{i}, \ldots, a_{i+5}, c_{i+1}, \ldots, c_{i+4}$; see Fig. 7c.

(a) $\widetilde{G}(p, q) ; p=16$ and $q=11$.

(b) $T_{a}(i)$

(c) $X_{a}(i)$

Fig. 7: Illustration of graph $\widetilde{G}(p, q)$ and its subgraphs $T_{a}(i)$ and $X_{a}(i)$.

The following lemma guarantees the existence of a 3 -rainbow, when there exists an edge, say $(u, v)$, that "nests" $T_{a}(i)$ in a linear extension of $G(p, q)$, that is, when $u \prec a_{i}<\cdots \prec a_{i+6} \prec v$. We denote this configuration by $\left[u, T_{a}(i), v\right]$.

Lemma 10. In every linear extension of $G(p, q)$, each of $T_{a}(i)$ and $T_{c}(i)$ requires 2 queues for all $i=1, \ldots, q-3$.

Proof. We give a proof only for $T_{a}(i)$, as the case with $T_{c}(i)$ is symmetric. Let $L$ be a linear extension of $G(p, q)$. Since $\left(a_{i}, c_{i+3}\right)$ and $\left(c_{i+3}, a_{i+6}\right)$ are edges of $G(p, q), a_{i} \prec c_{i+3} \prec a_{i+6}$ holds in L.If $a_{i+3} \prec c_{i+3}$, then $\left[a_{i} \ldots a_{i+2} \ldots a_{i+3} \ldots\right.$ $c_{i+3}$ ] holds in $L$ and thus $\left(a_{i}, c_{i+3}\right)$ and $\left(a_{i+2}, a_{i+3}\right)$ form a 2-rainbow. Otherwise, $\left[c_{i+3} \ldots a_{i+3} \ldots a_{i+4} \ldots a_{i+6}\right]$ holds and thus $\left(c_{i+3}, a_{i+6}\right)$ and $\left(a_{i+3}, a_{i+4}\right)$ form a 2-rainbow.

The next lemma establishes some properties of $X_{a}(i)$.
Lemma 11. In every linear extension of $G(p, q)$, in which one of the following holds, $X_{a}(i)$ requires 3 queues:
(i) $a_{i+1} \prec c_{i+1} \prec a_{i+2} \prec c_{i+2}$,
(ii) $c_{i+1} \prec a_{i+1} \prec c_{i+2} \prec a_{i+2}$,
(iii) $a_{i+3} \prec c_{i+3} \prec a_{i+4} \prec c_{i+4}$,
(iv) $c_{i+3} \prec a_{i+3} \prec c_{i+4} \prec a_{i+4}$,
(v) $c_{i} \prec a_{i+1} \prec c_{i+2} \prec a_{i+3} \prec c_{i+4}$.

Proof. Let $L$ be a linear extension of $G(p, q)$ satisfying one of (i)-(iv). We consider each of the cases of the proposition separately in the following.
(i) Assume $a_{i+1} \prec c_{i+1} \prec a_{i+2} \prec c_{i+2}$. Since $c_{i+2} \prec c_{i+3} \prec c_{i+4}$, if $c_{i+3} \prec a_{i+3}$, then the edges $\left(c_{i+1}, a_{i+4}\right),\left(a_{i+2}, a_{i+3}\right)$ and $\left(c_{i+2}, c_{i+3}\right)$ form a 3-rainbow, since $\left[c_{i+1} \ldots a_{i+2} \ldots c_{i+2} \ldots c_{i+3} \ldots a_{i+3} \ldots a_{i+4}\right]$ holds in $L$. Hence, we may assume that $a_{i+3} \prec c_{i+3}$ holds in $L$. We distinguish two cases depending on whether $a_{i+3} \prec c_{i+2}$ or $c_{i+2} \prec a_{i+3}$. In the former case, the edges $\left(a_{i+1}, c_{i+4}\right),\left(c_{i+1}, c_{i+2}\right)$ and $\left(a_{i+2}, a_{i+3}\right)$ form a 3 -rainbow, since $\left[a_{i+1} \ldots c_{i+1} \ldots a_{i+2} \ldots a_{i+3} \ldots c_{i+2} \ldots c_{i+4}\right]$ holds in $L$. In the latter case, in which $c_{i+2} \prec a_{i+3}$, the relative order in $L$ is $\left[a_{i+1} \ldots c_{i+1} \ldots a_{i+2} \ldots c_{i+2} \ldots\right.$ $a_{i+3} \ldots c_{i+3}$. Since $c_{i+3} \prec c_{i+4} \prec c_{i+5}$, we distinguish possible positions for $a_{i+4}$

- If $a_{i+3} \prec a_{i+4} \prec c_{i+3}$, then $\left(a_{i+2}, c_{i+5}\right),\left(c_{i+2}, c_{i+3}\right)$ and $\left(a_{i+3}, a_{i+4}\right)$ form a 3-rainbow, since $\left[a_{i+2} \ldots c_{i+2} \ldots a_{i+3} \ldots a_{i+4} \ldots c_{i+3} \ldots c_{i+5}\right]$ holds in $L$.
- If $c_{i+3} \prec a_{i+4} \prec c_{i+4}$, then $\left(a_{i+1}, c_{i+4}\right),\left(c_{i+1}, a_{i+4}\right)$ and $\left(c_{i+2}, c_{i+3}\right)$ form a 3-rainbow, since $\left[a_{i+1} \ldots c_{i+1} \ldots c_{i+2} \ldots c_{i+3} \ldots a_{i+4} \ldots c_{i+4}\right]$ holds in $L$.
- If $c_{i+4} \prec a_{i+4} \prec c_{i+5}$, then $\left(a_{i+2}, c_{i+5}\right),\left(a_{i+3}, a_{i+4}\right)$ and $\left(c_{i+3}, c_{i+4}\right)$ form a 3-rainbow, since $\left[a_{i+2} \ldots a_{i+3} \ldots c_{i+3} \ldots c_{i+4} \ldots a_{i+4} \ldots c_{i+5}\right]$ holds in $L$.
- If $c_{i+5} \prec a_{i+4}$, then $\left(c_{i+1}, a_{i+4}\right),\left(a_{i+2}, c_{i+5}\right)$ and $\left(c_{i+2}, c_{i+3}\right)$ form a 3-rainbow, since $\left[c_{i+1} \ldots a_{i+2} \ldots c_{i+2} \ldots c_{i+3} \ldots c_{i+5} \ldots a_{i+4}\right]$ holds in $L$.
(ii) Assume $c_{i+1} \prec a_{i+1} \prec c_{i+2} \prec a_{i+2}$. If $a_{i+3} \prec c_{i+3}$, then $\left(a_{i+1}, c_{i+4}\right)$, $\left(c_{i+2}, c_{i+3}\right)$ and $\left(a_{i+2}, a_{i+3}\right)$ form a 3-rainbow, since $\left[a_{i+1} \ldots c_{i+2} \ldots a_{i+2} \ldots\right.$ $\left.a_{i+3} \ldots c_{i+3} \ldots c_{i+4}\right]$ holds in $L$. Hence, we may assume $c_{i+3} \prec a_{i+3}$. On the other hand, if $a_{i+4} \prec c_{i+4}$, then $\left(a_{i+2}, c_{i+5}\right),\left(c_{i+3}, c_{i+4}\right)$ and $\left(a_{i+3}, a_{i+4}\right)$ form a 3 -rainbow, since $\left[a_{i+2} \ldots c_{i+3} \ldots a_{i+3} \ldots a_{i+4} \ldots c_{i+4} \ldots c_{i+5}\right]$ holds in $L$. Hence, we may further assume $c_{i+4} \prec a_{i+4}$, which together with our previous assumption implies that the underlying order in $L$ is $\left[c_{i+1} \ldots a_{i+1} \ldots\right.$ $\left.c_{i+2} \ldots c_{i+3} \ldots c_{i+4} \ldots a_{i+4}\right]$. The case is then concluded by the observation that $\left(c_{i+1}, a_{i+4}\right),\left(a_{i+1}, c_{i+4}\right)$ and $\left(c_{i+2}, c_{i+3}\right)$ form a 3 -rainbow, as desired.
(iii) It can be proved symmetrically to (i).
(iv) It can be proved symmetrically to (ii).
(v) Assume $c_{i} \prec a_{i+1} \prec c_{i+1}$. By Lemma 11.(i), $a_{i+2} \prec c_{i+1}$ or $a_{i+2} \succ c_{i+2}$. In the former case, edges $\left(a_{i+1}, c_{i+4}\right),\left(a_{i+2}, a_{i+3}\right)$ and $\left(c_{i+1}, c_{i+2}\right)$ form a 3-rainbow, since $\left[a_{i+1} \ldots a_{i+2} \ldots c_{i+1} \ldots c_{i+2} \ldots a_{i+3} \ldots c_{i+4}\right]$, holds in $L$ (recall $a_{i+3} \prec c_{i+4}$ ). In the latter case, a 3 -rainbow is formed by the edges $\left(c_{i}, a_{i+3}\right),\left(c_{i+1}, c_{i+2}\right)$ and $\left(a_{i+1}, a_{i+2}\right)$, since $\left[c_{i} \ldots a_{i+1} \ldots c_{i+1} \ldots c_{i+2} \ldots\right.$ $a_{i+2} \ldots a_{i+3}$ ] holds in $L$. Thus, we have $c_{i+1} \prec a_{i+1} \prec c_{i+2}$. Again by Lemma 11.(ii), $a_{i+2} \prec c_{i+2}$, which yields a 3 -rainbow formed by the edges $\left.\left(c_{i}, a_{i+3}\right]\right),\left(a_{i+1}, a_{i+2}\right)$ and $\left(c_{i+1}, c_{i+2}\right)$, since $\left[c_{i}, c_{i+1}, a_{i+1}, a_{i+2}, c_{i+2}, a_{i+3}\right]$ holds in $L$.

The above case analysis completes the proof.
In the following, we prove that for sufficiently large values of $p$ and $q$ graph $\widetilde{G}(p, q)$ does not admit a 3 -queue layout. For a contradiction, assume that $\widetilde{G}(p, q)$


Fig. 8: Illustrations for the proofs of Lemma 12 and Theorem 3.
admits a 3 -queue layout and let $L$ be its linear extension. Intuitively, we distinguish two cases depending on the length of edge $\left(b_{1}, c_{p}\right)$ in $L$. If the edge is "short" (that is, $b_{1}$ is close to $c_{p}$ in $L$ ), then we use Lemma 12 to show the existence of a 4-rainbow. In the opposite case, the edge ( $b_{1}, c_{p}$ ) nests a large subgraph of $\widetilde{G}(p, q)$. By Lemma 11 , the subgraph that is nested requires 3 queues, which together with the long edge $\left(b_{1}, c_{p}\right)$ yields a 4 -rainbow. Both cases contradict the assumption that $\widetilde{G}(p, q)$ admits a 3 -queue layout.

Lemma 12. $G(14,6)$ requires 4 queues in every linear extension with $c_{14} \prec b_{1}$.
Proof. Let $L$ be a linear extension of $G(14,6)$ with $c_{14} \prec b_{1}$; see Fig. 8. Since $c_{14} \prec$ $b_{1},\left[c_{1} \ldots c_{14} \ldots b_{1} \ldots b_{6}\right]$ holds in $L$. Consider vertex $a_{3}$. Since $\left(a_{3}, c_{6}\right)$ belongs to $G(14,6), a_{3} \prec c_{6}$. If $a_{3} \prec c_{2}$, then configuration $\left[a_{3}, c_{2}, T_{c}(3), b_{2}, b_{3}\right.$ ] follows; see Fig. 8a. In other words, $T_{c}(3)$ induced by the vertices $c_{3}, \ldots, c_{9}$ and $a_{6}$ is nested by two independent edges, which yields a 4 -rainbow by Lemma 10. Similarly, if $c_{4} \prec a_{3} \prec c_{6}$ then we have a 4 -rainbow by the configuration $\left[c_{4}, a_{3}, T_{c}(6), b_{3}, b_{4}\right]$; see Fig. 8b. Hence, only the case $c_{2} \prec a_{3} \prec c_{4}$ is left to be considered. Now consider vertex $a_{5}$. Since $\left(c_{2}, a_{5}\right)$ and $\left(a_{5}, c_{8}\right)$ belong to $G(14,6), c_{2} \prec a_{5} \prec c_{8}$. If $c_{2} \prec a_{5} \prec c_{4}$, then we have $\left[a_{5}, c_{4}, T_{c}(5), b_{4}, b_{5}\right]$; see Fig. 8c. If $c_{6} \prec a_{5} \prec c_{8}$, then we have $\left[c_{6}, a_{5}, T_{c}(8), b_{5}, b_{6}\right]$; see Fig. 8d. In both cases, a 4-rainbow is implied. Hence, only the case $c_{4} \prec a_{5} \prec c_{6}$ is left to be considered. This case together with the leftover case $c_{2} \prec a_{3} \prec c_{4}$ from above implies that Condition (v) of Lemma 11 is fulfilled for $X_{a}(2)$; see Fig. 8 e . But in this case configuration $\left[c_{1}, X_{a}(2), b_{1}\right]$ yields a 4 -rainbow, as desired.

Similarly, we prove the following property of $G(6,2)$.
Lemma 13. $G(6,2)$ requires 3 queues in every linear extension.
Proof. Assume to the contrary that $G(6,2)$ admits a queue layout with at most 2 queues and let $L$ be its liner extension. We distinguish the cases based on the relative order of $a_{2}$ with respect to $c_{1}, \ldots, c_{6}$. Since the roles of $a$ 's and $c$ 's in $G(6,2)$ are interchangeable, we can w.l.o.g. assume that $c_{2} \prec a_{2}$; hence, $c_{2} \prec a_{2} \prec c_{5}$.


Fig. 9: Illustration of graph $G(6,2)$ of Lemma 13.
(i) Consider first the case, in which $c_{2} \prec a_{2} \prec c_{3}$. It follows from Lemma 11.(ii) that $a_{3} \prec c_{3}$. Hence, $c_{2} \prec a_{1}$, as otherwise the edges $\left(a_{1}, c_{4}\right),\left(c_{2}, c_{3}\right)$ and $\left(a_{2}, a_{3}\right)$ form a 3 -rainbow, since $\left[a_{1} \ldots c_{2} \ldots a_{2} \ldots a_{3} \ldots c_{3} \ldots c_{4}\right]$ holds in $L$. Similarly, if $b_{2} \prec a_{4}$, then the edges $\left(c_{1}, a_{4}\right),\left(c_{2}, b_{2}\right)$ and $\left(a_{1}, a_{2}\right)$ form a 3 -rainbow, since $\left[c_{1} \ldots c_{2} \ldots a_{1} \ldots a_{2} \ldots b_{2} \ldots a_{4}\right]$ holds in $L$. Thus, $a_{4} \prec b_{2}$. Now, if $b_{2} \prec c_{4}$, then the edges $\left(a_{1}, c_{4}\right),\left(a_{2}, b_{2}\right)$ and $\left(a_{3}, a_{4}\right)$ form a 3-rainbow, since $\left[a_{1} \ldots a_{2} \ldots a_{3} \ldots a_{4} \ldots b_{2} \ldots c_{4}\right]$ holds in $L$; otherwise, [ $c_{2} \ldots a_{1} \ldots a_{2} \ldots a_{3} \ldots c_{4} \ldots b_{2}$ ] holds in $L$, which implies that the edges $\left(c_{2}, b_{2}\right),\left(a_{1}, c_{4}\right)$ and $\left(a_{2}, a_{3}\right)$ from a 3 -rainbow.
(ii) Consider now the case, in which $c_{3} \prec a_{2} \prec c_{4}$. In particular, consider the placement of $b_{2}$ :
(a) if $a_{2} \prec b_{2} \prec a_{4}$ then $b_{2} \prec c_{4}$ (otherwise $\left[c_{1} \ldots c_{2} \ldots c_{3} \ldots c_{4} \ldots b_{2} \ldots a_{4}\right.$ ] yields 3 -rainbow) and $a_{4} \prec c_{4}$ (otherwise $\left[c_{1} \ldots c_{3} \ldots a_{2} \ldots b_{2} \ldots c_{4} \ldots a_{4}\right.$ ] also yields 3 -rainbow). Hence, the relative order is $\left[c_{1} \ldots c_{2} \ldots c_{3} \ldots a_{2} \ldots\right.$ $\left.b_{2} \ldots a_{4} \ldots c_{4}\right]$. Consider the placement of $a_{1}$ in this relative order. If $a_{1} \prec c_{1}$, then the edges $\left(a_{1}, c_{4}\right),\left(c_{1}, a_{4}\right)$, and $\left(c_{2}, b_{2}\right)$ form a 3rainbow, since $\left[a_{1} \ldots c_{1} \ldots c_{2} \ldots b_{2} \ldots a_{4} \ldots c_{4}\right]$ holds in $L$; if $c_{1} \prec a_{1} \prec$ $c_{2}$, then the edges $\left(c_{1}, a_{4}\right),\left(a_{1}, a_{2}\right)$, and $\left(c_{2}, c_{3}\right)$ form a 3-rainbow, since $\left[c_{1} \ldots a_{1} \ldots c_{2} \ldots c_{3} \ldots a_{2} \ldots a_{4}\right]$ holds in $L$; finally, if $c_{2} \prec a_{1}$, then the edges $\left(c_{1}, a_{4}\right),\left(c_{2}, b_{2}\right)$, and $\left(a_{1}, a_{2}\right)$ form a 3 -rainbow, since $\left[c_{1} \ldots c_{2} \ldots a_{1} \ldots a_{2} \ldots b_{2} \ldots a_{4}\right]$ holds in $L$.
(b) if $a_{4} \prec b_{2} \prec a_{6}$, then the edges $\left(c_{3}, a_{6}\right),\left(a_{2}, b_{2}\right),\left(a_{3}, a_{4}\right)$ form a 3rainbow, since $\left[c_{3} \ldots a_{2} \ldots a_{3} \ldots a_{4} \ldots b_{2} \ldots a_{6}\right]$ holds in $L$;
(c) if $a_{6} \prec b_{2}$, then the edges $\left(c_{2}, b_{2}\right),\left(c_{3}, a_{6}\right),\left(a_{3}, a_{4}\right)$ form a 3-rainbow, since $\left[c_{2} \ldots c_{3} \ldots a_{3} \ldots a_{4} \ldots a_{6} \ldots b_{2}\right]$ holds in $L$.
(iii) Finally, consider the case, in which $c_{4} \prec a_{2} \prec c_{5}$. As above, consider the placement of $b_{2}$ :
(a) if $a_{2} \prec b_{2} \prec a_{4}$, then the edges $\left(c_{1}, a_{4}\right),\left(c_{2}, b_{2}\right)$, and $\left(c_{3}, c_{4}\right)$ form a 3-rainbow, since $\left[c_{1} \ldots c_{2} \ldots c_{3} \ldots c_{4} \ldots b_{2} \ldots a_{4}\right]$ holds in $L$;
(b) if $a_{4} \prec b_{2} \prec a_{6}$, then the edges $\left(c_{3}, a_{6}\right)$, $\left(a_{2}, b_{2}\right)$, and $\left(a_{3}, 4\right)$ form a 3 -rainbow, since $\left[c_{3} \ldots a_{2} \ldots a_{3} \ldots a_{4} \ldots b_{2} \ldots a_{6}\right]$ holds in $L$;
(c) if $a_{6} \prec b_{2}$, then the edges $\left(c_{2}, b_{2}\right),\left(c_{3}, a_{6}\right)$, and $\left(a_{2}, a_{3}\right)$ form a 3 -rainbow, since $\left[c_{2} \ldots c_{3} \ldots a_{2} \ldots a_{3} \ldots a_{6} \ldots b_{2}\right]$ holds in $L$.

Since all the cases above yield a 3-rainbow, we obtain a contradiction to the assumption that $G(6,2)$ admits a queue layout with at most 2 queues.

We are now ready to show that $G(p, q)$ with $p=31$ and $q=22$ is a counterexample to Conjecture 1 when $w=3$.

Theorem 3. $\widetilde{G}(31,22)$ requires 4 queues in every linear extension.
Proof. Assume for a contradiction that $\widetilde{G}(31,22)$ admits a 3 -queue layout and let $L$ be its linear extension. If $c_{14} \prec b_{1}$ in $L$, then the subgraph of $\widetilde{G}(31,22)$ induced by vertices $a_{1}, \ldots, a_{14}, c_{1}, \ldots, c_{14}, b_{1}, \ldots, b_{6}$ is isomorphic to $G(14,6)$ and by Lemma 12 requires 4 queues; a contradiction. Hence, $b_{1} \prec c_{14}$ holds in $L$.

Symmetric as above, if $c_{31} \prec b_{18}$, the subgraph of $\widetilde{G}(31,22)$ induced by vertices $a_{17}, \ldots, a_{30}, c_{17}, \ldots, c_{30}, b_{17}, \ldots, b_{22}$ is isomorphic to $G(14,6)$ and by Lemma 12 requires 4 queues; a contradiction. Hence, $b_{18} \prec c_{31}$ holds in $L$.

Consider the subgraph of $\widetilde{G}(31,22)$ induced by vertices $a_{17}, \ldots, a_{22}, c_{17}, \ldots, c_{22}$, $b_{17}, b_{18}$, which is isomorphic to $G(6,2)$; see Fig. 8f. We show that $b_{1}$ precedes all the vertices of this subgraph, while all the vertices of this subgraph precede $c_{31}$. Since $\left(b_{1}, c_{31}\right)$ is an edge of $\widetilde{G}(31,22)$, by Lemma 13 we derive a contradiction. In particular, $b_{1} \prec a_{17}$ (since $b_{1} \prec c_{14}$ and $\left(c_{14}, a_{17}\right)$ is an edge of $\widetilde{G}(31,22)$ ), $b_{1} \prec c_{17}$ (since $b_{1} \prec c_{14}$ ), and clearly $b_{1} \prec b_{17}$. Similarly, $a_{22} \prec c_{31}$ (since $\left(a_{22}, c_{25}\right.$ ) is an edge of $\widetilde{G}(31,22)$ and $\left.c_{25} \prec c_{31}\right), c_{22} \prec c_{31}, b_{18} \prec c_{31}$.

To prove that Conjecture 1 does not hold for $w>3$, we need an auxiliary lemma, which is implicitly used in [15].

Lemma 14. Let $\left\langle P_{w},<\right\rangle$ be a width-w poset with queue number at least $k$. Then, there exists a poset, $\left\langle P_{w+1},\left\langle^{\prime}\right\rangle\right.$, of width $w+1$ whose queue number is at least $k+1$.

Proof. Let $G\left(P_{w},<_{w}\right)$ be the cover graph of $\left\langle P_{w},<\right\rangle$. The cover graph $G\left(P_{w+1},<^{\prime}\right)$ of $\left\langle P_{w+1},\left\langle^{\prime}\right\rangle\right.$ is constructed from two copies of $G\left(P_{w},<_{w}\right)$ and three new vertices, $s, t$, and $v$. Namely, let $G_{1}$ and $G_{2}$ be two copies of $G\left(P_{w},<_{w}\right)$. We first add directed edges from the sinks of $G_{1}$ to the sources of $G_{2}$ which ensures that in any linear extension of $G\left(P_{w+1},<^{\prime}\right)$, all vertices of $G_{1}$ precede those of $G_{2}$. Afterwards, we connect vertex $s$ to all sources, and vertex $t$ to all sinks. Observe that the former belong to $G_{1}$, while the latter belong to $G_{2}$. Finally, we add two directed edges $(s, v)$ and $(v, t)$. By construction, $s$ is a global source, and $t$ is a global sink in $G\left(P_{w+1},<^{\prime}\right)$. It is not difficult to see that $G\left(P_{w+1},<^{\prime}\right)$ is a poset. Since $v$ is incomparable to all vertices defining the width of $G\left(P_{w},<_{w}\right)$ in both $G_{1}$ and $G_{2}$, poset $\left\langle P_{w+1},<^{\prime}\right\rangle$ has width $w+1$. As already observed, in any linear extension of $G\left(P_{w+1},<\right)$ all vertices of $G_{1}$ must precede all vertices of $G_{2}$. This implies that either edge $(s, v)$ nests all edges of $G_{1}$ or edge $(v, t)$ nests all edges of $G_{2}$. Thus, the queue number of $\left\langle P_{w+1},\left\langle^{\prime}\right\rangle\right.$ is at least $k+1$.

Theorem 3 and Lemma 14 imply the following:
Theorem 4. For every $w \geq 3$, there is a width-w poset with queue number $w+1$.


[^0]:    * Knauer et al. [15] also claim to reduce the queue number of posets of width $w$ from $w^{2}$ to $w^{2}-2\lfloor w / 2\rfloor$. However, as we discuss in Appendix C, their argument is incomplete.

