Morphological adjunctions represented by matrices in max-plus algebra for signal and image processing

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Abstract. In discrete signal and image processing, many dilations and erosions can be written as the max-plus and min-plus product of a matrix on a vector. Previous studies considered operators on symmetrical, unbounded complete lattices, such as Cartesian powers of the completed real line. This paper focuses on adjunctions on closed hypercubes, which are the complete lattices used in practice to represent digital signals and images. We show that this constrains the representing matrices to be doubly-0-astic and we characterise the adjunctions that can be represented by them. A graph interpretation of the defined operators naturally arises from the adjacency relationship encoded by the matrices, as well as a max-plus spectral interpretation.

Keywords: Morphological operators · Max-plus algebra · Graph theory.

1 Introduction

Like linear filters can be represented by matrices in discrete image and signal processing, many morphological dilations and erosions can be seen as applying a matrix product to a vector, but in the minimax algebra. This is in particular the case for those defined with structuring functions, either flat or not, local or non-local [13,14], translation invariant or spatially variant [4,7,10,16]. They are commonly known to be the vertical-shift-invariant dilations and erosions [9,11]. While the matrix point of view is not the most appropriate for the implementation of these operators, especially translation-invariant ones, it is a valuable insight for their theoretical understanding. In particular, it can help predict and control complex behaviours such as those of iterated operators based on adjunctions with non-flat, spatially variant and input-adapted structuring functions [2,3]. Indeed, it is a flexible and general framework which embraces a very broad part of morphological literature, and it is supported by the rich theory of Minimax algebra [1,6].

In the abundant literature on spatially-variant morphological image processing, only a few approaches explicitly used the matrix formulation [2,3,13,14], whereas most contributions were limited to flat structuring elements and focused on the local effects of the adaptive strategy. On the theoretical side, the representation of morphological adjunctions by matrices was studied in a setting that does not directly apply to digital signal and image processing, as the co-domain is usually an unbounded lattice, stable under any vertical translation [11]. Although a method was proposed to convert these adjunctions to new ones on bounded lattices [12], it is not practical and does not allow for the interpretations that are exposed here.

In the present paper, we focus on complete lattices of the type $[a, b]^n$, where a and b represent the minimal and maximal possible signal values (typically $a = 0$ and $b = 255$ for 8-bits images), and n is an integer representing the size of the signal (typically, the number of pixels of an image, reshaped as a column vector). This is a theoretical contribution that can be viewed as a companion paper to previous studies where this framework has been successfully applied to adaptive anisotropic filtering [2,3][3]. In Section 2 we introduce the matrix-based morphological setting and prove simple but fundamental results: in particular, we characterise the adjunctions that can be represented by matrices and show that these matrices need to be doubly-0-astic. By viewing matrices as encoding adjacency, we provide in Section 3 a graph interpretation of iterated operators and their associated granulometries. In Section 4 we draw a link between these operators and some results on the spectrum of matrices in the max-plus algebra, before concluding in Section 5.

2 Matrix-based morphological adjunctions

2.1 Notations

In this paper matrices will be denoted by capital letters, such as W, and their i-th row and j-th column coefficients by corresponding indexed lowercase letters w_{ij}. Similarly, vectors are written as boldface lowercase letters, such as $\mathbf{x}$, and their i-th component as x_i. Let $0 \leq a < b \in \mathbb{R}^+$ be two non-negative real numbers, $n \in \mathbb{N}^*$ a positive integer. The set $\{1, \dots, n\}$ will be denoted by $[\![1, n]\!]$. Let $\mathcal{L} = ([a, b]^n, \leq)$ be the complete lattice equipped with the usual product partial ordering (Pareto ordering): $\mathbf{x} \leq \mathbf{y} \iff x_i \leq y_i, \quad \forall i \in [\![1, n]\!]$. The supremum and infimum on $\mathcal{L}$ are induced by the Pareto ordering: for a family $(\mathbf{x}^{(k)})_{k \in K}$ of $\mathcal{L}$, $\bigvee_{k \in K} \mathbf{x}^{(k)}$ is the vector $\mathbf{y}$ defined by $y_i = \bigvee_{k \in K} x_i^{(k)}$, where K is any index set. Therefore $\mathbf{a} = (a, \dots, a)^T$ and $\mathbf{b} = (b, \dots, b)^T$ are respectively the smallest and largest elements in $\mathcal{L}$. For $\mathbf{x} \in \mathcal{L}$, we note $\mathbf{x}^c \doteq \mathbf{b} - \mathbf{x} + \mathbf{a}$, and for any $i \in \{1, \dots, n\}$, $\mathbf{e^{(i)}}$ is the "impulse" vector in $\mathcal{L}$ such that $e_i^{(i)} = b$ and $e_j^{(i)} = a$ for $j \neq i$.

We note $\mathbb{R}_{\max} \doteq \mathbb{R} \cup \{-\infty\}$, $\mathbb{R}_{\min} \doteq \mathbb{R} \cup \{+\infty\}$ and $\mathcal{M}_n$ the set of $n \times n$ square matrices with coefficients in $\mathbb{R}_{\max}$. Like $(\mathbb{R}_{\max}, \vee, +)$, $(\mathcal{M}_n, \vee, \otimes)$ is an idempotent semiring, with the addition $\vee$ and product $\otimes$ defined as follows. For $A, B \in \mathcal{M}_n$, $A \otimes B$ and $A \vee B$ are the $n \times n$ matrices defined respectively by $(A \otimes B)_{ij} = \bigvee_{k=1}^n a_{ik} + b_{kj}$ and $(A \vee B)_{ij} = a_{ij} \vee b_{ij} = \max(a_{ij}, b_{ij})$, for $1 \leq i, j \leq n$. Similarly, for $\mathbf{x} \in \mathbb{R}_{\max}^n$, $A \otimes \mathbf{x}$ is the vector such that $(A \otimes \mathbf{x})_i =$

[3] An online demo for [3] is available: `https://bit.ly/anisop_demo`.

$\bigvee_{j=1}^{n} a_{ij} + x_j$. Note that $\vee$ and $\otimes$ are associative and $\otimes$ is distributive over $\vee$. Finally, the product of a scalar $\lambda \in \mathbb{R}_{\max}$ by a vector $\mathbf{x} \in \mathbb{R}_{\max}^n$ is $\lambda \otimes \mathbf{x} \doteq \lambda + \mathbf{x}$, the vector in $\mathbb{R}_{\max}$ such that $(\lambda \otimes \mathbf{x})_i = \lambda + x_i$. In [6] and [14], special subsets of $\mathcal{M}_n$ are introduced, that we will show to be essential to represent morphological adjunctions on $\mathcal{L}$.

Definition 1 (0-asticity [6]) *A matrix $W \in \mathcal{M}_n$ is said* **row-0-astic** *if for any $1 \leq i \leq n$, $\bigvee_{j=1}^{n} w_{ij} = 0$. Similarly, it is said* **column-0-astic** *if the supremum of each column is 0, and* **doubly-0-astic** *if the matrix is both row-0-astic and column-0-astic. Finally, W is simply said* **0-astic** *if $\bigvee_{1\leq i,j\leq n} w_{ij} = 0$.*

A special kind of doubly-0-astic matrices are those with zeros on the diagonal and non-positive coefficients elsewhere.

Definition 2 (CMW matrices [14]) *A matrix $W \in \mathcal{M}_n$ is called a Conservative Morphological Weights (CMW) matrix if $\forall i, j \in [\![1, n]\!]$, $w_{ij} \leq 0$ and $w_{ii} = 0$.*

We now introduce the morphological framework on $\mathcal{L}$, based on the max-plus algebra product between matrices and vectors.

2.2 Dilations

For $W \in \mathcal{M}_n$, we consider the function δ_W from $\mathcal{L}$ to $\mathbb{R}_{\max}^n$ such that

$$\forall \mathbf{x} \in \mathcal{L}, \quad \delta_W(\mathbf{x}) = W \otimes \mathbf{x} = \left(\bigvee_{1\leq j\leq n} \{w_{ij} + x_j\} \right)_{1\leq i\leq n}. \tag{1}$$

In the processing of digital data such as images we usually want the input to be comparable with the output. Hence, we will constrain W such that $\delta_W(\mathcal{L}) \subseteq \mathcal{L}$. This has the following consequences:

$$\delta_W(\mathbf{b}) \leq \mathbf{b} \Rightarrow \forall i \in [\![1, n]\!], \;\; b + (\bigvee_{j=1}^{n} w_{ij}) \leq b \Rightarrow \forall i \in [\![1, n]\!], \;\; \bigvee_{j=1}^{n} w_{ij} \leq 0$$

since $b > -\infty$. Similarly, $\delta_W(\mathbf{a}) \geq \mathbf{a} \Rightarrow \forall i \in [\![1, n]\!], \;\; \bigvee_{j=1}^{n} w_{ij} \geq 0$. Hence a necessary condition to have $\delta_W(\mathcal{L}) \subseteq \mathcal{L}$ is that W be row-0-astic (Def. 1). Conversely, the row-0-asticity for W implies that $\delta_W(\mathbf{a}) = \mathbf{a}$ and $\delta_W(\mathbf{b}) = \mathbf{b}$, and therefore that $\delta_W(\mathcal{L}) \subseteq \mathcal{L}$ by increasingness of δ_W. This leads to the following result.

Proposition 1 *Let $W \in \mathcal{M}_n$ and δ_W be the function defined by (1). Then δ_W is a dilation mapping $\mathcal{L}$ to $\mathcal{L}$ if and only if W is row-0-astic.*

Proof. If δ_W is a dilation mapping $\mathcal{L}$ to $\mathcal{L}$, then $\delta_W(\mathcal{L}) \subseteq \mathcal{L}$ which, as we showed, implies that W is row-0-astic. Conversely, we saw that a row-0-astic W implies $\delta_W(\mathcal{L}) \subseteq \mathcal{L}$. Therefore, we only have to verify that δ_W is a dilation, or equivalently that it commutes with the supremum. This is straightforward from the definition of $W \otimes \mathbf{x}$. $\square$

2.3 Erosions and adjunctions

Now we suppose that $W \in \mathcal{M}_n$ is row-0-astic, hence δ_W is a dilation from $\mathcal{L}$ to $\mathcal{L}$, and we are interested in its adjoint erosion α_W defined for any $\mathbf{y} \in \mathcal{L}$ by $\alpha_W(\mathbf{y}) = \bigvee E_{\mathbf{y}}$ where $E_{\mathbf{y}} = \{\mathbf{x} \in \mathcal{L}, \delta_W(\mathbf{x}) \leq \mathbf{y}\}$. Let us denote by ε_W the function from $\mathcal{L}$ to $\mathbb{R}^n_{\min}$ such that for any $\mathbf{y} \in \mathcal{L}$

$$\varepsilon_W(\mathbf{y}) = \big(\delta_{W^T}(\mathbf{y}^c)\big)^c = \big(W^T \otimes \mathbf{y}^c\big)^c = \left(\bigwedge_{1\leq j\leq n} \{y_j - w_{ji}\}\right)_{1\leq i\leq n}. \quad (2)$$

Then we can check that $\forall \mathbf{y} \in \mathcal{L},\ \ \alpha_W(\mathbf{y}) = \varepsilon_W(\mathbf{y}) \wedge \mathbf{b}$. Indeed, from (1) we see that for any $\mathbf{x}, \mathbf{y} \in \mathcal{L}$, $\delta_W(\mathbf{x}) \leq \mathbf{y} \iff \mathbf{x} \leq \varepsilon_W(\mathbf{y})$. Therefore, since $\delta_W(\mathbf{a}) = \mathbf{a} \leq \mathbf{y}$ we get $\mathbf{a} \leq \varepsilon_W(\mathbf{y})$, which implies $\varepsilon_W(\mathbf{y}) \wedge \mathbf{b} \in \mathcal{L}$; furthermore, $\varepsilon_W(\mathbf{y}) \wedge \mathbf{b} \leq \varepsilon_W(\mathbf{y})$ so $\varepsilon_W(\mathbf{y}) \wedge \mathbf{b} \in E_{\mathbf{y}}$; finally, as both $\varepsilon_W(\mathbf{y})$ and $\mathbf{b}$ are upper-bounds of $E_{\mathbf{y}}$, so is $\varepsilon_W(\mathbf{y}) \wedge \mathbf{b}$. Hence, $\varepsilon_W(\mathbf{y}) \wedge \mathbf{b} = \bigvee E_{\mathbf{y}} = \alpha_W(\mathbf{y})$. By a similar reasoning as in Section 2.2, we get the following result.

Proposition 2 *Let $W \in \mathcal{M}_n$ and ε_W be the function defined by (2). Then ε_W is an erosion mapping $\mathcal{L}$ to $\mathcal{L}$ if and only if W is column-0-astic.*

If W is also row-0-astic, then $\varepsilon_W = \alpha_W$ is the adjoint of δ_W, as stated next.

Proposition 3 *Let $W \in \mathcal{M}_n$ and δ_W and ε_W be the functions defined by (1) and (2), respectively. Then $(\varepsilon_W, \delta_W)$ is an adjunction on $\mathcal{L}$ if and only if W is doubly-0-astic. Furthermore, $(\varepsilon_W, \delta_W)$ is an adjunction on $\mathcal{L}$ with δ_W extensive (and ε_W anti-extensive) if and only if W is a CMW matrix.*

Proof. Most of the points have already been addressed above or are straightforward from Proposition 1. To see that δ_W extensive implies $w_{ii} = 0$ for all i, just remark that $w_{ii} < 0$ would imply $\delta_W(\mathbf{e^{(i)}})_i < b = e^{(i)}_i$. □

2.4 Generality of $(\varepsilon_W,\ \delta_W)$

The dilation δ_W, already introduced in [2,11,15], can be viewed as a generalisation of the non-local and adaptive mathematical morphology [13,14] on signals and images. Each column $W_{:,j}$ of W represents the structuring function corresponding to pixel (or instant) j.

As pointed out in [9,11], the dilations that can be written as matrix-based max-plus products like Eq. (1) are the shift (or *vertical-translation*) invariant ones. However the result stated in [9,11] does not directly apply to our setting where the lattice $\mathcal{L}$ is different from the lattice of scalars which define vertical translation of signal values, usually $\mathbb{R} \cup \{-\infty, +\infty\}$. Still, the same idea holds here with some adaptation, as stated in the next proposition.

Proposition 4 *Let $\delta : \mathcal{L} \to \mathcal{L}$ be a dilation. Then there exists $W \in \mathcal{M}_n$ such that $\delta = \delta_W$ if and only if*

$$\forall \lambda \leq 0, \forall \mathbf{x} \in \mathcal{L},\ \ \delta\big((\lambda + \mathbf{x}) \vee \mathbf{a}\big) = \big(\lambda + \delta(\mathbf{x})\big) \vee \mathbf{a}. \quad (3)$$

In that case, the matrix W whose j-th column is $W_{:,j} = \delta(\mathbf{e^{(j)}}) - \mathbf{b}$ for $1 \leq j \leq n$, is such a representing matrix.

We see that this class of dilations is very broad and covers the most commonly used in morphological image and signal processing: dilations based on structuring functions, possibly non-local, varying in space and non-flat.

Proof (Proposition 4). If $\delta = \delta_W$ for some $W \in \mathcal{M}_n$, then it is straightforward to check that δ verifies Eq. (3).

Conversely, suppose δ verifies Eq. (3). Then we first remark that $\delta(\mathbf{b}) = \mathbf{b}$. Indeed, on the one hand, $\delta(\mathbf{a}) = \mathbf{a}$ as $\mathbf{a} = \bigwedge \mathcal{L}$ and δ is a dilation mapping $\mathcal{L}$ to $\mathcal{L}$. On the other hand, $\delta(\mathbf{a}) = \delta\big((a-b) + \mathbf{b}\big) = (a-b) + \delta(\mathbf{b})$ by Eq. (3). Hence $\mathbf{a} = (a-b) + \delta(\mathbf{b})$ which means that $\delta(\mathbf{b}) = \mathbf{b}$.
As a consequence: for any $i \in [\![1,n]\!]$, there is a $j_i \in [\![1,n]\!]$ such that $\delta(\mathbf{e^{(j_i)}})_i = b$. This is simply because $\mathbf{b} = \bigvee_{1\leq j\leq n} \mathbf{e^{(j)}}$ so $\mathbf{b} = \delta\big(\bigvee_{1\leq j\leq n} \mathbf{e^{(j)}}\big) = \bigvee_{1\leq j\leq n} \delta(\mathbf{e^{(j)}})$, which means that, for any i, $b = \bigvee_{1\leq j\leq n} \delta(\mathbf{e^{(j)}})_i$ and finally that $\delta(\mathbf{e^{(j_i)}})_i = b$ for some j_i, as the supremum is reached here.
Now, let $\mathbf{x} \in \mathcal{L}$. Then it can be decomposed as $\mathbf{x} = \bigvee_{1\leq j\leq n} \big[(\lambda_j + \mathbf{e^{(j)}}) \vee \mathbf{a}\big]$ with $\lambda_j = x_j - b \leq 0$. Hence, as δ is a dilation verifying Eq. (3), we get $\delta(\mathbf{x}) = \bigvee_{1\leq j\leq n} \big[\big(\lambda_j + \delta(\mathbf{e^{(j)}})\big) \vee \mathbf{a}\big]$. We now use the result stated just above: for any $i \in [\![1,n]\!]$ there is a $j_i \in [\![1,n]\!]$ such that $\lambda_{j_i} + \delta(\mathbf{e^{(j_i)}})_i = x_{j_i} - b + b = x_{j_i} \geq a$. Therefore, $\bigvee_{1\leq j\leq n} \big[\big(\lambda_j + \delta(\mathbf{e^{(j)}})\big) \vee \mathbf{a}\big] = \bigvee_{1\leq j\leq n} \lambda_j + \delta(\mathbf{e^{(j)}})$ from which we finally get

$$\delta(\mathbf{x}) = \bigvee_{1\leq j\leq n} \lambda_j + \delta(\mathbf{e^{(j)}}) = \bigvee_{1\leq j\leq n} (x_j - b) + \delta(\mathbf{e^{(j)}}) = \bigvee_{1\leq j\leq n} x_j + [\delta(\mathbf{e^{(j)}}) - \mathbf{b}] \quad (4)$$

which is exactly $W \otimes \mathbf{x}$ for W the matrix with columns $W_{:,j} = \delta(\mathbf{e^{(j)}}) - \mathbf{b}$ for $1 \leq j \leq n$. $\square$

Note that the dual of Proposition 4 obviously holds: the erosions $\varepsilon : \mathcal{L} \to \mathcal{L}$ which can be written as ε_W for some $W \in \mathcal{M}_n$ are those for which

$$\forall \lambda \geq 0, \forall \mathbf{x} \in \mathcal{L},\ \ \varepsilon\big((\lambda + \mathbf{x}) \wedge \mathbf{b}\big) = \big(\lambda + \varepsilon(\mathbf{x})\big) \wedge \mathbf{b}. \quad (5)$$

To show this it is sufficient to see that ε verifies (5) if and only if the dilation $\delta = \varepsilon(\cdot^c)^c$ verifies (3), and recall that $\delta_W(\cdot^c)^c = \varepsilon_{W^T}(\cdot)$.

2.5 Equivalent dilations and erosions

In Proposition 4 we exhibited one possible matrix $W \in \mathcal{M}_n$ that represents a dilation, but this matrix is not unique. In this section we characterise the set of such matrices and show that it is a complete lattice.

Since we are interested in adjunctions $(\varepsilon_W, \delta_W)$, following Proposition 3 we focus on the set of matrices in $\mathcal{M}_n$ that are doubly-0-astic, which we denote by $\mathcal{D}_0(n)$. Let the equivalence relation defined for any two matrices $A, B \in \mathcal{D}_0(n)$ by

$$A \sim B \iff \delta_A = \delta_B \iff \forall \mathbf{x} \in \mathcal{L},\ \delta_A(\mathbf{x}) = \delta_B(\mathbf{x}) \quad (6)$$

and note $\mathcal{C}_W = \{M \in \mathcal{D}_0(n),\ M \sim W\}$ the equivalence class of any $W \in \mathcal{D}_0(n)$. We provide an easy characterisation of $\mathcal{C}_W$ that will show useful in numerical computations of the morphological operators defined earlier. For any $u \in \mathbb{R}_{\max}$ let I_u denote the matrix in $\mathcal{M}_n$ whose coefficients are all equal to u. Then two equivalent matrices are characterised as follows.

Proposition 5 *Let $M, W \in \mathcal{D}_0(n)$. Then*

$$M \in \mathcal{C}_W \iff M \vee I_{a-b} = W \vee I_{a-b} \iff \begin{cases} m_{ij} = w_{ij} & \text{if } w_{ij} > a-b \\ m_{ij} \leq a-b & \text{otherwise.} \end{cases} \quad (7)$$

This means that if W has coefficients not larger than $a-b$, these can be set to any value not larger than $a-b$, including $-\infty$, and can therefore be ignored in the computation of $\delta_W(\mathbf{x})$.

Proof (Proposition 5). The second equivalence is just a matter of writing, so we prove the first one. Let us first notice that for any $\mathbf{x} \in \mathcal{L},\ I_{(a-b)} \otimes \mathbf{x} \leq \mathbf{a}$. Therefore $\forall \mathbf{x} \in \mathcal{L},\ \ (W \vee I_{(a-b)}) \otimes \mathbf{x} = (W \otimes \mathbf{x}) \vee (I_{(a-b)} \otimes \mathbf{x}) = W \otimes \mathbf{x}$, since $W \otimes \mathbf{x} \geq \mathbf{a}$, and this holds for M too. Hence, if $M \vee I_{a-b} = W \vee I_{a-b}$, then for any $\mathbf{x} \in \mathcal{L},\ W \otimes \mathbf{x} = (W \vee I_{a-b}) \otimes \mathbf{x} = (M \vee I_{a-b}) \otimes \mathbf{x} = M \otimes \mathbf{x}$, which means $M \in \mathcal{C}_W$.

Conversely, suppose that $M \sim W$ and that $w_{i_0 j_0} > a-b$ for some $i_0, j_0 \in [\![1,n]\!]$. Let $\mathbf{x} = \mathbf{e}^{(\mathbf{j_0})} \in \mathcal{L}$, *i.e.* $x_{j_0} = b$ and $x_j = a \quad \forall j \neq j_0$. The 0-asticity of W and M implies $(W \otimes \mathbf{x})_{i_0} = b + w_{i_0 j_0}$ and $(M \otimes \mathbf{x})_{i_0} = b + m_{i_0 j_0}$, hence $m_{i_0 j_0} = w_{i_0 j_0}$. We have just shown that $\forall i,j \in [\![1,n]\!],\ (w_{ij} > a-b \Rightarrow w_{ij} = m_{ij})$ and by symmetry of the equivalence relation $(m_{ij} > a-b \Rightarrow w_{ij} = m_{ij})$, which combined yields $\max(m_{ij},\ a-b) = \max(w_{ij},\ a-b)$. So finally $M \sim W \Rightarrow M \vee I_{a-b} = W \vee I_{a-b}$. ☐

While it is clear that if $A, B \in \mathcal{C}_W$ then $A \vee B \in \mathcal{C}_W$, the characterisation in Proposition 5 shows that $\mathcal{C}_W$ is also closed under infimum, that is: $A \wedge B \in \mathcal{C}_W$. This has the following straightforward consequence.

Proposition 6 *Let $W \in \mathcal{D}_0(n)$ and $\leq$ the partial ordering on $\mathcal{C}_W$ defined by $A \leq B \iff A \vee B = B \iff a_{ij} \leq b_{ij}\ \forall i,j \in [\![1,n]\!]$. Then*

- *$(\mathcal{C}_W, \leq)$ is a complete lattice (with coefficient-wise supremum and infimum);*
- *Its greatest element is $\overline{W} = W \vee I_{a-b}$;*
- *Its smallest element is $\underline{W}$, defined by $\underline{w}_{ij} = \begin{cases} w_{ij} & \text{if } w_{ij} > a-b \\ -\infty & \text{otherwise.} \end{cases}$*

2.6 Iterated operators and granulometries

In this section, given $W \in \mathcal{D}_0(n)$ and $p \in \mathbb{N}^*$, we focus on the iterated dilations and erosions δ_W^p and ε_W^p, as well as their sup and inf integrations, that we note respectively $D_W^{[p]} \doteq \bigvee_{k=1}^p \delta_W^k$ and $E_W^{[p]} \doteq \bigwedge_{k=1}^p \varepsilon_W^k$. One can easily check that both $(\varepsilon_W^p, \delta_W^p)$ and $(E_W^{[p]}, D_W^{[p]})$ are adjunctions. We note respectively $\gamma_W^{[p]} \doteq \delta_W^p \varepsilon_W^p$ and $G_W^{[p]} \doteq D_W^{[p]} E_W^{[p]}$ their corresponding openings.

Note that if δ_W is extensive, or equivalently if W is a CMW matrix (Prop. 3), then these adjunctions are equal: $(\varepsilon_W^p, \delta_W^p) = (E_W^{[p]}, D_W^{[p]})$. As this is not true in general, both adjunctions are worth studying. In particular, we shall examine whether $(\gamma_W^{[p]})_{p\in\mathbb{N}^*}$ and $(G_W^{[p]})_{p\in\mathbb{N}^*}$ define granulometries, that is to say families of openings that are decreasing with p. The answer is yes and it is a general result that does not depend on the representation of the adjunction.

Proposition 7 *Let $(\varepsilon,\ \delta)$ be an adjunction on a complete lattice. For any integer $p \in \mathbb{N}^*$, let us note $\gamma_p = \delta^p\varepsilon^p$ and $G_p = D_pE_p$ the openings associated to the adjunctions $(\varepsilon^p,\ \delta^p)$ and $\left(E_p = \bigwedge_{1\le k\le p}\varepsilon^k, D_p = \bigvee_{1\le k\le p}\delta^k\right)$, respectively. Then $(\gamma_p)_{p\in\mathbb{N}^*}$ and $(G_p)_{p\in\mathbb{N}^*}$ are granulometries.*

Proof. We first show that the family of openings $(\gamma_p)_{p\ge1}$ decreases with p, hence a granulometry. This is straightforward by writing $\gamma_{p+1} = \delta^{p+1}\varepsilon^{p+1} = \delta^p\gamma_1\varepsilon^p \le \delta^p\varepsilon^p = \gamma_p$. Secondly, regarding $(G_p)_{p\ge1}$, we show $G_{p+1} \le G_p$ by proving that $G_pG_{p+1} = G_{p+1}$. We obtain this by remarking that $D_{p+1} = D_p(id \bigvee \delta)$, which makes it an invariant of G_p: $G_pD_{p+1} = D_pE_pD_p(id\bigvee\delta) = D_p(id\bigvee\delta) = D_{p+1}$. Then we can conclude $G_pG_{p+1} = G_pD_{p+1}E_{p+1} = D_{p+1}E_{p+1} = G_{p+1}$. □

To conclude this section, let us write δ_W^p, ε_W^p, $D_W^{[p]}$ and $E_W^{[p]}$ as dilations and erosions represented by one suitable doubly-0-astic matrix. This will help in their graph interpretation of the next section. The associativity of $\otimes$ yields $\forall \mathbf{x} \in \mathcal{L},\ \delta_W^p(\mathbf{x}) = W \otimes \ldots \otimes W \otimes \mathbf{x} = W^p \otimes \mathbf{x}$, therefore $\delta_W^p = \delta_{W^p}$. We obtain similarly $\varepsilon_W^p = \varepsilon_{W^p}$. The distributivity of $\otimes$ over $\vee$ yields $D_W^{[p]}(\mathbf{x}) = \bigvee_{k=1}^p \delta_W^k(\mathbf{x}) = \bigvee_{k=1}^p (W^k \otimes \mathbf{x}) = (\bigvee_{k=1}^p W^k) \otimes \mathbf{x}$ therefore $D_W^{[p]} = \delta_{S_p(W)}$, with $S_p(W) \doteq \bigvee_{k=1}^p W^k$. Similarly, $E_W^{[p]} = \varepsilon_{S_p(W)}$. Note that by the same arguments and Proposition 3, we get that $\mathcal{D}_0(n)$ is closed under $\otimes$ and $\vee$.

3 Graph interpretations

3.1 Weighted graphs

Let $W \in \mathcal{M}_n$ and $\mathcal{G}(W) = (V, E)$ be a weighted and directed graph containing n vertices whose $n \times n$ adjacency matrix is W, with the convention that $w_{ij} > -\infty$ if and only if $(i, j) \in E$. We now recall that a *path* from vertex i to vertex j in $\mathcal{G}(W)$ is a tuple $\pi = (k_1, \ldots, k_l)$ of vertices such that $k_1 = i$, $k_l = j$, and $(k_p, k_{p+1}) \in E$ for $1 \le p \le l-1$. The *length* of the path, denoted by $\ell(\pi)$, is $l-1$ (the number of its edges). For $p \ge 1$, $\Gamma_{ij}^{(p)}(W)$ denotes the set of paths from i to j in $\mathcal{G}(W)$ of length p and $\Gamma_{ij}^{(\infty)}(W)$ the set of all paths from i to j. The *weight of a path* $\pi = (k_1, \ldots, k_l)$, denoted by $\omega(\pi)$, is the sum $\omega(\pi) = \sum_{p=1}^{l-1} w_{k_pk_{p+1}}$.

3.2 Iterated operators

Recall that for $W \in \mathcal{M}_n$ and $p \in \mathbb{N}^*$, W^p is the p-th power of W in the $\otimes$ sense, and $S_p(W)$ is the matrix defined in Section 2.6, denoted by S_p here for simplicity. We note respectively $w_{ij}^{(p)}$ and $s_{ij}^{[p]}$ their coefficients. The following result is well known in tropical algebra and graph theory [5,6], and will help interpret the operators defined earlier. It can be proved by induction.

Proposition 8 *Let $W \in \mathcal{M}_n$ and $p \in \mathbb{N}^*$. Then for any $1 \leq i, j \leq n$,*

$$w_{ij}^{(p)} = \max\left\{\omega(\pi),\ \pi \in \Gamma_{ij}^{(p)}(W)\right\} \ \textit{and}\ s_{ij}^{[p]} = \max\left\{\omega(\pi),\ \pi \in \bigcup_{1\leq k\leq p} \Gamma_{ij}^{(k)}(W)\right\} \tag{8}$$

with the convention $\max(\emptyset) = -\infty$.

The equations in (8) are equivalent to saying that

1. $w_{ij}^{(p)} > -\infty$ (resp. $s_{ij}^{[p]} > -\infty$) if and only if there is at least a path in $\mathcal{G}(W)$ from vertex i to vertex j of length exactly (resp. at most) p;
2. $w_{ij}^{(p)}$ (resp. $s_{ij}^{[p]}$) is the maximal weight over the set of paths from vertex i to vertex j of length exactly (resp. at most) p.

Therefore the graphs $\mathcal{G}(W^p)$ and $\mathcal{G}(S_p)$ have the same set of vertices as the original graph $\mathcal{G}(W)$, but an edge exists between vertices i and j in $\mathcal{G}(W^p)$ (resp. $\mathcal{G}(S_p)$) whenever there is a path of length exactly (resp. at most) p from i to j in $\mathcal{G}(W)$. The weights associated with this new edge are the maximal weights over the corresponding set of paths.

Now if $W \in \mathcal{D}_0(n)$, following Section 2.6 we get, for $\mathbf{x} \in \mathcal{L}$ and $i \in [\![1, n]\!]$:

$$\delta_W^p(\mathbf{x})_i = \bigvee_{j\in\mathcal{N}_i^p}\{x_j + w_{ij}^{(p)}\}\ , \quad \varepsilon_W^p(\mathbf{x})_i = \bigwedge_{j\in\check{\mathcal{N}}_i^p}\{x_j - w_{ji}^{(p)}\} \tag{9}$$

and

$$D_W^{[p]}(\mathbf{x})_i = \bigvee_{j\in N_i^p}\{x_j + s_{ij}^{[p]}\}\ , \quad E_W^{[p]}(\mathbf{x})_i = \bigwedge_{j\in\check{N}_i^k}\{x_j - s_{ji}^{[p]}\} \tag{10}$$

where $\mathcal{N}_i^p$ is the set of neighbours of vertex i in $\mathcal{G}(W^p)$ or, equivalently, the set of vertices in $\mathcal{G}(W)$ that can be reached from i through a path of length p; $\check{\mathcal{N}}_i^p = \left\{j \in \{1, \ldots, n\}, i \in \mathcal{N}_j^p\right\}$; $N_i^p = \cup_{1\leq k\leq p}\mathcal{N}_i^k$ and $\check{N}_i^p = \cup_{1\leq k\leq p}\check{\mathcal{N}}_i^k$. Hence these dilations and erosions are suprema and infima of "penalised" values over extended neighbourhoods induced by the original graph. The penalization is given by the strength of the connection between vertices: the closer the penalising weight to zero, the more the neighbours' value contributes to the result. The fact that we can restrict the supremum and infimum over graph neighbourhoods in (9) and (10) is due to the weight values being $-\infty$ outside these neighbourhoods, hence not contributing to the supremum and infimum.

3.3 Path interpretation of the opening $G_W^{[p]}$

The goal of this section is to show that $G_W^{[p]}$ can be interpreted similarly to a path opening [8], in the sense that it preserves bright values that are connected to other bright values forming long enough paths in a graph. We can first remark that for any $\mathbf{x} \in \mathcal{L}$, $i \in [\![1, n]\!]$ and $t \in [a, b]$:

$$G_W^{[p]}(\mathbf{x})_i \geq t \iff \exists j \in N_i^p, \text{ such that } \forall l \in \check{N}_j^p \;\; x_l \geq t - s_{ij}^{[p]} + s_{lj}^{[p]}, \tag{11}$$

which is straightforward from the expressions in (10), as $G_W^{[p]} = D_W^{[p]} E_W^{[p]}$. This directly yields

$$G_W^{[p]}(\mathbf{x})_i = \bigvee \left\{ t \in [a, b], \exists j \in N_i^p, \; \forall l \in \check{N}_j^p, \;\; x_l \geq t - s_{ij}^{[p]} + s_{lj}^{[p]} \right\}. \tag{12}$$

In the case of binary weights, *i.e.* $w_{ij} = 0$ if vertex j is neighbour of i in $\mathcal{G}$ and $w_{ij} = -\infty$ otherwise, which corresponds to a non-weighted graph, then $s_{ij}^{[p]} = s_{lj}^{[p]} = 0$ in (11) and (12). Therefore, if $G_W^{[p]}(\mathbf{x})_i \geq t$, then there is a vertex j which is at most p steps away from i, such that all paths of length at most p and ending in j, including those of length exactly p and passing through i (if they exist), show values larger than t. In the general case, the additional term $-s_{ij}^{[p]} + s_{lj}^{[p]}$ modulates this constraint in function of the strength of the connection of i and the other vertices of $\check{N}_j$, to j.

4 Links to the max-plus spectral theory

Now we present the consequences and interpretations of some results from the spectral theory in max-plus algebra. We first report definitions from [6] necessary to Theorem 1 (also from [6]). Then we draw the links to our setting and more particularly in the case of a symmetric matrix, corresponding to a non-directed graph. In all this section, $W \in \mathcal{M}_n$.

4.1 General definitions and results

Definition 3 (Eigenvector, eigenvalue [6]) *Let* $\mathbf{x} \in \mathbb{R}_{\max}^n$ *and* $\lambda \in \mathbb{R}_{\max}$. *Then* $\mathbf{x}$ *is an* eigenvector *of* W *with* λ *as corresponding eigenvalue if* $W \otimes \mathbf{x} = \lambda \otimes \mathbf{x} = \lambda + \mathbf{x}$. *If there exists finite* $\mathbf{x}$ *and* λ *solutions to this equation, we say that the eigenproblem is* finitely soluble*.*

In the graph $\mathcal{G}(W)$, a path $(k_1, \ldots, k_l)$ is called a *circuit* if $k_1 = k_l$. We will note $\mathcal{C}(W)$ the set of all circuits of $\mathcal{G}(W)$. Circuits allow us to distinguish another class of matrices in $\mathcal{M}_n$, called *definite* matrices. They are important to the present framework as they include the doubly-0-astic matrices.

Definition 4 (Definite matrix [6]) *W is said* definite *if* $\max_{c \in \mathcal{C}(W)} \omega(c) = 0$. *In other words, all the circuits of $\mathcal{G}(W)$ have non positive weights, and at least one circuit c^*, called a zero-weight circuit, achieves $\omega(c^*) = 0$.*

To see that if W is row or column-0-astic, then it is definite, it is sufficient to build an increasing path with zero-weight, until one vertex repeats. The path can be initialized with any vertex j_1. Then given the current path $(j_1, \dots, j_m)$, we extend it by adding a vertex j_{m+1} such that $w_{j_m j_{m+1}} = 0$. This is always possible thanks to the row or column-0-asticity of W. Since there are n distinct vertices in $\mathcal{G}(W)$, an index will repeat after at most n iterations.

Definition 5 (Eigen-node, equivalent eigen-nodes [6]) *Let W be a definite matrix. An* eigen-node *is any vertex in $\mathcal{G}(W)$ belonging to a zero-weight circuit. Two eigen-nodes are said* equivalent *if there is a zero-weight circuit passing through both of them.*

In [6], $S_n(W) = \bigvee_{1 \leq k \leq n} W^k$ is denoted by $\Delta(W)$ and called the **metric matrix**. Recall that for $i, j \in [\![1, n]\!]$, $\Delta(W)_{ij}$ is the maximal weight over the set of paths from vertex i to vertex j of length at most n, in $\mathcal{G}(W)$ (Prop. 8). If W is definite, circuits have non-positive weights in $\mathcal{G}(W)$ and therefore any path longer than n can be reduced to a shorter path with non larger weight. Hence, $\Delta(W)_{ij}$ is actually the maximal weight over the set of *all* paths from i to j. This provides an easy characterisation of eigen-nodes for W definite: j is an eigen-node of $\mathcal{G}(W)$ if and only if $\Delta(W)_{jj} = 0$. Furthermore, the j-th column ξ_j of $\Delta(W)$ is a map of the ancestors of j in $\mathcal{G}(W)$. It tells which vertices can reach j and at which cost.

Definition 6 (Fundamental eigenvectors, eigenspace [6]) *Let W be a definite matrix. Then a fundamental eigenvector of W is any j-th column ξ_j of $\Delta(W)$, where j is an eigen-node. Two fundamental eigenvectors are said* equivalent *if their associated eigen-nodes are equivalent (see Definition 5).*
Let $\mathcal{E} = \{\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_k}\}$ be a set of $k \geq 1$ fundamental eigenvectors of W, all pairwise non-equivalent. The set $\mathcal{E}$ is said to be a maximal set of non-equivalent fundamental eigenvectors if any other fundamental eigenvector of W is equivalent to one of the eigenvectors in $\mathcal{E}$.
In this case the set $\{\bigvee_{j=1}^{k} x_j + \xi_{i_j}, \mathbf{x} \in \mathbb{R}^k_{\max}\}$ is called the eigenspace of W and does not depend on $\mathcal{E}$ (see [6], Lemma 24-1).

Theorem 1 ([6]) *Let W be a doubly-0-astic (hence definite) matrix. Then the following statements are valid:*

- *For any fundamental eigenvector ξ_j of W (finite or not), $W \otimes \xi_j = \xi_j$.*
- *The eigenproblem is finitely soluble.*
- *If two fundamental eigenvectors are equivalent, then they are equal.*
- *Any finite eigenvector is associated to the eigenvalue $\lambda = 0$, and lies in the eigenspace of W.*

4.2 Consequences and interpretations

In general. As said, the results of the previous section apply to our setting since we consider adjunctions represented by doubly-0-astic matricesa. For $W \in \mathcal{D}_0(n)$, $\Delta(W)$ is also in $\mathcal{D}_0(n)$ and the corresponding opening $\delta_{\Delta(W)}\varepsilon_{\Delta(W)}$ is

$G_W^{[n]}$. By definition, $G_W^{[n]}(\mathbf{x})$ projects $\mathbf{x} \in \mathcal{L}$ onto $\delta_{\Delta(W)}(\mathcal{L})$, which is the set $\{\bigvee_{j=1}^n y_j + \xi_j, \mathbf{y} \in \mathcal{L}\}$ of max-plus combinations of columns of $\Delta(W)$. Theorem 1 tells that this decomposition can be split as $G_W^{[n]}(\mathbf{x}) = \mathbf{u} \vee \mathbf{v}$, where $\mathbf{u}$ lies in the eigenspace of W and $\mathbf{v}$ is a max-plus combination of the ξ_js which are not fundamental eigenvectors. This decomposition may be sparser than the original one, as the dimension of the eigenspace of W, *i.e.* $Card(\mathcal{E})$, can be lower than the number of fundamental eigenvectors.

The case of symmetric $W \in \mathcal{D}_0(n)$. This case corresponds to considering a non-directed graph supporting the signal $\mathbf{x}$. As the adjacency relationship is often based on a symmetrical function on pairs of vertex values, this assumption covers many practical cases (e.g. [2,3]). The main consequence of $W \in \mathcal{D}_0(n)$ symmetric is that *every vertex* j is an eigen-node: for any $j \in [\![1,n]\!]$ there is i such that $w_{ij} = 0 = w_{ji}$ and therefore (j,i,j) is a zero-weight circuit. This entails three other consequences.

First, $\Delta(W)_{jj} = 0$ for every $j \in [\![1,n]\!]$, following the characterisation of eigen-nodes described earlier, which implies that $\delta_{\Delta(W)} = D_W^{[n]}$ is extensive and $\varepsilon_{\Delta(W)} = E_W^{[n]}$ anti-extensive (Prop. 3). Secondly, $W \otimes \xi_j = \xi_j$ for every column ξ_j of $\Delta(W)$, which implies $W^k \otimes \xi_j = \xi_j$ for $1 \leq k \leq n$, hence $\Delta(W) \otimes \xi_j = \xi_j$ and finally $\Delta(W) \otimes \Delta(W) = \Delta(W)$. This means $D_W^{[n]}$ and $E_W^{[n]}$ are idempotent. They are therefore a closing and an opening respectively and $E_W^{[n]} = G_W^{[n]}$, since an adjunction $(\varepsilon,\ \delta)$ for which ε is an opening and δ a closing verifies $\varepsilon = \delta\varepsilon$ (and $\delta = \varepsilon\delta$). The third consequence is the following.

Corollary 1 *If $W \in \mathcal{D}_0(n)$ is symmetric, then the set of invariants of $G_W^{[n]}$ is exactly the eigenspace of W.*

When W is symmetric, a maximal set of k non-equivalent fundamental eigenvectors $\{\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k}\}$, $k \leq n$, can be seen as negative distance maps to the k corresponding eigen-nodes $\mathcal{G}(W)$, as they contain the optimal cost (maximal weight) between any vertex and the eigen-nodes[4]. Hence we can picture the aspect of $G_W^{[n]}(\mathbf{x})$, for $\mathbf{x} \in \mathcal{L}$: it is the upper-envelope of the largest vertical translations of these distance maps that are dominated by $\mathbf{x}$. Therefore, adapting $\mathcal{G}(W)$ to $\mathbf{x}$ by well connecting vertices within relevant structures preserves these structures under the filter $G_W^{[n]}$, as shown in [2,3]. In practice, n might be large, such as the number of pixels of an image. Since $(G_W^{[p]})_{1 \leq p \leq n}$ is a granulometry, we know that $G_W^{[n]}$ can be approximated by $G_W^{[p]}$ with increasing p.

5 Conclusion

In this paper we consolidated the basis of the representation of adjunctions by matrices in max-plus algebra. We showed that it is a very flexible framework that

[4] Note that $\Delta(W)$ is a metric, not exactly between vertices, but between their equivalence classes induced by Def. 5, as all vertices are eigen-nodes when W is symmetric.

generalises many types of morphological adjunctions. In particular, it allows describing precisely the behaviour of iterated operators based on spatially-variant, non-flat structuring functions. This is made possible by their graph interpretation and spectral results in max-plus algebra. Future works shall investigate further the insights that max-plus algebra can bring to mathematical morphology through this framework.

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