Shades of Iteration: from Elgot to Kleene

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Abstract. Notions of iteration range from the arguably most general *Elgot iteration* to a very specific *Kleene iteration*. The fundamental nature of Elgot iteration has been extensively explored by Bloom and Esik in the form of *iteration theories*, while *Kleene iteration* became extremely popular as an integral part of (untyped) formalisms, such as automata theory, regular expressions and Kleene algebra. Here, we establish a formal connection between Elgot iteration and Kleene iteration in the form of Elgot monads and Kleene monads, respectively. We also introduce a novel class of *while-monads*, which like Kleene monads admit a relatively simple description in algebraic terms. Like Elgot monads, while-monads cover a large variety of models that meaningfully support while-loops, but may fail the Kleene algebra laws, or even fail to support a Kleen iteration operator altogether.

1 Introduction

Iteration is fundamental in many areas of computer science, such as semantics, verification, theorem proving, automata theory, formal languages, computability theory, compiler optimisation, etc. An early effort to identifying a generic notion of iteration is due to Elgot [7], who proposed to consider an *algebraic theory* induced by a notion of abstract machine (motivated by *Turing machines*, and their variants) and regard iteration as an operator over this algebraic theory.

Roughly speaking, an algebraic theory carries composable spaces of morphisms $L(n, m)$, indexed by natural numbers n and m and including all functions from n to m[1], called *base morphisms*. For example, following Elgot, one can consider as $L(n, m)$ the space of all functions $n \times S \to m \times S$ representing transitions from a machine state ranging over n to a machine state ranging over m, and updating the background store over S (e.g. with S being the Turing machine's tape) in the meanwhile. In modern speech, $L(n, m)$ is essentially the space of Kleisli morphisms $n \to Tm$ of the *state monad* $T = (- \times S)^S$. Then a machine over m halting states and n non-halting states is represented by a morphism in $L(n, m + n)$, and the iteration operator is meant to compute a morphism in $L(n, m)$, representing a run of the machine, obtained by feedbacking all non-halting states. This perspective has been extensively elaborated by Bloom and Esik [4] who identified the ultimate equational theory of Elgot iteration together with plenty other examples of algebraic theories induced by existing semantic models, for which the theory turned out to be sound and complete.

[1] Here we identify numbers $n \in \mathbb{N}$ with finite ordinals $\{0, \dots, n-1\}$.

By replacing natural numbers with arbitrary objects of a category with finite coproducts and by moving from purely equational to a closely related and practically appealing quasi-equational theory of iteration, one arrives at *(complete) Elgot monads* [2,16], which are monads T, equipped with an iteration operator

$$\frac{f\colon X \to T(Y+X)}{f^{\dagger}\colon X \to TY} \tag{$\dagger$}$$

In view of the connection between computational effects and monads, pioneered by Moggi [33], Elgot monads provide arguably the most general model of iteration w.r.t. functions carrying computational effects, such as mutable store, non-determinism, probability, exceptional and abnormal termination, input-output actions of process algebra. The standard way of semantics via *domain theory* yields a general (least) fixpoint operator, which sidelines Elgot iteration and overshadows its fundamental role. This role becomes material again when it comes to the cases when the standard scenario cannot be applied or is difficult to apply, e.g. in constructive setting [12], for deterministic hybrid system semantics [13], and infinite trace semantics [31].

In contrast to Elgot iteration, *Kleene iteration*, manifested by *Kleene algebra*, is rooted in logic and automata theory [23], and crucially relies on non-determinism. The laws of Kleene algebra are from the outset determined by a rather conservative observation model, describing discrete events, coming one after another in linear order and in finite quantities. Nevertheless, Kleene algebra and thus Kleene iteration proved to be extremely successful (especially after the celebrated complete algebraic axiomatization of Kleene algebra by Kozen [26]) and have been accommodated in various formalizations and verification frameworks from those for concurrency [19] to those for modelling hybrid systems [35]. A significant competitive advantage of Kleene iteration is that it needs no (even very rudimental) type grammar for governing well-definedness of syntactic constructs, although this cannot be avoided when extending Kleene algebra with standard programming features [27,1,28]. Semantically, just as Elgot iteration, Kleene iteration can be reconciled with computational effects, leading to *Kleene monads* [11], which postulate Kleene iteration with the type profile:

$$\frac{f\colon X \to TX}{f^{*}\colon X \to TX} \tag{$*$}$$

Given f, f^* self-composes it non-deterministically indefinitely many times. In contrast to Elgot monads, the stock of computational effects modelled by Kleene monads is rather limited, which is due to the fact that many computational effects are subject to laws, which contradict the Kleene algebra laws. For a simple example, consider the computational effect of exception raising, constrained by the law, stating that postcomposing an exception raising program by another program is ineffective. Together with the Kleene algebra laws, we obtain a havoc:

$$\mathsf{raise}\, e_1 = \mathsf{raise}\, e_1;\ \bot = \bot = \mathsf{raise}\, e_2;\ \bot = \mathsf{raise}\, e_2,$$

where $\perp$ is the unit of non-deterministic choice. This and similar issues led to a number of proposals to weaken Kleene algebra laws [9,34,32,10] (potentially leading to other classes of monads, somewhere between Elgot and Kleene), although not attempting to identify the weakest set of such laws from the foundational perspective. At the same time, it seems undebatable that Kleene iteration and the Kleene algebra laws yield the most restricted notion of iteration.

We thus obtain a spectrum of potential notions of iteration between Elgot monads and Kleene monads. The goal of the present work is, on the one hand to explore this spectrum, and on the other hand to contribute into closing the conceptual gap between Kleene iteration and Elgot iteration. To that end, we introduce *while-monads*, which capture iteration in the conventional form of while-loops. Somewhat surprisingly, despite extensive work on axiomatizing iteration in terms of (†), a corresponding generic axiomatization in terms of "while" did not seem to be available. We highlight the following main technical contributions of the present work:

- We provide a novel axiomatization of Kleene algebra laws, which is effective both for Kleene algebras and Kleene monads (Proposition 4);
- We show that the existing axiomatization of Elgot monads is minimal (Proposition 12);
- We establish a connection between Elgot monads and while-monads (Theorem 18);
- We render Kleene monads as Elgot monads with additional properties (Theorem 22).

2 Preliminaries

We rely on rudimentary notions and facts of category theory, as used in semantics, most notably *monads* [3]. For a (locally small) category $\mathbf{C}$ we denote by $|\mathbf{C}|$ the class of its objects and by $\mathbf{C}(X, Y)$ the set of morphisms from $X \in |\mathbf{C}|$ to $Y \in |\mathbf{C}|$. We often omit indices at components of natural transformations to avoid clutter. **Set** will denote the category of classical sets and functions, i.e. sets and functions formalized in a classical logic with the law of excluded middle (we will make no use of the axiom of choice). By $\langle f, g\rangle\colon X \to Y \times Z$ we will denote the *pairing* of two morphisms $f\colon X \to Y$ and $g\colon X \to Z$ (in a category with binary products), and dually, by $[f, g]\colon X + Y \to Z$ we will denote the *copairing* of $f\colon X \to Z$ and $g\colon Y \to Z$ (in a category with binary coproducts). By $!\colon X \to 1$ we will denote terminal morphisms (if 1 is an terminal object).

An (F-)algebra for an endofunctor $F\colon \mathbf{C} \to \mathbf{C}$ is a pair $(A, a\colon FA \to A)$. Algebras form a category under the following notion of morphism: $f\colon A \to B$ if a morphism from (A, a) to (B, b) if $bf = (Ff)\,a$. The *initial algebra* is an initial object of this category (which may or may not exit). We denote this object $(\mu F, \mathsf{in})$. (F-)coalgebras are defined dually as pairs of the form $(A,\ a\colon A \to FA)$. The final coalgebra will be denoted $(\nu F,\ \mathsf{out})$. By Lambek's Lemma [30], both in and out are isomorphisms, and we commonly make use of their inverses in^{-1} and out^{-1}.

3 Monads for Computation

We work with monads represented by *Kleisli triples* $(T, \eta, (-)^\sharp)$ where T is a map $|\mathbf{C}| \to |\mathbf{C}|$, η is the family $(\eta_X : X \to TX)_{X \in |\mathbf{C}|}$ and $(-)^\sharp$ sends $f : X \to TY$ to $f^\sharp : TX \to TY$ in such a way that the standard *monad laws*

$$\eta^\sharp = \mathsf{id}, \qquad f^\sharp \eta = f, \qquad (f^\sharp g)^\sharp = f^\sharp g^\sharp$$

hold true. It is then provable that T extends to a functor with $Tf = (\eta f)^\sharp$ and η to a *unit* natural transformation. Additionally, we can define the *multiplication* natural transformation $\mu : TT \to T$ with $\mu_X = \mathsf{id}^\sharp$ (thus extending T to a monoid in the category of endufunctors). We preferably use bold letters, e.g. $\mathbf{T}$, for monads, to contrast with the underlying functor T. The axioms of monads entail that the morphisms of the form $X \to TY$ determine a category, called *Kleisli category*, and denoted $\mathbf{C}_{\mathbf{T}}$, under *Kleisli composition* $f \cdot g = f^\sharp g$ with η as the identity morphism. Intuitively, Kleisli category is the category of (generalized) effectful programs w.r.t. $\mathbf{C}$ as the category of "pure", or effectless, programs. More precisely, we will call *pure* those morphisms in $\mathbf{C}_{\mathbf{T}}$ that are of the form $\eta\, f$. We thus use diagrammatic composition $g; f$ alongside and equivalently to functional composition $f \cdot g$, as the former fits with the sequential composition operators of traditional programming languages.

A monad $\mathbf{T}$ is *strong* if it comes with a natural transformation $\tau_{X,Y} : X \times TY \to T(X \times Y)$ called *strength* and satisfying a number of coherence conditions [33]. Any monad on **Set** is canonically strong [24].

Example 1 (Monads). Recall some computationally relevant monads on **Set** (all monads on **Set** are strong [33]).

1. *Maybe-monad*: $TX = X+1$, $\eta(x) = \mathsf{inl}\, x$, $f^\sharp(\mathsf{inl}\, x) = f(x)$, $f^\sharp(\mathsf{inr}\, \star) = \mathsf{inr}\, \star$.
2. *Powerset monad*: $TX = \mathcal{P}X$, $\eta(x) = \{x\}$, $f^\sharp(S \subseteq X) = \{y \in f(x) \mid x \in S\}$.
3. $TX = \{S \mid S \in \mathcal{P}^+(X + 1), \text{if } S \text{ is infinite then } \mathsf{inr}\, \star \in S\}$ where $\mathcal{P}^+$ is the non-empty powerset functor, $\eta(x) = \{\mathsf{inl}\, x\}$, $f^\sharp(S \subseteq X) = \{y \in f(x) \mid \mathsf{inl}\, x \in S\} \cup (\{\mathsf{inr}\, \star\} \cap S)$.
4. *Exception monad*: $TX = X + E$ where E is a fixed (unstructured) non-empty set of exceptions, $\eta(x) = \mathsf{inl}\, x$, $f^\sharp(\mathsf{inl}\, x) = f(x)$, $f^\sharp(\mathsf{inr}\, e) = \mathsf{inr}\, e$.
5. *Non-deterministic writer monad*: $TX = \mathcal{P}(M \times X)$ where $(M, \epsilon, \bullet)$ is any monoid, $\eta(x) = \{(e, x)\}$, $f^\sharp(S \subseteq M \times X) = \{(n \bullet m, y) \mid (m, x) \in S, (n, y) \in f(x)\}$.
6. *Discrete sub-distribution monad*: $TX = \{d : [0, 1] \to X \mid \sum_{x \in X} d(x) \leqslant 1\}$ (the *supports* of d, $\{x \in X \mid d(x) > 0\}$ are necessarily countable – otherwise the sum $\sum_{x \in X} d(x)$ would diverge), $\eta(x)$ is the *Dirac distribution* δ_x, centred in x, i.e. $\delta_x(y) = 1$ if $x = y$, $\delta_x(y) = 0$ otherwise, $(f : X \to \mathcal{D}Y)^\sharp(d)(y) = \sum_{x \in X} f(x)(y) \cdot d(x)$.
7. *Partial state monad:* $TX = (X \times S + 1)^S$, where S is a fixed set of global states, $\eta(x)(s) = \mathsf{inl}(x, s)$, $f^\sharp(g : S \to X \times S + 1)(s) = \mathsf{inr}\, \star$ if $g(s) = \mathsf{inr}\, \star$ and $f^\sharp(g : S \to Y \times S + 1)(s) = f(x)(s')$ if $g(s) = \mathsf{inl}(x, s')$.

8. *Partial interactive input:* $TX = \nu\gamma.\,((X + \gamma^I) + 1)$, where I is a set of input values, $\eta(x) = \mathsf{out}^{\text{-}1}(\mathsf{inl}\,\mathsf{inl}\,x)$, $(f\colon X \to TY)^\sharp$ is the unique such morphism $f^\sharp\colon TX \to TY$ that (eliding the isomorphisms $T \cong (\text{–} + T^I) + 1$)

$$f^\sharp(\mathsf{inl}\,\mathsf{inl}\,x) = f(x), \qquad f^\sharp(\mathsf{inl}\,\mathsf{inr}\,h) = \mathsf{inl}\,\mathsf{inr}(f^\sharp\,h), \qquad f^\sharp(\mathsf{inr}\,\star) = \mathsf{inr}\,\star.$$

Intuitively, $p \in TX$ is a computation that either finishes and gives a result in X, or takes an input from I and continues recursively, or (unproductively) diverges.

9. *Partial interactive output:* $TX = \nu\gamma.\,((X + \gamma \times O) + 1)$, where O is a set of output values, $\eta(x) = \mathsf{out}^{\text{-}1}(\mathsf{inl}\,\mathsf{inl}\,x)$, $(f\colon X \to TY)^\sharp$ is the unique such morphism $f^\sharp\colon TX \to TY$ that (eliding the isomorphisms $T \cong (\text{–} + T \times O) + 1$)

$$f^\sharp(\mathsf{inl}\,\mathsf{inl}\,x) = f(x), \quad f^\sharp(\mathsf{inl}\,\mathsf{inr}(p,o)) = \mathsf{inl}\,\mathsf{inr}(f^\sharp(p),o), \quad f^\sharp(\mathsf{inr}\,\star) = \mathsf{inr}\,\star.$$

The behaviour of $p \in TX$ is as in the previous case, except that it outputs to O instead of expecting an input from I in the relevant branch.

Kleisli categories are often equivalent to categories with more familiar independent descriptions. For example, the Kleisli category of the maybe-monad is equivalent to the category of partial functions and the Kleisli category of the powerset monad is equivalent to the category of relations. Under the monads-as-effects metaphor, partial functions can thus be regarded as possibly non-terminating functions and relations as non-deterministic functions.

The above examples can often be combined. E.g. non-deterministic stateful computations are obtained as $TA = S \to \mathcal{P}(A \times S)$. The Java monad of [21],

$$TX = S \to (X \times S + E \times S) + 1$$

with S the set of states and E the set of exceptions.

4 Kleene Monads

A Kleene algebra can be concisely defined as an idempotent semiring $(S, \bot, \eta, \vee, ;)$ equipped with an operator $(\text{–})^*\colon S \to S$, such that

- $g; f^*$ is the least (pre-)fixpoint of $g \vee (\text{–}); f$,
- $f^*; h$ is the least fixpoint of $h \vee f; (\text{–})$,

where the order is induced by $\vee$: $f \leqslant g$ if $f \vee g = g$. We assume here and henceforth that sequential composition ; binds stronger than $\vee$.

More concretely, a Kleene algebra $(S, \bot, \eta, \vee, ;, (\text{–})^*)$ is an algebraic structure, satisfying the laws in Figure 1. A categorical version of Kleene algebra emerges as a class of monads, called *Kleene monads* [15], which can be used for interpreting effectful languages with iteration and non-determinism.

Definition 2 (Kleene-Kozen Category/Kleene Monad). We say that a category $\mathbf{C}$ is a *Kleene-Kozen category* if $\mathbf{C}$ is enriched over bounded (i.e. possessing a least element) join-semilattices and strict join-preserving morphisms, and there is *Kleene iteration* operator

$$(\text{–})^*\colon \mathbf{C}(X,X) \to \mathbf{C}(X,X),$$

Idempotent semiring laws:

idempotence:	$f \vee f = f$
commutativity:	$f \vee g = g \vee f$
neutrality of $\bot$*:*	$f \vee \bot = f$
associativity of $\vee$*:*	$f \vee (g \vee h) = (f \vee g) \vee h$
associativity of ; *:*	$f; (g; h) = (f; g); h$
right strictness:	$f; \bot = \bot$
right neutrality of η*:*	$f; \eta = f$
right distributivity:	$(f \vee g); h = f; h \vee g; h$
left strictness:	$\bot; f = \bot$
left neutrality of η*:*	$\eta; f = f$
left distributivity:	$f; (g \vee h) = f; g \vee f; h$

Iteration laws:

right unfolding:	$f^* = \eta \vee f; f^*$
right induction:	$f; g \leqslant f \implies f; g^* \leqslant f$
left unfolding:	$f^* = \eta \vee f^*; f$
left induction:	$f; g \leqslant g \implies f^*; g \leqslant g$

Fig. 1. Axioms Kleene algebras/monads.

such that, given $f\colon Y \to Y$, $g\colon Y \to Z$ and $h\colon X \to Y$, $g\, f^*$ is the least (pre-)fixpoint of $g \vee (-)\, f$ and $f^*\, h$ is the least (pre-)fixpoint of $h \vee f\, (-)$.

A monad $\mathbf{T}$ is a *Kleene monad* if $\mathbf{C_T}$ is a Kleene-Kozen category.

Recall that a monoid is nothing but a single-object category, whose morphisms are identified with monoid elements, and whose identity morphisms and morphism composition are identified with monoidal unit and composition. This suggests a connection between Kleene-Kozen categories and Kleene algebras.

Proposition 3. *A Kleene algebra is precisely a Kleene-Kozen category with one object.*

We record the following characterization of Kleene-Kozen categories (hence, also of Kleene algebras by Proposition 3).

Proposition 4. *A category* $\mathbf{C}$ *is a Kleene-Kozen category iff*

- $\mathbf{C}$ *is enriched over bounded join-semilattices and strict join-preserving morphisms;*

- *there is an operator* $(-)^*\colon \mathbf{C}(X,X) \to \mathbf{C}(X,X)$*, such that*
  1. $f^* = \mathsf{id} \vee f^*\, f$*;*
  2. $\mathsf{id}^* = \mathsf{id}$*;*
  3. $f^* = (f \vee \mathsf{id})^*$*;*
  4. $h\, f = f\, g$ *implies* $h^*\, f = f\, g^*$*.*

Proof. Let us show *necessity.*

1. The law $f^* = \mathsf{id} \vee f^*\, f$ holds by assumption.
2. Since $\mathsf{id} = \mathsf{id} \vee \mathsf{id}\,\mathsf{id}$, id is a fixpoint of $f \mapsto \mathsf{id} \vee f\,\mathsf{id}$, and thus $\mathsf{id}^* \leqslant \mathsf{id}$. Also $\mathsf{id}^* = \mathsf{id} \vee \mathsf{id}^*\,\mathsf{id} \geqslant \mathsf{id}$. Hence $\mathsf{id} = \mathsf{id}^*$ by mutual inequality.
3. To show that $(f \vee \mathsf{id})^* \leqslant f^*$, note that $f^* = \mathsf{id} \vee f^*\, f = \mathsf{id} \vee f^*\, f \vee f^* = \mathsf{id} \vee f^*\,(f \vee \mathsf{id})$, and use the fact that $(f \vee \mathsf{id})^*$ is the least fixpoint. The opposite inequality is shown analogously, by exploiting the fact that f^* is a least fixpoint.
4. Suppose that $h\, f = f\, g$, and show that $h^*\, f = f\, g^*$. Note that

$$f \vee (h^*\, f)\, g = f \vee h^*\, f\, g = f \vee h^*\, h\, f = h^*\, f,$$

i.e. $h^*\, f$ satisfies the fixpoint equation for $f\, g^*$, and therefore $h^*\, f \geqslant f\, g^*$. By a symmetric argument, $h^*\, f \leqslant f\, g^*$, hence $h^*\, f = f\, g^*$.

We proceed with *sufficiency.* Suppose that $(-)^*$ is as described in the second clause of the present proposition. Observe that by combining assumptions 1 and 4 we immediately obtain the dual version of 1, which is $f^* = \mathsf{id} \vee f\, f^*$. Now, fix $f\colon Y \to TY$ and $g\colon Y \to TZ$, and show that $f^*\, g$ is the least fixpoint of $g \vee f\,(-)$ – we omit proving the dual property, since it follows by a dual argument. From $f^* = \mathsf{id} \vee f^*\, f$ we obtain $f^*\, g = g \vee f\,(f^*\, g)$, i.e. $f^*\, g$ is a fixpoint. We are left to show that it is the least one. Suppose that $h = g \vee f\, h$ for some h, which entails

$$(\mathsf{id} \vee f)\, h = h \vee f\, h = g \vee f\, h \vee f\, h = h = h\,\mathsf{id}.$$

Since $g \leqslant h$, using assumptions 2, 3 and 4, we obtain

$$f^*\, g \leqslant (\mathsf{id} \vee f)^*\, g \leqslant (\mathsf{id} \vee f)^*\, h = h\,\mathsf{id}^* = h,$$

as desired. □

The axioms of Kleene monads do not in fact need a monad, and can be interpreted in any category. We focus on Kleisli categories for two reasons: (i) in practice, Kleene-Kozen categories are often realized as Kleisli categories, and monads provide a compositional mechanism for constructing more Kleene-Kozen categories by generalities; (ii) we will relate Kleene monads and Elgot monads, and the latter are defined by axioms, which do involve both general Kleisli morphisms and the morphisms of the base category.

Example 5. Let us revisit Example 1. Many monads therein fail to be Kleene simply because they fail to support binary non-determinism. Example 1.6 is an interesting case, since we can define the operation of probabilistic choice $+_p\colon \mathbf{C}(X,TY) \times \mathbf{C}(X,TY) \to \mathbf{C}(X,TY)$ indexed by $p \in [0,1]$, meaning that

$x +_p y$ is resolved to x with probability p and to y with probability $1 - p$. For every $x \in X$, $f(x) +_p g(x)$ is a *convex sum* of the distributions $f(x)$ and $g(x)$. This operation satisfies the axioms of *barycentric algebras* (or, *abstract convex sets* [37]), which are somewhat similar to those of a monoid, but with the multiplication operator indexed over $[0, 1]$. To get rid of this indexing, one can remove the requirement that probabilities sum up to at most 1 and thus obtain spaces of *valuations* [38] instead of probability distributions. Valuations can be conveniently added pointwise, and thus defined addition satisfies monoidal laws, but fails idempotence, hence still does not yield a Kleene monad. Given two valuations v and w, we also can define $v \vee w$ as the pointwise maximum. This satisfies the axioms of semilattices, but fails both distributivity laws.

Example 1.3 is the **Set**-reduct or *Plotkin powerdomain* [36] over a flat domain. It supports proper non-deterministic choice, but the only candidate for $\bot$ is not a unit for it.

Kleene monads of Example 1 are only 2 and 5. The non-deterministic state monad over $TX = (\mathcal{P}(X \times S))^S$ obtained by adapting Example 1.7 in the obvious way is also Kleene.

Except for the powerset monad, our examples of Kleene monads are in fact obtained by generic patterns.

Proposition 6. *Let* T *be a Kleene monad. Then so are*

1. *the state monad transformer* $(T(- \times S))^S$ *for every* S*;*
2. *the writer monad transformer* $T(M \times -)$ *for every monoid* $(M, \epsilon, \bullet)$ *if* T *is strong and strength* $\tau_{X,Y} \colon X \times TX \to T(X \times Y)$ *respects the Kleene monad structure, as follows:*

$$\begin{gathered}\tau\,(\mathsf{id} \times \bot) = \bot, \qquad \tau\,(\mathsf{id} \times f^*) = (\tau\,(\mathsf{id} \times f))^*,\\ \tau\,(\mathsf{id} \times (f \vee g)) = \tau\,(\mathsf{id} \times f) \vee \tau\,(\mathsf{id} \times g).\end{gathered} \tag{1}$$

Proof. 1. By definition, the Kleisli category of the state transformer is equivalent to the full subcategory of $\mathbf{C}_{\mathsf{T}}$ over the objects of the form $X \times S$ (using the isomorphism $\mathbf{C}(X, (T(Y \times S))^S) \cong \mathbf{C}(X \times S, T(Y \times S))$). The enrichment, the iteration operator and the axioms are clearly restricted along the induced inclusion functor.

2. The semilattice structure for every $\mathbf{C}(X, T(M \times Y))$ is inherited from $\mathbf{C}_{\mathsf{T}}$, but to show enrichment, the strictness and the distributivity laws must be verified manually. For every $f \colon X \to T(M \times Y)$, let $f^\circ \colon M \times X \to T(M \times Y)$ be as follows

$$M \times X \xrightarrow{\tau\,(\mathsf{id} \times f)} T(M \times (M \times Y)) \cong T((M \times M) \times Y) \xrightarrow{T(\bullet \times \mathsf{id})} T(M \times Y).$$

The assumptions (1) entail the following identities:

$$\bot^\circ = \bot \quad (2) \qquad (f \vee g)^\circ = f^\circ \vee g^\circ \quad (3) \qquad (f \langle \epsilon, \mathsf{id}\rangle)^\circ = f \quad (4)$$

$$(f^\circ \cdot g)^\circ = f^\circ \cdot g^\circ \quad (5) \qquad f^\circ \langle \epsilon, \mathsf{id}\rangle = f \quad (6)$$

Kleisli composition of the transformed monad sends $f\colon X \to T(M \times Y)$ and $g\colon Y \to T(M \times Z)$ to the Kleisli composition $g^\circ \cdot f$ of T. Left strictness and right distributivity are then obvious, while right strictness and left distributivity follow too by (2),(3): $\bot^\circ \cdot f = \bot \cdot f = \bot$, $(f \vee g)^\circ \cdot h = (f^\circ \vee g^\circ) \cdot h = f^\circ \cdot h \vee g^\circ \cdot h$. Kleene star for the transformed monad is defined as $(f^\circ)^*\langle \epsilon, \mathsf{id}\rangle$ for every $f\colon X \to T(M \times X)$ where $\epsilon\colon 1 \to M$ is the monoid unit.

$$\begin{aligned}
(f^\circ)^*\langle \epsilon, \mathsf{id}\rangle &= (\eta \vee f^\circ \cdot (f^\circ)^*)\langle \epsilon, \mathsf{id}\rangle \\
&= (\eta \vee f^\circ \cdot (f^\circ)^*) \cdot \eta\langle \epsilon, \mathsf{id}\rangle \\
&= \eta\langle \epsilon, \mathsf{id}\rangle \vee f^\circ \cdot (f^\circ)^* \cdot \eta\langle \epsilon, \mathsf{id}\rangle \\
&= \eta\langle \epsilon, \mathsf{id}\rangle \vee f^\circ \cdot (f^\circ)^*\langle \epsilon, \mathsf{id}\rangle,
\end{aligned}$$

$$\begin{aligned}
(f^\circ)^*\langle \epsilon, \mathsf{id}\rangle &= (\eta \vee (f^\circ)^* \cdot f^\circ)\langle \epsilon, \mathsf{id}\rangle \\
&= (\eta \vee (f^\circ)^* \cdot f^\circ) \cdot \eta\langle \epsilon, \mathsf{id}\rangle \\
&= \eta\langle \epsilon, \mathsf{id}\rangle \vee (f^\circ)^* \cdot f^\circ \cdot \eta\langle \epsilon, \mathsf{id}\rangle \\
&= \eta\langle \epsilon, \mathsf{id}\rangle \vee (f^\circ)^* \cdot f && /\!/ \ (6) \\
&= \eta\langle \epsilon, \mathsf{id}\rangle \vee ((f^\circ)^*\langle \epsilon, \mathsf{id}\rangle)^\circ \cdot f. && /\!/ \ (4)
\end{aligned}$$

If $f^\circ \cdot g \leqslant g$ then by (4), $((f^\circ)^*\langle \epsilon, \mathsf{id}\rangle)^\circ \cdot g = (f^\circ)^* \cdot g \leqslant g$. Analogously, if $f^\circ \cdot g \leqslant f$ then by (5) $f^\circ \cdot g^\circ = (f^\circ \cdot g)^\circ \leqslant f^\circ$, and therefore $f^\circ \cdot (g^\circ)^*\langle e, \mathsf{id}\rangle \leqslant f^\circ\langle e, \mathsf{id}\rangle = f$. Thus, the axioms of iteration are all satisfied. □

Example 7. Note that the powerset monad $\mathcal{P}$ is a Kleene monad with f^* calculated as a least fixpoint of $\eta \vee (\text{–}) \cdot f$.

1. By Proposition 6.1, $(\mathcal{P}(\text{–} \times S))^S$ is a Kleene monad.
2. By applying Proposition 6.2, to the free monoid $A^\star$ of finite strings over an alphabet A we obtain that $\mathcal{P}(A^\star \times \text{–})$.

It is easy to see that for every Kleene monad T, $\mathsf{Hom}(1, T1)$ is a Kleene algebra. By applying this to the above clauses we obtain correspondingly the standard *relational* and *language-theoretic* models of Kleene algebra [29].

5 Elgot Monads

A general approach to monad-based iteration is provided by Elgot monads. We continue under the assumption that $\mathbf{C}$ supports finite coproducts. This, in particular, yields an if-the-else operator sending $b \in \mathbf{C}(X, X+X)$ and $f, g \in \mathbf{C}(X, Y)$ to $\mathsf{if}\ b\ \mathsf{then}\ p\ \mathsf{else}\ q = [q, p] \cdot b \in \mathbf{C}(X, Y)$. Note that for any monad T on $\mathbf{C}$, $\mathbf{C}_{\mathsf{T}}$ inherits finite coproducts.

Definition 8 (Elgot monad). An *Elgot monad* in a category with binary coproducts is a monad T equipped with an *Elgot iteration* operator

$$(\text{–})^\dagger\colon \mathbf{C}(X, T(Y + X)) \to \mathbf{C}(X, TY),$$

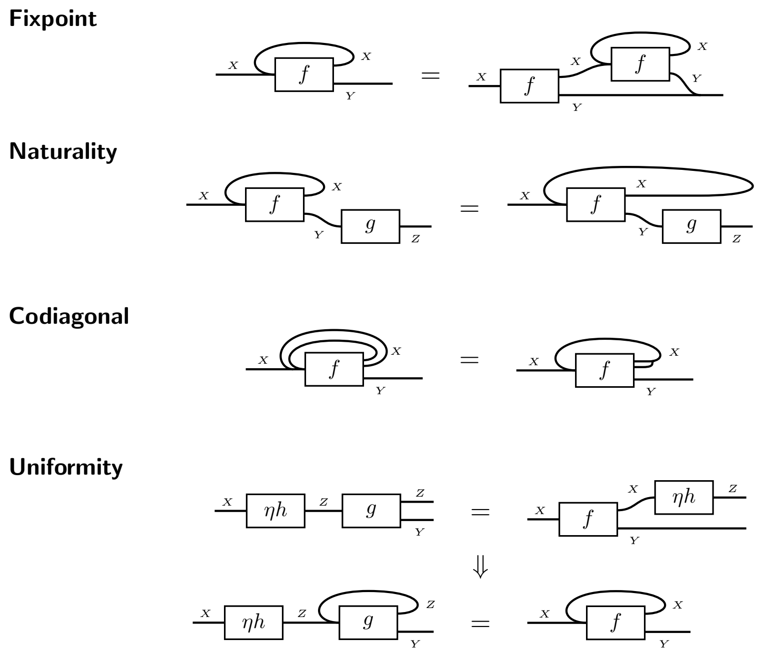


Fig. 2. Axioms of Elgot monads.

subject to the following principles:

Fixpoint : $[\eta, f^\dagger] \cdot f = f^\dagger$ $\qquad (f : X \to T(Y + X))$

Naturality : $g \cdot f^\dagger = ([\eta\, \mathsf{inl} \cdot g, \eta\, \mathsf{inr}] \cdot f)^\dagger$ $\qquad (g : Y \to TZ, f : X \to T(Y + X))$

Codiagonal : $f^{\dagger\dagger} = ([\eta, \eta\, \mathsf{inr}] \cdot f)^\dagger$ $\qquad (f : X \to T((Y + X) + X))$

Uniformity : $\dfrac{g \cdot \eta h = \eta(\mathsf{id} + h) \cdot f}{g^\dagger \cdot \eta h = f^\dagger}$ $\qquad (h : X \to Z, g : Z \to T(Y + Z), f : X \to T(Y + X))$

These laws are easier to grasp by depicting them graphically (Figure 2), more precisely speaking, as *string diagrams* (cf. [22,18] for a rigorous treatment in terms of monoidal categories). Iterating f is depicted as a feedback loop. It is then easy to see that while **Fixpoint** expresses the basic fixpoint property of iteration, **Naturality** and **Codiagonal** are essentially rearrangements of wires. The **Uniformity** law is a form of induction: the premise states that ηh can be pushed over g, so that at the same time g is replaced by f, and the conclusion is essentially the result of closing this transformation under iteration. **Uniformity** is therefore the only law, which alludes to pure morphisms. Intuitively, the morphisms f and g can be seen as programs operating correspondingly on X and Y as their state spaces, and $h : X \to Y$ is a map between these state spaces.

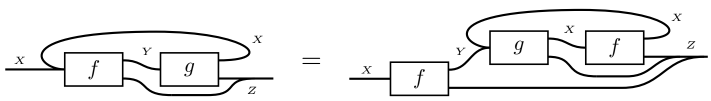
Uniformity thus ensures that the behaviour of iteration does not depend on the shape of the state space. It is critical for this view that h is pure, i.e. does not trigger any side-effects.

Remark 9 (Divergence). Every Elgot monad comes together with the definable (unproductive) divergence constant $\delta = (\eta\,\mathsf{inr})^\dagger$. Graphically, $\delta\colon X \to T\emptyset$ will be depicted as , symmetrically to the depiction of the initial morphism $!\colon \emptyset \to TX$ as .

Example 10 (Elgot Monads). Clauses 1–9 of Example 1 all define Elgot monads. A standard way of introducing Elgot iteration is enriching the Kleisli category over pointed complete partial orders and defining $(f\colon X \to T(Y+X))^\dagger$ as a least fixpoint of the map $[\eta, \text{–}] \cdot f\colon \mathbf{C}(X, TY) \to \mathbf{C}(X, TY)$ by the *Kleene fixpoint theorem*. This scenario covers 1–7. In all these cases, we inherit complete partial order structures on $\mathbf{Set}(X, TY)$ by extending canonical complete partial order structures from TY pointwise. In particular, in 4, we need to chose the divergence element $\delta \in E$. This choice induces a *flat domain* structure on $X + E$: $x \sqsubseteq y$ if $x = y$ or $x = \delta$. The induced divergence constant in the sense of Remark 9 then coincides with δ, and hence there are at least as many distinct Elgot monad structures on the exception monad as exceptions.

Clauses 8 and 9 fit a different pattern. For every Elgot monad $\mathbf{T}$ and every endofunctor H, if all final coalgebras $T_H X = \nu\gamma. T(X + H\gamma)$ exist then T_H extends to an Elgot monad [13], called the *coalgebraic generalized resumption transform* of $\mathbf{T}$. This yields 8 and 9 by taking $\mathbf{T}$ to be the maybe-monad in both cases and $HX = X^I$ and $HX = O \times X$ respectively.

Remark 11 (Dinaturality). A classical law of iteration, which is not included in Definition 8, is the **Dinaturality** law, which has the following graphical representation:



This law has been used in one of the equivalent axiomatization of iteration theories [4] (under the name *"composition identity"*) and thus was initially inherited in the definition of Elgot monads [2,16]. However, Ésik and Goncharov [8] latter discovered that **Dinaturality** is derivable in presence of **Uniformity**.

Remark 11 poses the question, if the present axiomatization of Elgot monads possibly contains further derivable laws. Here, we resolve it in the negative.

Proposition 12. *The axiomatization in Definition 8 is minimal.*

Proof. For every axiom, we construct a separating example that fails that axiom, but satisfies the other three. Every example is a monad on **Set**.

- **Fixpoint**: For any monad $\mathbf{T}$, equipped with a natural transformation $p\colon 1 \to TX$, we can define $f^\dagger = [\eta, p\,!] \cdot f$ for a given $f\colon X \to T(Y + X)$. It is easy to see that **Naturality**, **Codiagonal** and **Uniformity** are satisfied, but **Fixpoint** need not to, e.g. with $\mathbf{T}$ being the non-deterministic writer monad (Example 1.5) over the additive monoid of natural numbers $\mathbb{N}$.

• **Naturality**: Let $\mathbf{T} = \mathcal{P}$ and let $f^\dagger(x) = Y$ for every $f\colon X \to T(Y + X)$ and every $x \in X$. Note that every $f\colon X \to T(Y + Z)$ is equivalent to a pair $(f_1\colon X \to TY, f_2\colon X \to TZ)$ and $[g, h] \cdot f = g \cdot f_1 \cup h \cdot f_2$ for any $g\colon Y \to TV$, $h\colon Z \to TV$. This helps one to see that all the axioms, except **Naturality** hold true, e.g. $([\eta, f^\dagger] \cdot f)(x) = f_1(x) \cup Y = Y = f^\dagger(x)$. **Naturality** fails, because $g \cdot \delta = g \cdot (\eta\,\mathsf{inr})^\dagger \neq ([\eta\,\mathsf{inl} \cdot g, \eta\,\mathsf{inr}] \cdot \eta\,\mathsf{inr})^\dagger = \delta$, since the image of $\delta\colon X \to TY$ is $\{Y\}$, while the image of $g \cdot \delta$, aka the image of g, need not be $\{Y\}$.

• **Codiagonal**: Consider the exception monad transform $TX = \mathcal{P}(2^\star \times X \cup 2^\omega)$ of the non-deterministic writer monad over the free monoid $2^\star$. This is canonically an Elgot monad, and let us denote by $(-)^\ddagger$ the corresponding iteration operator. Every $f\colon X \to T(Y + X)$, using the isomorphism $T(Y + X) \cong \mathcal{P}(2^\star \times Y \cup 2^\omega) \times \mathcal{P}(2^\star \times X)$, induces a map $\hat{f}\colon X \to \mathcal{P}(2^\star \times X)$. Let $f^\dagger\colon X \to TY$ be as follows: $f^\dagger(x)$ is the union of $f^\ddagger(x)$ and the set

$$\{w \in 2^\omega \mid \exists u \in 2^\star.\, uw = w_1 w_2 \ldots, (w_1, x_1) \in \hat{f}(x), (w_2, x_2) \in \hat{f}(x_1), \ldots\}.$$

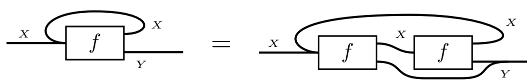
That $(-)^\dagger$ satisfies **Fixpoint**, **Naturality** and **Uniformity** follows essentially from the fact that so does $(-)^\ddagger$. To show that $(-)^\dagger$ fails **Codiagonal**, consider $g\colon 1 \to \mathcal{P}((2^\star + 2^\star) \cup 2^\omega)$, with $g(\star) = \{\mathsf{inl}\, 0, \mathsf{inr}\, 1\}$. Let f be the composition of g with the obvious isomorphism $\mathcal{P}((2^\star + 2^\star) \cup 2^\omega) \cong T((0 + 1) + 1)$. Now $([\eta, \eta\,\mathsf{inr}] \cdot f)^\dagger(\star) = 2^\omega \neq \{0^\omega, 1^\omega\} = f^{\dagger\dagger}(\star)$.

• **Uniformity**: Consider the exception monad on $TX = X + \{0, 1\}$. This can be made into an Elgot monad in two ways: by regarding either 0 or 1 as the divergence element. Given $f\colon X \to T(Y + X)$, we let $f^\dagger$ be computed as a least fixpoint according to the first choice if X is a singleton and according to the second choice otherwise. The axioms except **Uniformity** are clearly satisfied. To show that **Uniformity** fails, let $|X| > 2$, $|Z| = 1$, $g = \eta\,\mathsf{inr}$, $f = \eta\,\mathsf{inr}$, $h = !$. The premise of **Uniformity** is thus satisfied, while the conclusion is not, since $f^\dagger$ is constantly 1 and $g^\dagger$ is constantly 0. ◻

Although we cannot lift any of the Elgot monad laws, **Naturality** can be significantly restricted.

Proposition 13. *In the definition of Elgot monad,* **Naturality** *can be equivalently replaced by its instance with g of the form $\eta\,\mathsf{inr}\colon Y \to T(Y' + Y)$.*

Proof. We use the fact that **Dinaturality** and the following law, called **Squaring** are derivable from **Fixpoint**, **Codiagonal** and **Uniformity** [14, Lemma 31]:



Let us fix $g\colon Y \to TZ$, $f\colon X \to T(Y+X)$ and proceed to show that $g \cdot f^\dagger = ([\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f)^\dagger$. Let $w = [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}\cdot f]\colon Y+X \to T(Z+(Y+X))$, and note that **Fixpoint** entails

$$w^\dagger \cdot \eta\,\mathsf{inl} = g \tag{7}$$

The goal will follow from the identities

$$w^\dagger \cdot \eta\,\mathsf{inr} = g \cdot f^\dagger \tag{8}$$
$$w^\dagger \cdot \eta\,\mathsf{inr} = ([\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f)^\dagger \tag{9}$$

Let us show (8), using an allowed instance of **Naturality**:

$$\begin{aligned}
w^\dagger \cdot \eta\,\mathsf{inr} &= ([\eta, \eta\,\mathsf{inr}] \cdot [\eta\,\mathsf{inl}\,\mathsf{inl}\cdot g, T(\mathsf{inr}\,\mathsf{inl} + \mathsf{inr})\, f])^\dagger \cdot \eta\,\mathsf{inr} \\
&= ([\eta\,\mathsf{inl}\,\mathsf{inl}\cdot g, T(\mathsf{inr}\,\mathsf{inl} + \mathsf{inr})\, f])^{\dagger\dagger} \cdot \eta\,\mathsf{inr} && /\!\!/\ \textbf{Codiagonal} \\
&= [\eta, w^\dagger] \cdot ([\eta\,\mathsf{inl}\,\mathsf{inl}\cdot g, T(\mathsf{inr}\,\mathsf{inl} + \mathsf{inr})\, f])^\dagger \cdot \eta\,\mathsf{inr} && /\!\!/\ \textbf{Fixpoint} \\
&= [\eta, w^\dagger] \cdot (T(\mathsf{inr}\,\mathsf{inl} + \mathsf{id})\, f)^\dagger \cdot && /\!\!/\ \textbf{Uniformity} \\
&= [\eta, w^\dagger] \cdot \eta\,\mathsf{inr}\,\mathsf{inl}\cdot f^\dagger && /\!\!/\ \textbf{Naturality} \\
&= w^\dagger \cdot \eta\,\mathsf{inl}\cdot f^\dagger \\
&= g \cdot f^\dagger. && /\!\!/\ (7)
\end{aligned}$$

Finally, let us show (9):

$$\begin{aligned}
w^\dagger \cdot \mathsf{inr} &= [\eta, w^\dagger] \cdot w \cdot \mathsf{inr} && /\!\!/\ \textbf{Fixpoint} \\
&= [\eta, w^\dagger] \cdot [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}\cdot f] \cdot \mathsf{inr} \\
&= w^\dagger \cdot f \\
&= ([\eta\,\mathsf{inl}, w] \cdot w)^\dagger \cdot f && /\!\!/\ \textbf{Squaring} \\
&= ([\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}\cdot f]] \cdot [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}\cdot f])^\dagger \cdot f \\
&= [\eta, ([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot f] \\
&\qquad \cdot [\eta\,\mathsf{inl}\cdot g, [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f])^\dagger] \cdot \eta\,\mathsf{inr}\cdot f \\
&= ([\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}\cdot g, [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f]] \cdot \eta\,\mathsf{inr}\cdot f)^\dagger && /\!\!/\ \textbf{Dinaturality} \\
&= ([\eta\,\mathsf{inl}\cdot g, [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f] \cdot f)^\dagger \\
&= ([\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f] \cdot [\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f)^\dagger \\
&= ([\eta\,\mathsf{inl}\cdot g, \eta\,\mathsf{inr}] \cdot f)^\dagger. && /\!\!/\ \textbf{Squaring}
\end{aligned}$$

□

6 While-Monads

We proceed to develop a novel alternative characterization of Elgot monads in more conventional for computer science terms of while-loops.

Definition 14 (Decisions). Given a monad $\mathbf{T}$ on $\mathbf{C}$, we call any family $(\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X) \subseteq \mathbf{C}(X, T(X+X)))_{X \in |\mathbf{C}|}$, a family of *decisions* if every $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X)$ contains $\eta\,\mathsf{inl}$, $\eta\,\mathsf{inr}$, and is closed under if-then-else.

We encode logical operations on decisions as follows:

$$\begin{aligned} \mathsf{ff} &= \eta\,\mathsf{inl}, & b \;\&\&\; c &= \mathsf{if}\ b\ \mathsf{then}\ c\ \mathsf{else}\ \mathsf{ff}, & \sim b &= \mathsf{if}\ b\ \mathsf{then}\ \mathsf{ff}\ \mathsf{else}\ \mathsf{tt}, \\ \mathsf{tt} &= \eta\,\mathsf{inr}, & b \;||\; c &= \mathsf{if}\ b\ \mathsf{then}\ \mathsf{tt}\ \mathsf{else}\ c. \end{aligned}$$

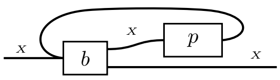
By definition, decisions can range from the smallest family with $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X) = \{\mathsf{ff}, \mathsf{tt}\}$, to the greatest one with $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X) = \mathbf{C}(X, T(X+X))$.

Remark 15. Our notion of decision is maximally simple and general. An alternative are morphisms of the form $b\colon X \to T2$, from which we can obtain $X \xrightarrow{\langle \mathsf{id}, b\rangle} X \times T2 \xrightarrow{\tau} T(X \times 2) \cong T(X+X)$ if $\mathbf{T}$ is strong, with τ being the strength. The resulting decision d would satisfy many properties we are not assuming generally, e.g. $\mathsf{if}\ d\ \mathsf{then}\ \mathsf{tt}\ \mathsf{else}\ \mathsf{ff} = \eta$. Both morphisms of the form $X \to T2$ and $X \to T(X+X)$ are relevant in semantics as decision making abstractions – this is explained in detail from the perspective of categorical logic by Jacobs [20], who uses the names *predicates* and *instruments* correspondingly (alluding to physical, in particular, quantum experiments).

Elgot monads are essentially the semantic gadgets for effectful while-languages. In fact, we can introduce a semantic while-operator and express it via Elgot iteration. Given $b \in \mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X)$ and $p \in \mathbf{C}(X, TX)$, let

$$\mathsf{while}\ b\ p = (\mathsf{if}\ b\ \mathsf{then}\ p; \mathsf{tt}\ \mathsf{else}\ \mathsf{ff})^{\dagger}, \tag{10}$$

or diagrammatically, $\mathsf{while}\ b\ p$ is expressed as



It is much less obvious that, conversely, Elgot iteration can be defined via while, and moreover that the entire class of Elgot monads can be rebased on while. We dub the corresponding class of monad *while-monads*.

Definition 16 (While-Monad). A *while-monad* is a monad $\mathbf{T}$, equipped with an operator

$$\mathsf{while}\colon \mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X) \times \mathbf{C}(X, TX) \to \mathbf{C}(X, TX),$$

such that the following axioms are satisfied

W-Fix $\quad \mathsf{while}\ b\ p = \mathsf{if}\ b\ \mathsf{then}\ p; (\mathsf{while}\ b\ p)\ \mathsf{else}\ \eta$

W-Or $\quad \mathsf{while}\,(b \;||\; c)\ p = (\mathsf{while}\ b\ p); \mathsf{while}\ c\ (p; \mathsf{while}\ b\ p)$

W-And $\quad \dfrac{\eta h; b = \eta u; \mathsf{ff}}{\mathsf{while}\,(b \;\&\&\; (c \;||\; \eta u; \mathsf{ff}))\ p = \mathsf{while}\ b\ (\mathsf{if}\ c\ \mathsf{then}\ p\ \mathsf{else}\ \eta h)}$

W-Uni $\quad \dfrac{\eta h; b = \mathsf{if}\ c\ \mathsf{then}\ \eta h'; \mathsf{tt}\ \mathsf{else}\ \eta u; \mathsf{ff} \qquad \eta h'; p = q; \eta h}{\eta h; \mathsf{while}\ b\ p = (\mathsf{while}\ c\ q); \eta u}$

The laws of while-monads roughly correspond to **Fixpoint**, **Codiagonal**, **Naturality** and **Uniformity**. This correspondence is somewhat allusive for **W-And**, which under $u = \mathsf{id}$ instantiates to the nicer looking

$$\frac{\eta h; b = \mathsf{ff}}{\mathsf{while}\,(b \;\texttt{\&\&}\; c)\; p = \mathsf{while}\; b\; (\mathsf{if}\; c\; \mathsf{then}\; p\; \mathsf{else}\; \eta h)}$$

However, this instance generally seems to be insufficient. Let us still consider it in more detail. The while-loop $\mathsf{while}\,(b \;\texttt{\&\&}\; c)\; p$ repeats p as long as both b and c are satisfied, and $\mathsf{while}\; b\; (\mathsf{if}\; c\; \mathsf{then}\; p\; \mathsf{else}\; \eta h)$ repeats $(\mathsf{if}\; c\; \mathsf{then}\; p\; \mathsf{else}\; \eta h)$ as long as b is satisfied, but the latter program still checks c before running p and triggers ηh only if c fails. The equality in the conclusion of the rule is thus due to the premise, which ensures that once ηh is triggered, the loop is exited at the beginning of the next iteration.

Note that using the following equations

$$\mathsf{do}\; p\; \mathsf{while}\; b = p; \mathsf{while}\; b\; p \tag{11}$$

$$\mathsf{while}\; b\; p = \;\mathsf{if}\; b\; \mathsf{then}\; (\mathsf{do}\; p\; \mathsf{while}\; b)\; \mathsf{else}\; \eta \tag{12}$$

we can define $(\mathsf{do}\; p\; \mathsf{while}\; b)$ from $(\mathsf{while}\; b\; p)$ and conversely obtain the latter from the former. Unsurprisingly, while-monads can be equivalently defined in terms of do-while.

Lemma 17. *Giving a while-monad structure on* $\mathbf{T}$ *is equivalent to equipping* $\mathbf{T}$ *with an operator, sending every* $b \in \mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X)$ *and every* $p \in \mathbf{C}(X, TX)$ *to* $(\mathsf{do}\; p\; \mathsf{while}\; b) \in \mathbf{C}(X, TX)$*, such that the following principles hold true:*

DW-Fix $\quad \mathsf{do}\; p\; \mathsf{while}\; b = p; \mathsf{if}\; b\; \mathsf{then}\; (\mathsf{do}\; p\; \mathsf{while}\; b)\; \mathsf{else}\; \eta$

DW-Or $\quad \mathsf{do}\; p\; \mathsf{while}\; (b \;||\; c) = \mathsf{do}\; (\mathsf{do}\; p\; \mathsf{while}\; b)\; \mathsf{while}\; c$

DW-And
$$\frac{\eta h; b = \eta u; \mathsf{ff}}{\begin{array}{r}\mathsf{if}\; c\; \mathsf{then}\;\; \mathsf{do}\; p\;\; \mathsf{while}\; (b \;\texttt{\&\&}\; (c \;||\; \eta u; \mathsf{ff}))\; \mathsf{else}\; \eta u \\ = \mathsf{do}\; (\mathsf{if}\; c\; \mathsf{then}\; p\; \mathsf{else}\; \eta h)\; \mathsf{while}\; b\end{array}}$$

DW-Uni
$$\frac{\eta h; p = q; \eta h' \qquad \eta h'; b = \mathsf{if}\; c\; \mathsf{then}\; \eta h; \mathsf{tt}\; \mathsf{else}\; \eta u; \mathsf{ff}}{\eta h; \mathsf{do}\; p\; \mathsf{while}\; b = (\mathsf{do}\; q\; \mathsf{while}\; c); \eta u}$$

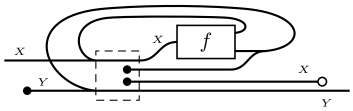
The relevant equivalence is witnessed by the equations (11) *and* (12)*.*

Finally, we can prove the equivalence of while-monads and Elgot monads, under an expressivity assumption, stating that sets of decisions $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}$ are sufficiently non-trivial. Such an assumption is clearly necessary, for, as we indicated above, the smallest family of decisions is the one with $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X) = \{\mathsf{ff}, \mathsf{tt}\}$, and it is not enough to express meaningful while-loops.

Theorem 18. *Suppose that for all* $X, Y \in |\mathbf{C}|$, $\eta(\mathsf{inl} + \mathsf{inr}) \in \mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X + Y)$*. Then* $\mathbf{T}$ *is and Elgot monad iff it is a while-monad w.r.t.* $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}$*. The equivalence is witnessed by mutual translations:* (10) *and*

$$f^{\dagger} = \eta\,\mathsf{inr}; (\mathsf{while}\; \eta(\mathsf{inl} + \mathsf{inr})\;\; [\eta\,\mathsf{inl}, f]); [\eta, \delta]. \tag{13}$$

Diagrammatically, $f^\dagger$ is expressed as



for $f\colon X \to T(Y+X)$.

Proof. Note that (13) is equivalent to

$$f^\dagger = \eta\,\mathsf{inr}; \big(\mathsf{do}\ [\eta\,\mathsf{inl}, f]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\big); [\eta, \delta]. \tag{14}$$

First, we show that the indicated translations are mutually inverse.

(i) $(\text{–})^\dagger \to \mathsf{while} \to (\text{–})^\dagger$: We need to show that

$$\eta\,\mathsf{inr}; (\mathsf{if}\ (\mathsf{inl}+\mathsf{inr})\ \mathsf{then}\ [\eta\,\mathsf{inl}, f]; \eta\,\mathsf{inr}\ \mathsf{else}\ \eta\,\mathsf{inl})^\dagger; [\eta, \delta] = f^\dagger.$$

Indeed,

$$\begin{aligned}
&\eta\,\mathsf{inr}; (\mathsf{if}\ (\mathsf{inl}+\mathsf{inr})\ \mathsf{then}\ [\eta\,\mathsf{inl}, f]; \eta\,\mathsf{inr}\ \mathsf{else}\ \eta\,\mathsf{inl})^\dagger; [\eta, \delta]\\
&\quad = [\eta,\delta]\cdot([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot[\eta\,\mathsf{inl}, f]]\,(\mathsf{inl}+\mathsf{inr}))^\dagger\cdot\eta\,\mathsf{inr}\\
&\quad = [\eta,\delta]\cdot[\eta\,\mathsf{inl}\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot f]^\dagger\cdot\eta\,\mathsf{inr} && /\!\!/\ \textbf{Naturality}\\
&\quad = ([\eta\,\mathsf{inl}\cdot[\eta,\delta], \eta\,\mathsf{inr}]\cdot[\eta\,\mathsf{inl}\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot f])^\dagger\cdot\eta\,\mathsf{inr}\\
&\quad = ([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot f])^\dagger\cdot\eta\,\mathsf{inr}\\
&\quad = ([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot f]\cdot[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}])^\dagger\cdot\eta\,\mathsf{inr}\\
&\quad = [\eta, ([\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}, \eta\,\mathsf{inr}]]\cdot\eta\,\mathsf{inr}\cdot f)^\dagger]\cdot[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}]\cdot\eta\,\mathsf{inr} && /\!\!/\ \textbf{Dinaturality}\\
&\quad = f^\dagger.
\end{aligned}$$

(ii) $\mathsf{while} \to (\text{–})^\dagger \to \mathsf{while}$: We need to show that

$$\eta\,\mathsf{inr}; \big(\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ \ [\eta\,\mathsf{inl}, \mathsf{if}\ b\ \mathsf{then}\ p; \mathsf{tt}\ \mathsf{else}\ \mathsf{ff}]\big); [\eta,\delta] = \mathsf{while}\ b\ p. \tag{15}$$

Observe that

$$\begin{aligned}
&[\eta\,\mathsf{inl}, \mathsf{if}\ b\ \mathsf{then}\ p; \mathsf{tt}\ \mathsf{else}\ \mathsf{ff}]\\
&\qquad = [[\mathsf{ff},\mathsf{ff}], [\mathsf{ff}, p;\mathsf{tt}]]\ [\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]\\
&\qquad = \ \mathsf{if}\ [\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]\ \mathsf{then}\ [\mathsf{ff}, p;\mathsf{tt}]\ \mathsf{else}\,[\mathsf{ff},\mathsf{ff}],
\end{aligned}$$

$$[\mathsf{ff},\mathsf{ff}]; \eta(\mathsf{inl}+\mathsf{inr}) = \eta[\mathsf{inl},\mathsf{inl}]; \mathsf{ff},$$

and

$$\begin{aligned}
&\eta(\mathsf{inl}+\mathsf{inr})\ \&\&\ ([\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]\ \,||\,\ \eta[\mathsf{inl},\mathsf{inl}];\mathsf{ff})\\
&\qquad = [\eta\,\mathsf{inl};\mathsf{ff}, \eta\,\mathsf{inr};\mathsf{tt}]\ \&\&\ [\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]
\end{aligned}$$

$$= [\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]$$

Hence, using **W-And**,

$$\begin{aligned}&\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl}, \mathsf{if}\ b\ \mathsf{then}\ p;\mathsf{tt}\ \mathsf{else}\ \mathsf{ff}]\\ &\qquad = \mathsf{while}\,[\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]\ [\mathsf{ff}, p;\mathsf{tt}].\end{aligned}$$

Now,

$$\begin{aligned}&\eta\,\mathsf{inr}; [\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]\\ &\qquad = \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff},\end{aligned}$$

$$\eta\,\mathsf{inr}; [\,\mathsf{ff}, p;\mathsf{tt}] = p;\eta\,\mathsf{inr}\,.$$

Hence, using **W-Uni**,

$$\begin{aligned}&\eta\,\mathsf{inr}; (\mathsf{while}\,[\eta\,\mathsf{inl};\mathsf{ff}, \mathsf{if}\ b\ \mathsf{then}\ \eta\,\mathsf{inr};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}]\ [\mathsf{ff}, p;\mathsf{tt}]); [\eta,\delta]\\ &\qquad = (\mathsf{while}\ b\ p);\eta\,\mathsf{inl};[\eta,\delta]\\ &\qquad = \mathsf{while}\ b\ p,\end{aligned}$$

and we are done with the proof of (15).

(iii) Let us check that the laws of Elgot monads follow from the laws of while-monads under the encoding (13).

Fixpoint: Suppose $f\colon X \to T(Y+X)$ and let $p = \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl}, f]$. Note that

$$\eta\,\mathsf{inl}; p = \eta\,\mathsf{inl}, \tag{16}$$

which follows by applying **W-Uni** to the identities $\eta\,\mathsf{inl};\eta(\mathsf{inl}+\mathsf{inr}) = \mathsf{if}\ \mathsf{ff}\ \mathsf{then}\ \eta\,\mathsf{inl};\mathsf{tt}\ \mathsf{else}\ \eta\,\mathsf{inl};\mathsf{ff}$ and $\eta\,\mathsf{inl};[\eta\,\mathsf{inl}, f] = \eta\,\mathsf{inl}$ and noting that $\mathsf{while}\ \mathsf{ff}\ (\eta\,\mathsf{inl}) = \eta$. Then

$$\begin{aligned}f^\dagger &= \eta\,\mathsf{inr}; p; [\eta,\delta]\\ &= \eta\,\mathsf{inr}; (\mathsf{if}\ (\mathsf{inl}+\mathsf{inr})\ \mathsf{then}\ [\eta\,\mathsf{inl}, f]; p\ \mathsf{else}\ \eta); [\eta,\delta] && /\!\!/\ \textbf{W-Fix}\\ &= [\eta,\delta]\cdot[\eta, p\cdot[\eta\,\mathsf{inl}, f]](\mathsf{inl}+\mathsf{inr})\cdot\eta\,\mathsf{inr}\\ &= [\eta,\delta]\cdot p\cdot f\\ &= [\eta,\delta]\cdot[p\cdot\eta\,\mathsf{inl}, p\cdot\eta\,\mathsf{inr}]\cdot f\\ &= [\eta,\delta]\cdot[\eta\,\mathsf{inl}, p\cdot\eta\,\mathsf{inr}]\cdot f && /\!\!/\ (16)\\ &= [\eta, [\eta,\delta]\cdot p\cdot\eta\,\mathsf{inr}]\cdot f\\ &= f; [\eta, \eta\,\mathsf{inr}; p; [\eta,\delta]]\\ &= [\eta, f^\dagger]\cdot f.\end{aligned}$$

Naturality: By Proposition 13, it suffices to show that

$$\eta h\cdot f^\dagger = (\eta(h+\mathsf{id})\cdot f)^\dagger$$

for all $f\colon X \to T(Y+X)$ and $h\colon Y \to TZ$. Note that

$$\eta(h+\mathsf{id});\eta(\mathsf{inl}+\mathsf{inr}) = \mathsf{if}\ \eta(\mathsf{inl}+\mathsf{inr})\ \mathsf{then}\ \eta(h+\mathsf{id});\mathsf{tt}\ \mathsf{else}\ \eta(h+\mathsf{id});\mathsf{ff},$$
$$\eta(h+\mathsf{id});[\eta\,\mathsf{inl},\eta(h+\mathsf{id})\cdot f] = [\eta\,\mathsf{inl},f];\eta(h+\mathsf{id}),$$

hence, by **W-Uni**,

$$\begin{aligned}\eta(h+\mathsf{id});&\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl},\eta(h+\mathsf{id})\cdot f]\\ &= (\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl},f]);\eta(h+\mathsf{id}).\end{aligned}$$

Therefore, Then

$$\begin{aligned}\eta h\cdot f^\dagger &= \eta\,\mathsf{inr};(\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl},f]);[\eta,\delta];\eta h\\ &= \eta\,\mathsf{inr};(\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl},f]);\eta(h+\mathsf{id});[\eta,\delta]\\ &= \eta\,\mathsf{inr};\eta(h+\mathsf{id});(\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ [\eta\,\mathsf{inl},\eta(h+\mathsf{id})\cdot f]);[\eta,\delta]\\ &= (\eta(h+\mathsf{id})\cdot f)^\dagger,\end{aligned}$$

and we are done.

Codiagonal: Let $f\colon X \to T((Y+X)+X)$. We will work with the translation (14). Let $Z=(Y+X)+X$, fix the following morphisms:

$$\begin{aligned}p &= [[\eta\,\mathsf{inl}\,\mathsf{inl},f],f]\colon Z \to TZ,\\ b &= \eta(\mathsf{inl}+\mathsf{inr})\colon Z \to T(Z+Z),\\ c &= [\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{inl}\,\mathsf{inr}),\eta\,\mathsf{inr}\,\mathsf{inr}]\colon Z \to T(Z+Z),\\ d &= [\eta,\eta\,\mathsf{inr}]\colon Z \to T(Y+X),\end{aligned}$$

observe that $b \mathrel{||} c = c$ and that

$$\begin{aligned}d;[\eta\,\mathsf{inl},f;d] &= p;d,\\ d;\eta(\mathsf{inl}+\mathsf{inr}) &= \mathsf{if}\ c\ \mathsf{then}\ d;\mathsf{tt}\ \mathsf{else}\ d;\mathsf{ff},\end{aligned} \tag{17}$$

which by **DW-Uni** entails

$$d;\mathsf{do}\ [\eta\,\mathsf{inl},f;d]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr}) = (\mathsf{do}\ p\ \mathsf{while}\ c);d. \tag{18}$$

It is easy to see by **DW-Uni** that

$$f^\dagger = \eta\,\mathsf{inr};(\mathsf{do}\ [\eta\,\mathsf{inl},f]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr}));d. \tag{19}$$

Next, observe that $p=[\eta(\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inr}];[\eta\,\mathsf{inl},f]$, and hence

$$\begin{aligned}(\mathsf{do}\ p\ \mathsf{while}\ b);d &= [\eta(\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inr}];(\mathsf{do}\ [\eta\,\mathsf{inl},f]\ \mathsf{while}\ b);d && /\!/\ \textbf{DW-Uni}\\ &= [[\eta\,\mathsf{inl}\,\mathsf{inl},\eta\,\mathsf{inr}],\eta\,\mathsf{inr}];(\mathsf{do}\ [\eta\,\mathsf{inl},f]\ \mathsf{while}\ b);d\\ &= [[\eta\,\mathsf{inl},f^\dagger],f^\dagger] && /\!/\ (19)\\ &= d;[\eta\,\mathsf{inl},f^\dagger],\end{aligned}$$

which, together with (17) by **DW-Uni** yields

$$d; (\mathsf{do}\ [\eta\,\mathsf{inl}, f^{\dagger}]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})) = (\mathsf{do}\ (\mathsf{do}\ p\ \mathsf{while}\ b)\ \mathsf{while}\ c); d \tag{20}$$

Finally, we can proceed with the proof of **Codiagonal**:

$$\begin{aligned}
&([\eta, \eta\,\mathsf{inr}] \cdot f)^{\dagger} \\
&\quad = \eta\,\mathsf{inr}; (\mathsf{do}\ [\eta\,\mathsf{inl}, f \cdot d]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})); [\eta, \delta] \\
&\quad = \eta\,\mathsf{inr}; d; (\mathsf{do}\ [\eta\,\mathsf{inl}, d \cdot f]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})); [\eta, \delta] \\
&\quad = \eta\,\mathsf{inr}; (\mathsf{do}\ p\ \mathsf{while}\ c); d; [\eta, \delta] && /\!\!/\ (18) \\
&\quad = \eta\,\mathsf{inr}; (\mathsf{do}\ p\ \mathsf{while}\ (b \mathrel{||} c)); d; [\eta, \delta] \\
&\quad = \eta\,\mathsf{inr}; (\mathsf{do}\ (\mathsf{do}\ p\ \mathsf{while}\ b)\ \mathsf{while}\ c); d; [\eta, \delta] && /\!\!/\ \textbf{DW-Or} \\
&\quad = \eta\,\mathsf{inr}; d; (\mathsf{do}\ [\eta\,\mathsf{inl}, f^{\dagger}]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})); [\eta, \delta] && /\!\!/\ (20) \\
&\quad = \eta\,\mathsf{inr}; (\mathsf{do}\ [\eta\,\mathsf{inl}, f^{\dagger}]\ \mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})); [\eta, \delta] \\
&\quad = f^{\dagger\dagger}.
\end{aligned}$$

Uniformity: Suppose that $g \cdot \eta h = \eta(\mathsf{id} + h) \cdot f$. This entails

$$\begin{aligned}
\eta(\mathsf{id} + h); [\eta\,\mathsf{inl}, g] &= [\eta\,\mathsf{inl}, g \cdot \eta h] \\
&= [\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}, \eta\,\mathsf{inr} \cdot \eta h] \cdot f] \\
&= [\eta\,\mathsf{inl}, f]; \eta(\mathsf{id} + h),
\end{aligned}$$

and note that

$$\begin{aligned}
&\eta(\mathsf{id} + h); \eta(\mathsf{inl}+\mathsf{inr}) \\
&\qquad = \eta(\mathsf{inl} + \mathsf{inr}\,h) \\
&\qquad = \eta(\mathsf{inl}+\mathsf{inr}); \eta((\mathsf{id} + h) + (\mathsf{id} + h)) \\
&\qquad = \ \mathsf{if}\ (\mathsf{inl}+\mathsf{inr})\ \mathsf{then}\ \eta(h + \mathsf{id}); \mathsf{tt}\ \mathsf{else}\ \eta(h + \mathsf{id}); \mathsf{ff}\,.
\end{aligned}$$

Therefore, using **W-Uni**,

$$\begin{aligned}
\eta h; g^{\dagger} &= \eta h; \eta\,\mathsf{inr}; (\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ \ [\eta\,\mathsf{inl}, g]); [\eta, \delta] \\
&= \eta\,\mathsf{inr}\,h; (\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ \ [\eta\,\mathsf{inl}, g]); [\eta, \delta] \\
&= \eta\,\mathsf{inr}; \eta(\mathsf{id} + h); (\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ \ [\eta\,\mathsf{inl}, g]); [\eta, \delta] \\
&= \eta\,\mathsf{inr}; (\mathsf{while}\ \eta(\mathsf{inl}+\mathsf{inr})\ \ [\eta\,\mathsf{inl}, f]); [\eta, \delta] \\
&= f^{\dagger}.
\end{aligned}$$

(iv) Finally, we check that the laws of while-monads follow from those of Elgot monads. To that end, we verify the properties, listed in Lemma 17, which is equivalent. First of all, note that by (11) and (10),

$$\begin{aligned}
\mathsf{do}\ p\ \mathsf{while}\ b &= p; \mathsf{while}\ b\ p && /\!\!/\ (11) \\
&= p; (\mathsf{if}\ b\ \mathsf{then}\ p; \mathsf{ff}\ \mathsf{else}\ \mathsf{ff})^{\dagger} && /\!\!/\ (10)
\end{aligned}$$

$$= ([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot p]\cdot b)^\dagger \cdot p$$
$$= [\eta, ([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\cdot p]\cdot b)^\dagger]\cdot \eta\,\mathsf{inr}\cdot p$$
$$= ([\eta\,\mathsf{inl}, b]\cdot \eta\,\mathsf{inr}\cdot p)^\dagger \qquad /\!/ \ \textbf{Dinaturality}$$
$$= (b\cdot p)^\dagger.$$

We will use the resulting encoding expansion of $\mathsf{do}\ p\ \mathsf{while}\ b$ throughout.

DW-Fix:

$$\mathsf{do}\ p\ \mathsf{while}\ b = (b\cdot p)^\dagger$$
$$= [\eta, (b\cdot p)^\dagger]\cdot b\cdot p \qquad /\!/ \ \textbf{Fixpoint}$$
$$= [\eta, \mathsf{do}\ p\ \mathsf{while}\ b]\cdot b\cdot p$$
$$= p; \mathsf{if}\ b\ \mathsf{then}\ (\mathsf{do}\ p\ \mathsf{while}\ b)\ \mathsf{else}\ \eta.$$

DW-Or:

$$\mathsf{do}\ p\ \mathsf{while}\ (b\ ||\ c)$$
$$= ((b\ ||\ c)\cdot p)^\dagger$$
$$= ([c, \eta\,\mathsf{inr}]\cdot b\cdot p)^\dagger$$
$$= ([\eta, \eta\,\mathsf{inr}]\cdot [\eta\,\mathsf{inl}\cdot c, \eta\,\mathsf{inr}]\cdot b\cdot p)^\dagger$$
$$= ([\eta\,\mathsf{inl}\cdot c, \eta\,\mathsf{inr}]\cdot b\cdot p)^{\dagger\dagger} \qquad /\!/ \ \textbf{Codiagonal}$$
$$= (c\cdot (b\cdot p)^\dagger)^\dagger \qquad /\!/ \ \textbf{Naturality}$$
$$= \mathsf{do}\ (\mathsf{do}\ p\ \mathsf{while}\ b)\ \mathsf{while}\ c.$$

DW-And: Assuming that $\eta h; b = \eta u; \mathsf{ff}$,

$$\mathsf{if}\ c\ \mathsf{then}\ \mathsf{do}\ p\ \mathsf{while}\ (b\ \&\&\ (c\ ||\ \eta u; \mathsf{ff}))\ \mathsf{else}\ \eta u$$
$$= [\eta u, ((b\ \&\&\ (c\ ||\ \eta u; \mathsf{ff}))\cdot p)^\dagger]\cdot c$$
$$= [\eta u, ([\eta\,\mathsf{inl}, T(u+\mathsf{id})\cdot c]\cdot b\cdot p)^\dagger]\cdot c$$
$$= [\eta, ([\eta\,\mathsf{inl}, T(u+\mathsf{id})\cdot c]\cdot b\cdot p)^\dagger]\cdot T(u+\mathsf{id})\cdot c$$
$$= ([\eta\,\mathsf{inl}, b\cdot p]\cdot T(u+\mathsf{id})\cdot c)^\dagger \qquad /\!/ \ \textbf{Dinaturality}$$
$$= ([\eta\,\mathsf{inl}\ u, b\cdot p]\cdot c)^\dagger$$
$$= ([b\cdot \eta h, b\cdot p]\cdot c)^\dagger \qquad /\!/ \ \text{assumption}$$
$$= (b\cdot [\eta h, p]\cdot c)^\dagger$$
$$= (b\cdot (\mathsf{if}\ c\ \mathsf{then}\ p\ \mathsf{else}\ \eta h))^\dagger$$
$$= \mathsf{do}\ (\mathsf{if}\ c\ \mathsf{then}\ p\ \mathsf{else}\ \eta h)\ \mathsf{while}\ b.$$

DW-Uni: Assuming that $\eta h; p = q; \eta h'$ and $\eta h'; b = \mathsf{if}\ c\ \mathsf{then}\ \eta h; \mathsf{tt}\ \mathsf{else}\ \eta u; \mathsf{ff}$,

$$\eta h; \mathsf{do}\ p\ \mathsf{while}\ b$$

$$
\begin{aligned}
&= (b \cdot p)^{\dagger} \cdot \eta h \\
&= ([\eta\, \mathsf{inl}\, u, \eta\, \mathsf{inr}] \cdot c \cdot q)^{\dagger} && /\!\!/ \ \textbf{Uniformity} \\
&= \eta\, \mathsf{inl}\, u \cdot (c \cdot q)^{\dagger} && /\!\!/ \ \textbf{Naturality} \\
&= (\mathsf{do}\ q\ \mathsf{while}\ c); \eta u.
\end{aligned}
$$

This concludes the proof. □

7 Kleene Monads as Elgot Monads

If hom-sets of the Kleisli category of a while-monad $\mathbf{T}$ are equipped with a semilattice structure and every $\mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X)$ is closed under that structure, we can define Kleene iteration as follows:

$$p^{*} = \mathsf{while}\,(\mathsf{ff} \vee \mathsf{tt})\ p.$$

That is, at each iteration we non-deterministically decide to finish or to continue.

Given a decision $b \in \mathbf{C}^{\mathsf{d}}_{\mathbf{T}}(X)$, let $b? = (\mathsf{if}\ b\ \mathsf{then}\ \eta\ \mathsf{else}\ \delta) \in \mathbf{C}(X, TX)$. The standard way to express while-loops via Kleene iteration is as follows:

$$\mathsf{while}\ b\ p = (b?; p)^{*}; (\sim b)?$$

If the composite translation $\mathsf{while} \to (\text{–})^{*} \to \mathsf{while}$ was a provable identity, this would essentially mean equivalence of Kleene iteration and while with non-determinism. This is generally not true, unless we postulate more properties that connect while and nondeterminism. We leave for future work the problem of establishing a minimal set of such laws. Here, we only establish the equivalence for the case when the induced Kleene iteration satisfies Kleene monad laws. To start off, we note an alternative to **Uniformity**, obtained by replacing the reference to pure morphisms with the reference to a larger class consisting of those h, for which $\delta \cdot h = \delta$. We need this preparatory step to relate Elgot iteration and Kleene iteration, since the latter does not hinge on a postulated class of pure morphisms, while the former does.

Definition 19 (Strong Uniformity). Given an Elgot monad $\mathbf{T}$, the *strong uniformity* law is as follows:

$$\textbf{Uniformity}^{\bigstar}: \qquad \frac{\delta \cdot h = \delta \qquad g \cdot h = [\eta\, \mathsf{inl}, \eta\, \mathsf{inr} \cdot h] \cdot f}{g^{\dagger} \cdot h = f^{\dagger}}$$

where $h\colon X \to TZ$, $g\colon Z \to T(Y + Z)$, and $f\colon X \to T(Y + X)$.

Clearly, **Uniformity** is an instance of **Uniformity**$^{\bigstar}$.

Example 20. An example of Elgot monad that fails **Uniformity**$^{\bigstar}$ can be constructed as follows. Let $\mathbf{S}$ be the reader monad transform of the maybe-monad on **Set**: $SX = (X + 1)^2$, which is an Elgot monad, since the maybe-monad is so and Elgotness is preserved by the reader monad transformer. Let $TX = X \times (X + 1) + 1$ and note that T is a retract of S under

$$\rho\colon (X + 1) \times (X + 1) \cong X \times (X + 1) + (X + 1) \xrightarrow{\mathsf{id}+!} X \times (X + 1) + 1.$$

It is easy to check that ρ is a congruence w.r.t. the Elgot monad structure, and it thus induces an Elgot monad structure on $\mathbf{T}$ [17, Theorem 20].

Now, let $T_E = T(-+E)$ for some non-empty E. The Elgot monad structure of $\mathbf{T}$ induces an Elgot monad structure on $\mathbf{T}_E$. However, $\mathbf{T}_E$ fails **Uniformity**$^\bigstar$. Indeed, let $h\colon X \to (X+E) \times ((X+E)+1)+1$ and $f\colon X \to ((1+X)+E) \times (((1+X)+E)+1)+1$ be as follows:

$$h(x) = \mathsf{inl}(\mathsf{inl}\, x, \mathsf{inl}\,\mathsf{inr}\, e) \qquad\qquad f(x) = \mathsf{inl}(\mathsf{inr}\, e, \mathsf{inl}\,\mathsf{inl}\,\mathsf{inr}\, x)$$

where $e \in E$. Then $h \cdot \delta = \delta$, $f^\dagger(x) = \mathsf{inl}(\mathsf{inr}\, e, \mathsf{inr}\, \star)$, and $f \cdot h = [\eta\,\mathsf{inl}, \eta\,\mathsf{inr} \cdot h] \cdot f$, but $(f^\dagger \cdot h)(x) = \mathsf{inl}(\mathsf{inr}\, e, \mathsf{inl}\,\mathsf{inr}\, e) \neq \mathsf{inl}(\mathsf{inr}\, e, \mathsf{inr}\, \star) = f^\dagger(x)$.

Remark 21. Example 20 indicates that it is hard to come up with a general and robust notion of Elgot iteration, which would confine to a single category, without referring to another category of "well-behaved" (e.g. pure) morphisms. While the class of Elgot monads is closed under various monad transformers, the example shows that Elgot monads with strong uniformity are not even closed under the exception monad transformer.

We are in a position to relate Kleene monads and Elgot monads.

Theorem 22. *A monad $\mathbf{T}$ is a Kleene monad iff*

1. *$\mathbf{T}$ is an Elgot monad;*
2. *the Kleisli category of $\mathbf{T}$ is enriched over join-semilattices (without least elements) and join-preserving morphisms;*
3. *$\mathbf{T}$ satisfies $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger = \eta$;*
4. *$\mathbf{T}$ satisfies* **Uniformity**$^\bigstar$.

To prove the theorem, we need to mutually encode Kleene iteration and Elgot iteration. These encodings go back to Căzănescu and Ştefănescu [5]. Some preparatory steps are needed. The following is a standard property of Kleene algebra, which carries over to Kleene monads straightforwardly.

Lemma 23. $(f \vee g)^* = f^* \cdot (g \cdot f^*)^*$.

Next, observe the following.

Lemma 24. *For any monad $\mathbf{T}$, whose Kleisli category is enriched over join-semilattices and join-preserving morphisms, $[f_1, g_1] \vee [f_2, g_2] = [f_1 \vee f_2, g_1 \vee g_2]$ where $f_1, f_2\colon X \to TZ$, $g_1, g_2\colon Y \to TZ$.*

Proof. The goal is entailed by the equations

$$([f_1, g_1] \vee [f_2, g_2])\, \mathsf{inl} = [f_1 \vee f_2, g_1 \vee g_2]\, \mathsf{inl},$$
$$([f_1, g_1] \vee [f_2, g_2])\, \mathsf{inr} = [f_1 \vee f_2, g_1 \vee g_2]\, \mathsf{inr},$$

of which, we prove the first one. Indeed,

$$([f_1, g_1] \vee [f_2, g_2])\, \mathsf{inl} = ([f_1, g_1] \vee [f_2, g_2]) \cdot \eta\,\mathsf{inl}$$

$$
\begin{aligned}
&= [f_1, g_1] \cdot \eta \,\mathsf{inl} \vee [f_2, g_2] \cdot \eta \,\mathsf{inl} \\
&= f_1 \vee f_2 \\
&= [f_1 \vee f_2, g_1 \vee g_2] \,\mathsf{inl}
\end{aligned}
$$

The second equation is shown analogously. □

Proof (of Theorem 22). We modify the claim slightly by replacing Clause 2. with the stronger

2′. The Kleisli category of T is enriched over bounded join-semilattices and strict join-preserving morphisms, and $\delta = (\eta \,\mathsf{inr})^{\dagger} \colon X \to TY$ is the least element of $\mathbf{C}(X, TY)$.

Let us show that 1.–4. entail 2′.

- *Right strictness of Kleisli composition*: $f \cdot \delta = \delta$. Using naturality, $f \cdot \delta = f \cdot (\eta \,\mathsf{inr})^{\dagger} = ([\eta \,\mathsf{inl} \cdot f, \eta \,\mathsf{inr}] \cdot \eta \,\mathsf{inr})^{\dagger} = (\eta \,\mathsf{inr})^{\dagger} = \delta$.
- *Left strictness of Kleisli composition*: $\delta \cdot f = \delta$. Since $\eta \,\mathsf{inr} \cdot f = [\eta \,\mathsf{inl}, \eta \,\mathsf{inr} \cdot f] \cdot \eta \,\mathsf{inr}$, by strong uniformity, $\delta \cdot f = (\eta \,\mathsf{inr})^{\dagger} \cdot f = (\eta \,\mathsf{inr})^{\dagger} = \delta$.
- *δ is the least element*, equivalently, $f \vee \delta = f$ for all suitably typed f. It suffices to consider the special case $f = \eta$, for then $f \vee \delta = f \cdot (\eta \vee \delta) = f \cdot \eta = f$ for a general f.

Note that $(\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr}) \cdot (\eta \vee \delta) = [\eta \,\mathsf{inl}, \eta \,\mathsf{inr} \cdot (\eta \vee \delta)] \cdot (\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr})$, which by 3. and 4. entails $\eta \vee \delta = (\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr})^{\dagger} \cdot (\eta \vee \delta) = (\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr})^{\dagger} = \eta$.

Now, given $(-)^{\dagger}$ of an Elgot monad, whose Kleisli category is enriched over join-semilattices, let

$$
(f \colon X \to TX)^{*} = \big(\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr} \cdot f \colon X \to T(X + X)\big)^{\dagger}.
$$

Conversely, given $(-)^{*}$ of a Kleene monad, let

$$
(f \colon X \to T(Y + X))^{\dagger} = ([\eta, \delta] \cdot f) \cdot \big([\delta, \eta] \cdot f \colon X \to TX\big)^{*}.
$$

We are left to check that these transformations are mutually inverse and that the expected properties of defined operators are satisfied.

(i) $(-)^{\dagger} \to (-)^{*} \to (-)^{\dagger}$: Given $f \colon X \to T(Y + X)$, we need to show that

$$
([\eta, \delta] \cdot f) \cdot (\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr} \cdot [\delta, \eta] \cdot f)^{\dagger} = f^{\dagger}.
$$

Indeed,

$$
\begin{aligned}
&([\eta, \delta] \cdot f) \cdot (\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr} \cdot [\delta, \eta] \cdot f)^{\dagger} \\
&\quad = \big([\eta \,\mathsf{inl} \cdot [\eta, \delta] \cdot f, \eta \,\mathsf{inr}]^{\sharp} (\eta \,\mathsf{inl} \vee \eta \,\mathsf{inr} \cdot [\delta, \eta] \cdot f)\big)^{\dagger} && /\!\!/ \ \textbf{Naturality} \\
&\quad = ([\eta \,\mathsf{inl}, \delta] \cdot f \vee [\delta, \eta \,\mathsf{inr}] \cdot f)^{\dagger} \\
&\quad = ([\eta \,\mathsf{inl} \vee \delta, \delta \vee \eta \,\mathsf{inr}] \cdot f)^{\dagger} && /\!\!/ \ \text{Lemma 24}
\end{aligned}
$$

$$= ([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}] \cdot f)^\dagger$$
$$= f^\dagger.$$

(ii) $(–)^* \to (–)^\dagger \to (–)^*$: Given $f\colon X \to TX$, we need to show that

$$([\eta, \delta] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f)) \cdot ([\delta, \eta] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f))^* = f^*.$$

Indeed, $[\eta, \delta] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f) = \eta \vee \delta = \eta$, and $[\delta, \eta] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f) = \delta \vee \eta \cdot f = f$, and therefore the right-hand side reduces to $\eta \cdot f^* = f^*$.

Next, we show that from an Elgot monad we obtain a Kleene monad and back.

(iii) *From Elgot to Kleene*: We verify conditions from Proposition 4. Since enrichment in semilattices is assumed, it suffices to check properties 1.–4.

1. We have

$$\begin{aligned}
\eta \vee f^* \cdot f &= [\eta, f^*] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f) \\
&= [\eta, (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f)^\dagger] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f) \\
&= (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f)^\dagger && /\!\!/\ \textbf{Fixpoint} \\
&= f^*.
\end{aligned}$$

2. $\eta^* = (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger = \eta$ by the global assumption of the theorem.
3. We prove a stronger property $(f \vee \eta)^* = f^*$:

$$\begin{aligned}
(f \vee \eta)^* &= (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f \vee \eta\,\mathsf{inr})^\dagger \\
&= (T[\mathsf{id}, \mathsf{inr}]\,(\eta\,\mathsf{inl}\,\mathsf{inl} \vee \eta\,\mathsf{inl}\,\mathsf{inr} \cdot f \vee \eta\,\mathsf{inr}))^\dagger \\
&= (\eta\,\mathsf{inl}\,\mathsf{inl} \vee \eta\,\mathsf{inl}\,\mathsf{inr} \cdot f \vee \eta\,\mathsf{inr})^{\dagger\dagger} && /\!\!/\ \textbf{Codiagonal} \\
&= ([\eta\,\mathsf{inl} \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f), \eta\,\mathsf{inr}] \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr}))^{\dagger\dagger} \\
&= ((\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f) \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger)^\dagger && /\!\!/\ \textbf{Naturality} \\
&= (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot f)^\dagger && /\!\!/\ \text{assumption} \\
&= f^*.
\end{aligned}$$

4. Suppose that $f \cdot h = g \cdot f$. Then $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot g) \cdot f = \eta\,\mathsf{inl} \cdot f \vee \eta\,\mathsf{inr}\, f \cdot h = [\eta\,\mathsf{inl}, \eta\,\mathsf{inr} \cdot f] \cdot (\eta\,\mathsf{inl} \cdot f \vee \eta\,\mathsf{inr} \cdot h)$, which entails $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot g)^\dagger \cdot f = (\eta\,\mathsf{inl} \cdot f \vee \eta\,\mathsf{inr} \cdot h)^\dagger$, by strong uniformity.

$$\begin{aligned}
f \cdot h^* &= f \cdot (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot h)^\dagger \\
&= (\eta\,\mathsf{inl} \cdot f \vee \eta\,\mathsf{inr} \cdot h)^\dagger && /\!\!/\ \textbf{Naturality} \\
&= (\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr} \cdot g)^\dagger \cdot f && /\!\!/\ \textbf{Uniformity}^\star \\
&= g^* \cdot f
\end{aligned}$$

(iv) *From Kleene to Elgot*: We need to verify the axioms of Elgot monads, with **Uniformity** replaced by **Uniformity**$^{\bigstar}$.

Fixpoint : Given $f\colon X \to T(Y+X)$,

$$
\begin{aligned}
[\eta, f^{\dagger}] \cdot f &= ([\eta \vee \delta, \delta \vee f^{\dagger}] \cdot f) \\
&= [\eta, \delta] \cdot f \vee [\delta, f^{\dagger}] \cdot f && /\!\!/ \text{ Lemma 24} \\
&= [\eta, \delta] \cdot f \vee f^{\dagger} \cdot [\delta, \eta] \cdot f \\
&= ([\eta, \delta] \cdot f) \cdot (\eta \vee ([\delta, \eta] \cdot f)^{*} \cdot [\delta, \eta] \cdot f) \\
&= ([\eta, \delta] \cdot f) \cdot ([\delta, \eta] \cdot f)^{*} \\
&= f^{\dagger}.
\end{aligned}
$$

Naturality: Given $g\colon Y \to TZ$, $f\colon X \to T(Y+X)$,

$$
\begin{aligned}
g \cdot f^{\dagger} &= g \cdot ([\eta, \delta] \cdot f) \cdot ([\delta, \eta] \cdot f)^{*} \\
&= [g, \delta] \cdot f \cdot ([\delta, \eta] \cdot f)^{*} \\
&= [\eta, \delta] \cdot [\eta \,\mathsf{inl} \cdot g, \eta \,\mathsf{inr}] \cdot f \cdot ([\delta, \eta] \cdot [\eta \,\mathsf{inl} \cdot g, \eta \,\mathsf{inr}] \cdot f)^{*} \\
&= ([\eta \,\mathsf{inl} \cdot g, \eta \,\mathsf{inr}] \cdot f)^{\dagger}.
\end{aligned}
$$

Codiagonal: Let $f\colon X \to T((Y+X)+X)$ and show $(T[\mathsf{id}, \mathsf{inr}]\, f)^{\dagger} = f^{\dagger\dagger}$. Let $g = [[\delta, \eta], \delta] \cdot f\colon X \to TX$, $h = [\delta, \eta] \cdot f\colon X \to TX$. Then

$$
\begin{aligned}
(T[\mathsf{id}, \mathsf{inr}]\, f)^{\dagger} &= ([\eta, \delta] \cdot T[\mathsf{id}, \mathsf{inr}]\, f) \cdot ([\delta, \eta] \cdot T[\mathsf{id}, \mathsf{inr}]\, f)^{*} \\
&= [[\eta, \delta], \delta] \cdot f \cdot ([[\delta, \eta], \eta] \cdot f)^{*} \\
&= [[\eta, \delta], \delta] \cdot f \cdot ([\delta, \eta] \cdot f \vee [[\delta, \eta], \delta] \cdot f)^{*} \\
&= [[\eta, \delta], \delta] \cdot f \cdot (h \vee g)^{*} \\
&= [[\eta, \delta], \delta] \cdot f \cdot h^{*} \cdot (g \cdot h^{*})^{*} && /\!\!/ \text{ Lemma 23} \\
&= [\eta, \delta] \cdot [\eta, \delta] \cdot f \cdot h^{*} \cdot ([\delta, \eta] \cdot [\eta, \delta] \cdot f \cdot h^{*})^{*} \\
&= ([\eta, \delta] \cdot f^{\dagger}) \cdot ([\delta, \eta] \cdot f^{\dagger})^{*} \\
&= f^{\dagger\dagger}.
\end{aligned}
$$

Uniformity$^{\bigstar}$: Assume $f \cdot h = [\eta \,\mathsf{inl}, \eta \,\mathsf{inr} \cdot h] \cdot g$. This entails $([\delta, \eta] \cdot f) \cdot h = ([\delta, \eta] \cdot [\eta \,\mathsf{inl}, \eta \,\mathsf{inr} \cdot h])\, g = [\delta, h] \cdot g = h \cdot [\delta, \eta] \cdot g$. By Lemma 4,

$$([\delta, \eta] \cdot f)^{*} \cdot h = h \cdot ([\delta, \eta] \cdot g)^{*}.$$

We now have

$$
\begin{aligned}
f^{\dagger} \cdot h &= ([\eta, \delta] \cdot f) \cdot ([\delta, \eta] \cdot f)^{*} \cdot h \\
&= [\eta, \delta] \cdot f \cdot h \cdot ([\delta, \eta] \cdot g)^{*} \\
&= [\eta, \delta] \cdot [\eta \,\mathsf{inl}, \eta \,\mathsf{inr} \cdot h] \cdot g \cdot ([\delta, \eta] \cdot g)^{*} \\
&= ([\eta, \delta] \cdot g) \cdot ([\delta, \eta] \cdot g)^{*} \\
&= g^{\dagger}.
\end{aligned}
$$

This concludes the proof. □

In presence of assumptions 1.–3., the distinction between **Uniformity** and **Uniformity**$^\bigstar$ becomes very subtle.

Example 25 (Filter Monad). There is an Elgot monad $\mathbf{T}$, whose Kleisli category is enriched over bounded semilattices, $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger = \eta$, but $\mathbf{T}$ fails strong uniformity. We prove it by adapting Kozen's separating example for left-handed and right-handed Kleene algebras [25, Proposition 7].

Recall that the *filter monad* [6] sends every X to the set of all filters on X, equivalently to those maps $h\colon (X \to 2) \to 2$, which preserve $\top$ and $\wedge$: $h(\top) = \top$, $h(f \wedge g) = h(f) \wedge h(g)$ where $\top$ and $\wedge$ on $X \to 2$ are computed pointwise. For us, it will be more convenient to use the equivalent formulation, obtained by flipping the order on 2 (so, the resulting monad $\mathbf{T}$ could be actually called the *ideal monad*). Every TX is then the set of those $h\colon \mathcal{P}X \to 2$, for which

$$f(\emptyset) = \bot, \qquad\qquad f(s \cup t) = f(s) \vee f(t).$$

1. Note that Kleisli category $\mathbf{Set}_{\mathbf{T}}$ is dually isomorphic to a category $\mathbf{C}$, for which every $\mathbf{C}(X, Y)$ consists of functions $\mathcal{P}X \to \mathcal{P}Y$, preserving finite joins (in particular, monotone). This category has finite products: $\mathcal{P}\emptyset$ is the terminal object and $\mathcal{P}X \times \mathcal{P}Y = \mathcal{P}(X + Y)$, by definition.
2. Under this dual isomorphism, every morphism $f\colon X \to T(Y + X)$ corresponds to a morphism $\hat{f}\colon \mathcal{P}Y \times \mathcal{P}X \to \mathcal{P}X$ in $\mathbf{C}$ where we compute a fixpoint $\mathcal{P}Y \to \mathcal{P}X$ using the *Knaster-Tarski theorem*, and transfer it back to $\mathbf{C}$ as $f^\dagger\colon X \to TY$.
3. The construction of $f^\dagger$ entails both $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger \leqslant \eta$ and $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger \geqslant \eta$, hence $(\eta\,\mathsf{inl} \vee \eta\,\mathsf{inr})^\dagger = \eta$.
4. Enrichment in semilattices is obvious in view of the dual isomorphism of $\mathbf{Set}_{\mathbf{T}}$ and $\mathbf{C}$.
5. The **Fixpoint** law follows by construction. The remaining Elgot monad laws follow by *transfinite induction*.
6. If $\mathbf{T}$ was a Kleene monad, any $\mathbf{C}(X, X)$ would be a Kleene algebra, but Kozen showed that it is not, hence $\mathbf{T}$ is not a Kleene monad.
7. By Theorem 22, $\mathbf{T}$ fails **Uniformity**$^\bigstar$.

8 Conclusions

When it comes to modelling and semantics, many issues can be framed and treated in terms of universal algebra and coalgebra. However, certain phenomena, such as recursion, partiality, extensionality, require additional structures, often imported from the theory of complete partial orders, by enriching categories and functors, and devising suitable structures, such as recursion and more specifically iteration. In many settings though, iteration is sufficient, and can be treated as a self-contained ingredient whose properties matter, while a particular construction behind it does not. From this perspective, Elgot monads present a base fundamental building block in semantics.

We formally compared Elgot monads with Kleene monads, which are a modest generalization of Kleene algebras. In contrast to inherently categorical Elgot monads, Kleene algebra is a simple notion, couched in traditional algebraic terms. The price of this simplicity is a tight pack of laws, which must be accepted altogether, but which are well-known to be conflicting with many models of iteration. We proposed a novel notion of *while-monad*, which in the categorical context are essentially equivalent to Elgot monads, and yet while-monads are morally a three-sorted algebra over (Boolean) decisions, programs and certain well-behaved programs (figuring in the so-called uniformity principle). This is somewhat similar to the extension of Kleene algebra with *tests* [29]. The resulting *Kleene algebra with tests* is two-sorted, with tests being a subsort of programs, and forming a Boolean algebra. Our *decisions* unlike tests do not form a subsort of programs, but they do support operations of Boolean algebra, without however complying with all the Boolean algebra laws. We have then related Elgot monads (and while-monads) with Kleene monads, and as a side-effect produced a novel axiomatization of Kleene algebra (Proposition 4), based on a version of the uniformity principle. We regard the present work as a step towards bringing the gap between Elgot iteration and Kleene iteration, not only in technical sense, but also in the sense of concrete usage scenarios. We plan to further explore algebraic axiomatizations of iteration, based on the current axiomatization of while-monads.

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