Optimally Computing Compressed Indexing Arrays Based on the Compact Directed Acyclic Word Graph

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Abstract. In this paper, we present the first study of the computational complexity of converting an automata-based text index structure, called the Compact Directed Acyclic Word Graph (CDAWG), of size e for a text T of length n into other text indexing structures for the same text, suitable for highly repetitive texts: the run-length BWT of size r, the $irreducible\ PLCP\ array$ of size r, and the quasi- $irreducible\ LPF\ array$ of size e, as well as the lex-parse of size O(r) and the LZ77-parse of size z, where $r, z \le e$. As main results, we showed that the above structures can be optimally computed from either the CDAWG for T stored in read-only memory or its self-index version of size e without a text in O(e) worst-case time and words of working space. To obtain the above results, we devised techniques for enumerating a particular subset of suffixes in the lexicographic and text orders using the forward and backward search on the CDAWG by extending the results by Belazzougui $et\ al.$ in 2015.

Keywords: Highly-repetitive text \cdot suffix tree \cdot longest common prefix

1 Introduction

Backgrounds. Compressed indexes for repetitive texts, which can compress a text beyond its entropy bound, have attracted a lot of attention in the last decade in information retrieval [13]. Among them, the most popular and powerful compressed text indexing structures [13] are the run-length Burrows-Wheeler transformation (RLBWT) [13] of size r, the Lempel-Ziv-parse (LZ-parse) [14] of size r, and finally the Compact Directed Acyclic Word Graph (CDAWG) [5] of size r. It is known [13] that their size parameters r, r, and r can be much smaller than the information theoretic upperbound of a text for highly-repetitive texts such as collections of genome sequences and markup texts [13]. Among these repetition-aware text indexes, we focus on the CDAWG for a text r, which is a minimized compacted finite automaton with r transitions for the set of all

suffixes of T [5]; It is the edge-labeled DAG obtained from the suffix tree for T by merging all isomorphic subtrees [8], and can be constructed from T in linear time and space [13]. The relationships between the size parameters r, z, and e of the RLBWT, LZ-parse, and CDAWG has been studied by, e.g. [2,4,6,11,12,15]; However, it seems that the actual complexity of conversion the CDAWG into the other structures in sublinear time and space has not been explored yet [13].

Research goal and main results. In this paper, we study for the first time the conversion problem from the CDAWG for T into the following compressed indexing structures for T:

- (i) the run-length BWT (RLBWT) [13] of size $r \leq e$;
- (ii) the irreducible permuted longest common prefix (PLCP) array [9] of size r;
- (iii) the quasi-irreducible longest previous factor (LPF) array [7] of size e (Sec. 2);
- (iv) the *lex-parse* [14] with size at most 2r = O(r); and
- (v) LZ-parse [14] with size $z \leq e$.

After introducing some notions and techniques, we present in Sec. 4 and 5 efficient algorithms for solving the conversion problem from the CDAWG into the aforementioned compressed indexing structures. We obtain the following results.

Main results (Thm. 4.1, 5.1, and 5.2). For any text T of length n over an integer alphabet Σ , we can solve the conversion problems from the CDAWG G of size e for T into the above compressed index array structures (i)–(v) for the same text in O(e) worst-case time using O(e) words of working space, where an input G is given in the form of either the CDAWG of size e for T stored in read-only memory, or its self-index version [3,16] of size O(e) without a text.

Techniques. To obtain the above results, we devise in Sec. 3 techniques for enumerating a *canonical subset of suffixes* in the lexicographic and text orders using the *forward and backward DFS* on the CDAWG by extending by [4].

Related Work. On the relationships between parameters r, z, and zagainst the text length n, Belazzougui and Cunial [4] have shown that $r \leq e$ and $z \leq e$ hold. Kempa [10] showed that the compressed PLCP and CSA and LZ-parse can be computed in $O(n/\log_{\sigma} n + r \operatorname{polylog}(n))$ time and space from RLBWT-based index of size r and T. It is shown by [11] that the RLBWT of size r for T can be computed from the LZ77-parse of size z for the same text in $r = O(z \operatorname{polylog}(n))$ time and space and $r = O(z \log^2 n)$. Concerning to conversion from the CDAWG G for T, we observe that G can be converted into the LZ78-parse of size $z_{78} \ge z$ in $O(e + z_{78} \log z_{78})$ time and space via an O(e)-sized grammar [3] on G [2]. Discussions. For some texts, e can be as small as r or zalthough e can be polynomially larger than z for other texts [11]. For the class of Thue-Morse words, ³ Radoszewski and Rytter [15] showed that $e = O(\log n)$, while Brlek et al. [6, Theorem 2] showed that $r = \Theta(\log n)$. Hence, for such a class, there is a chance that our O(e)-time method can run as fast as other $O(r \operatorname{polylog}(n))$ -time methods for some conversion problem. On the contrary, Mantaci et al. [12] showed that $e = \Theta(\log n)$ and r = O(1) for Fibonacci words.

The *n*-th Thue-Morse word is $\tau_n = \varphi^n(0)$ for the morphism $\varphi(0) = 01$ and $\varphi(1) = 10$.

2 Preliminaries

We prepare the necessary notation and definitions in the following sections. For precise definitions, see the literature [8,13] or the full paper [1].

Basic definitions and notation. For any integers $i \leq j$, the notation [i..j] or i..j denotes the interval $\{i,i+1,\ldots,j\}$ of integers, and [n] denotes $\{1,\ldots,n\}$. For a string $S[1..n] = S[1]\cdots S[n]$ of length n and any $i \leq j$, we denote $S[i..j] = S[i]S[i+1]\cdots S[j]$. Then, S[1..j], S[i..j], and S[i..|S|] are a prefix, a factor, and a suffix of S, resp. The reversal of S is $S^{-1} = S[n]\cdots S[1]$. Throughout, we assume a string $T[1..n] \in \Sigma^n$, called a text, over an alphabet Σ with symbol order \leq_{Σ} , which is terminated by the end-marker T[n] = `\$' such that $\$ \preccurlyeq_{\text{lex}} a$ for $\forall a \in \Sigma$. Suf $(T) = \{T_1, \ldots, T_n\} \subseteq \Sigma^+$ denotes the set of all of n non-empty suffixes of T, where $T_p := T[p..n]$ is the p-th suffix with position p. For any suffix $S \in \Sigma^*$ in Suf(T), we define: (i) pos(S) := n + 1 - |S| gives the starting position S. (ii) rnk(S) gives the lexicographic rank of S in Suf(T). lcp(X,Y) denotes the length of the longest common prefix of strings X and Y. In what follows, we refer to any suffix as S, any factors of T as X,Y,U,L,P,\ldots , nodes of a graph as v, w, \ldots , and edges as f, g, \ldots , which are possibly subscripted.

String order and extension. A string order is any total order \preccurlyeq over strings in Σ^* . Its co-relation \preccurlyeq^{co} is defined by $X \preccurlyeq^{\text{co}} Y \overset{\text{def}}{\Leftrightarrow} X^{-1} \preccurlyeq Y^{-1}$. The order \preccurlyeq is said to be extensible if $\forall a \in \Sigma, \forall X, Y \in \Sigma^*, aX \preccurlyeq aY \Leftrightarrow X \preccurlyeq Y$, and co-extensible if its co-order \preccurlyeq^{co} is extensible. We denote by $\preccurlyeq_{\text{lex}}$ the lexicographic order over Σ^* extending \preccurlyeq_{Σ} over Σ , and by $\preccurlyeq_{\text{pos}}$ the text order defined as: $X \preccurlyeq_{\text{pos}} Y \Leftrightarrow |X| \geq |Y|$. Both of $\preccurlyeq_{\text{pos}}$ and $\preccurlyeq_{\text{lex}}$ are extensible [14], while $\preccurlyeq_{\text{pos}}$ is co-extensible. A factor X in T is left-maximal if we can prepend some symbols to X without changing the set of its end-positions in T [4, 5].

Compact directed acyclic word graph. We assume that the reader is familiar with the suffix tree and the CDAWG [5,8]. The suffix tree [8] for a text T[1..n], denoted by Stree(T), is the compacted trie for the set Suf(T) of all suffixes of T. The CDAWG [5] for a text T, denoted CDAWG(T), is an edge-labeled DAG $G = (\mathcal{V}, \mathcal{E}, suf, root, sink)$ obtained from Stree(T) by merging all the isomorphic subtrees, where \mathcal{V}, \mathcal{E} , and suf are sets of nodes, labeled edges, and suffix links, $N_-(v)$ and $N_+(v)$, resp., denotes the sets of incoming and outgoing edges at node v. root and $sink \in \mathcal{V}$ are the distinguished nodes with $|N_-(v)| = 0$ and $|N_+(v)| = 0$, resp. Each edge f = (v, X, w) goes from node src(f) = v to node dst(f) = w with string label $lab(v) = X \in \Sigma^+$. Path(u, v) denotes the set of all paths from node u to node v, whose elements are called u-to-v paths. We denote the size of G by $e := e_R + e_L$, where $e_R := |\mathcal{E}(G)|$ and $e_L := |suf_G|$. The CDAWG can be stored in O(e) words of space by representing each edge label by its length and a pointer to T. In Fig. 1, we show examples of the suffix tree and the CDAWG for the same text T = aabaababb over $\Sigma = \{a, b, \$\}$.

Indexing arrays. $SA, PLCP, LPF \in [n]^n$ and $BWT \in \Sigma^n$ denote the suffix [13], permuted longest common prefix [9], longest previous factor [7], and the BWT arrays for a text T[1..n]: for any rank k and position $p \in [n]$, SA[k]

⁴ Our extensible order seems slightly different from [14], but essentially the same.

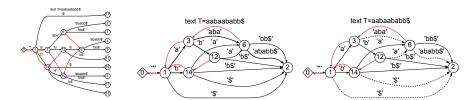


Fig. 1: Suffix tree (left) and CDAWG (middle, right) for T = aabaababb\$. Thick and dashed lines indicate (-)- and (+)-primary edges forming the forward and backward search trees \mathcal{T}_{-} and \mathcal{T}_{+} in Sec. 3, resp. Red lines indicate suffix links.

stores the position p of the suffix with rank k; BWT[k] is T[n] if SA[k] = 1 and T[SA[k] - 1] otherwise; PLCP[p] is 0 if p = SA[1] and $lcp(T_p, T_q)$ for $q = SA[SA^{-1}[p] - 1]$ otherwise; LPF[p] is $\max(\{lcp(T_p, T_q) \mid T_q \preccurlyeq_{pos} T_p, q \in [n]\} \cup \{0\})$. RLBWT is the run-length encoded BWT. The irreducible PLCP is obtained from PLCP by sampling such rank-value pairs (p, PLCP[p]) that the rank $i = SA^{-1}[p]$ satisfies $BWT[i] \neq BWT[k-1]$. The lex-parse [14] and LZ-parse [14] of T are obtained from PLCP and LPF, resp., as the partition $T = F_1 \dots F_u$ of T with u phrases $F_i = T[p_i \dots p_i + \ell_i - 1]$ such that $p_1 = 1$ and $\ell_i = \max(L[p_i], 1), \forall i \in [u]$, where L is either PLCP or LPF.

3 Techniques

We introduce novel techniques for generating elements of a compressed indexing array using so-called *canonical suffixes* in \leq_{lex} and \leq_{pos} based on the forward and backward DFS on the CDAWG by extending the results by *et al.* [3,4].

Our approach. We employ the one-to-one correspondence between all of n root-to-sink paths in $\operatorname{Path}(G)$ and all of n non-empty suffixes of T in $\operatorname{Suf}(T) \subseteq \Sigma^*$. Due to the determinism of $\operatorname{CDAWG}(T)$ as a DFA [5], we can interchangeably use a path $\pi = (f_1, \ldots, f_k)$ in $\operatorname{Path}(G)$ and a factoring $S = X_1 \ldots X_k = \operatorname{str}(\pi)$ of a suffix S in $\operatorname{Suf}(T)$, where $X_i = \operatorname{lab}(f_i)$ for all $i \in [k]$.

A basic idea of our approach for computing a sparse indexing array $\widetilde{A}:Dom\to Range$ with domain Dom is to represent \widetilde{A} as the graph

$$\widetilde{A} = \{ \, (\mathrm{idx}(S), \mathrm{val}(S)) \mid S \in \mathcal{S} \, \} \subseteq Dom \times Range \tag{1}$$

of array \widetilde{A} with the set of indexes $\{ \operatorname{idx}(S) \mid S \in \mathcal{S} \}$ using a combination of

- (i) a subset $S \subseteq Path(G)$ of root-to-sink paths,
- (ii) a mapping $idx : S \to Dom$ that assigns the index, and
- (iii) a mapping val : $S \to Range$ that assigns the value.

For actual computation of \widetilde{A} on the CDAWG G for T, we make the DFS based on a pair $\Pi = (\preccurlyeq_-, \preccurlyeq_+) = (\preccurlyeq_{pos}, \preccurlyeq_{lex})$ of ordering over paths of G. Two

⁵ The correspondence between Path(G) and Suf(T) can be extended to that between all root-to-node paths in G and all right-maximal factors [5] of T, but not used here.

ordering have different roles. For example, if we want to compute the run-length BWT for T (Sec. 4), the set of primary edges (in the sense of [5]) w.r.t. the first ordering \leq_{pos} defines a spanning tree T over G from the root, whereas the second ordering \leq_{lex} specifies the order of traversal. Finally, the set of secondary edges w.r.t. \leq_{pos} provides a collection C of target values to seek in the DFS. Actually, we can extract an equal-letter run from each secondary edge in constant time.

On the contrary, if we want to compute the sparse version of LPF array (Sec. 5), we make the backward DFS of G from the sink based on the pair $\Pi = (\preccurlyeq_{pos}, \preccurlyeq_{pos})$. Then, the set of primary edges w.r.t. the second ordering \preccurlyeq_{pos} defines a spanning tree, the first ordering \preccurlyeq_{pos} specifies the text order, and the set of secondary edges w.r.t. the first ordering \preccurlyeq_{pos} provides a collection of target values, which are the LCP values of neighboring suffixes. The PLCP array can be computed in a similar manner, but with the pair $\Pi = (\preccurlyeq_{pos}, \preccurlyeq_{lex})$.

Ordered CDAWGs. We assume a pair $\Pi = (\preccurlyeq -, \preccurlyeq_+)$ of co-extensible and extensible string orders, where $\preccurlyeq -$ and \preccurlyeq_+ are called *upper* and *lower path orders*, resp. The ordered CDAWG for T under a pair $\Pi = (\preccurlyeq -, \preccurlyeq_+)$ of path orders, denoted by $G = CDAWG(T; \preccurlyeq -, \preccurlyeq_+)$, is the CDAWG G for T whose incoming and outgoing edges are ordered by edge orderings $(\leqslant_-^E, \leqslant_+^E)$ compatible with path orders defined as follows. In what follows, for any node v, each sign $\delta \in \{-, +\}$ indicates the *side* of G, where the prefix (-)- reads "upper," while (+)- reads "upper" in what follows.

Upper/lower sets and their representatives. Consider the sets $\mathcal{U}_{-}(v) := \{ \operatorname{str}(\pi) \mid \pi \in \operatorname{Path}(root(G), v) \}$ and $\mathcal{U}_{+}(v) := \{ \operatorname{str}(\pi) \mid \pi \in \operatorname{Path}(v, sink(G)) \} \subseteq \mathcal{E}^*$ of prefixes and suffixes of all root-to-sink paths. Each member of \mathcal{U}_{-} and \mathcal{U}_{+} are called *upper* and *lower paths*, resp. For any $\delta \in \{-, +\}$, we define the δ -representative of the set \mathcal{U}_{δ} by the smallest element $\operatorname{repr}_{\delta}(v)$ of $\mathcal{U}_{\delta}(v)$ under \preccurlyeq_{δ} , i.e., $\operatorname{repr}_{\delta}(v) := \min_{\preccurlyeq_{\delta}} \mathcal{U}_{\delta}(v)$; For example, $\operatorname{repr}_{-}(v)$ is the longest strings in $\mathcal{U}_{-}(v)$ and $\operatorname{repr}_{+}(v)$ is the lex-first string in $\mathcal{U}_{+}(v)$ under $\mathcal{I}(\preccurlyeq_{\operatorname{pos}}, \preccurlyeq_{\operatorname{lex}})$.

Remark 3.1. For any $\delta \in \{-,+\}$, for any $P \in \mathcal{U}_{\delta}(v)$, $P = \operatorname{repr}_{\delta}(v)$, if and only if P consists of $\mathcal{E}_{\delta}^{\star}$ -edges only. Furthermore, any factor P is left-maximal if and only if $P = \operatorname{repr}_{-}(v)$ for some node v under $\preccurlyeq_{-} = (\preccurlyeq_{\operatorname{pos}})$.

Compatible Edge orderings. Under the pair $\Pi_{\text{lex}}^{\text{pos}} = (\preccurlyeq_{\text{pos}}, \preccurlyeq_{\text{lex}})$ of path orderings, we define the pair $\Gamma_{\text{lex}}^{\text{pos}} = (\leqslant_{-,\text{pos}}^E, \leqslant_{+,\text{lex}}^E)$ of edge orderings by

$$\begin{split} f_1 \leqslant^E_{-,\mathrm{pos}} f_2 &\overset{\mathrm{def}}{\Leftrightarrow} |\mathtt{repr}_-(v_1)| + |X_1| \geq |\mathtt{repr}_-(v_2)| + |X_2| \\ f_1 \leqslant^E_{+,\mathrm{lex}} f_2 &\overset{\mathrm{def}}{\Leftrightarrow} \mathtt{lab}(f_1)[1] <_{\Sigma} \mathtt{lab}(f_2)[1], \end{split}$$

 $f_i = (v_i, X_i, w_i) \in \mathcal{E}$ be an edge for i = 1, 2. Under $\Pi_{\text{pos}}^{\text{pos}} = (\preccurlyeq_{\text{pos}}, \preccurlyeq_{\text{pos}})$, we define the pair $\Gamma_{\text{pos}}^{\text{pos}} = (\leqslant_{-,\text{pos}}^E, \leqslant_{+,\text{pos}}^E)$ of edge orderings, where $f_1 \leqslant_{+,\text{pos}}^E f_2 \stackrel{\text{def}}{\Leftrightarrow} |X_1| + |\text{repr}_{-}(w_1)| \geq |X_2| + |\text{repr}_{-}(w_2)|$.

⁶ The set $\mathcal{U}_{-}(v)$ has appeared as the equivalence class $[\overrightarrow{X}]_R$ of all factors with the same end positions in [5], while $\mathcal{U}_{+}(v)$ was recently introduced by [3, 4]. Indeed, $\mathcal{U}_{-}(v)$ encodes the node v itself, while $\mathcal{U}_{+}(v)$ encodes all end positions of such factors [5].

Classification of edges. We classify edges in $N_{\delta}(v)$ using the representative $\operatorname{repr}_{\delta}(v)$ under \leqslant^E_{δ} as follows. For $\delta \in \{-, +\}$, δ -edge $f \in N_{\delta}(v)$ is said to be δ -primary if $\operatorname{repr}_{\delta}(v)$ goes through f. We denote by $\mathcal{E}^{\star}_{\delta}$ the set of all δ -primary edges, and by $\overline{\mathcal{E}^{\star}_{\delta}} := \mathcal{E} - \mathcal{E}^{\star}_{\delta}$ the set of all δ -secondary edges. The same edge can be both (-)-primary and (+)-secondary, and vice versa. We remark that it gives the partition $\mathcal{E} = \mathcal{E}^{\star}_{\delta} \uplus \overline{\mathcal{E}^{\star}_{\delta}}$ and equivalence $|\mathcal{E}^{\star}_{-}| = |\mathcal{E}^{\star}_{+}|$ and $|\overline{\mathcal{E}^{\star}_{-}}| = |\overline{\mathcal{E}^{\star}_{+}}|$. Any suffix $S \in \operatorname{Suf}(T)$ is δ -trivial if it consists only of $\mathcal{E}^{\star}_{\delta}$ -edges, and δ -nontrivial otherwise, where the δ -trivial one is unique and denoted by S_{δ} . We assume that S_{δ} has an imaginary edge f_{δ} at the bottom if $\delta = (-)$ and at the top if $\delta = (+)$.

Preprocessing. We observe that preprocessing of G = CDAWG(T) for the information necessary in Sec. 4 and 5 can be efficiently done as follows [3, 4].

Lemma 3.1 (preprocessing). Under a pair $\preccurlyeq_- = \preccurlyeq_{pos}$ and \preccurlyeq_+ of co-extensible and extensible string orders, we can preprocess CDAWG(T) in O(e) worst-case time and words of space to support the following operations in O(1) time for $\forall v \in \mathcal{V}$:

- (i) $|\text{repr}_{\delta}(v)|$ returns the length $\ell \in 0..n$ of $\text{repr}_{\delta}(v)$, and $\text{is-primary}_{\delta}(f) \in \{0,1\}$ indicates if f is δ -primary for $\delta \in \{-,+\}$ under $\preccurlyeq_{+} \in \{\preccurlyeq_{\text{lex}}, \preccurlyeq_{\text{pos}}\}$;
- (ii) $|\text{shortest}_{-}(v)|$ returns the length $|U| \in [n]$ and fstsym-shortest(v) returns the start symbol $U[1] \in \Sigma$ of the shortest string U in $U_{-}(v)$.
- (iii) $nleaves(v) \in \mathbb{N}$ returns the number $|\mathcal{U}_{+}(v)|$ of lower paths below v.

As usual, edges of the CDAWG are assumed to be sorted according to $\leq_{+,\text{lex}}^E$ and $\leq_{-,\text{pos}}^E$. If needed, it can be done in O(e) time and space; the sorting with $\leq_{+,\text{lex}}^E$ is done by transposing an incident matrix between nodes and edges, while the sorting with $\leq_{-,\text{pos}}^E$ is done by traversing either the suffix links with a readonly text or suffix links of type-ii nodes [16] in a self-index.

Canonical suffixes. Next, we introduce a set $\mathcal{CS}_{\delta}(G)$ of canonical suffixes, a set $\mathcal{SP}_{\delta}(G)$ of search paths, and a mapping $\operatorname{cano}_{\delta}: \overline{\mathcal{E}_{\delta}^{\star}} \cup \{f_{\delta}\} \to \mathcal{CS}_{\delta}(G)$.

Definition 3.1 (canonical suffix and search path). For $\delta \in \{-,+\}$, we define a δ -canonical suffix S, its δ -certificate f, and its δ -search path P, where $\pi = (f_1, \ldots, f_\ell), \ell \geq 1$ is any path in $\mathsf{Path}(G)$ spelling a suffix S in $\mathsf{Suf}(T)$:

- (a) In the case that S is trivial. Then, $S = \mathbf{S}_{\delta} = \operatorname{repr}_{\delta}(end_{\delta})$ and it is always canonical, where $end_{-} = sink$ and $end_{+} = root$. Then, the δ -certificate is $f = \mathbf{f}_{\delta}$, and the δ -search path for \mathbf{f}_{δ} is $P = \mathbf{S}_{\delta}$ itself. Let $\operatorname{cano}_{\delta}(f) = \mathbf{S}_{\delta}$.
- (b) In the case that S is non-trivial. S is δ -canonical if it has a δ -canonical factoring defined below with some index $k \in [\ell]$ of an edge in π :
 - (i) $f_{\delta} = f_k \in \overline{\mathcal{E}_{\delta}^{\star}}$, and moreover, if $\delta = (-)$ then f_k is the highest $\overline{\mathcal{E}_{-}^{\star}}$ -edge in S, and if $\delta = (-)$ then f_k is the lowest $\overline{\mathcal{E}_{+}^{\star}}$ -edge in S;
 - (ii) the upper path $U_{\delta} = (f_1, \dots, f_{k-1})$ consists only of \mathcal{E}_{-}^{\star} -edges;

⁷ This is because any nonempty set $N_{\delta}(v)$ has at least one δ -primary edge.

⁸ We assume to add imaginary edges f_{-} and f_{+} , resp., which are attached above root and below sink, into the sets $\overline{\mathcal{E}_{+}^{*}}$ and $\overline{\mathcal{E}_{+}^{*}}$.

- (iii) the lower path $L_{\delta} = (f_{k+1}, \dots, f_{\ell})$ consists only of \mathcal{E}_{+}^{\star} -edges;
- (iv) the factoring is $S = \mathsf{str}(U_\delta) \cdot X_\delta \cdot \mathsf{str}(D_\delta)$, where X_δ is the label of the edge $f_k = (v_\delta, X_\delta, w_\delta)$.

Then, the δ -certificate is $f = f_k$, and the δ -search path for f is the path $P = U \cdot X$ for $\delta = (-)$ and the path $P = X \cdot U$ for $\delta = (+)$, where $X = \mathsf{lab}(f)$. Let $\mathsf{cano}_{\delta}(f) = S$.

In what follows, we denote by $\mathcal{CS}_{\delta}(G) \subseteq \operatorname{Suf}(T)$ and $\mathcal{SP}_{\delta}(G) \subseteq (\mathcal{E})^*$ the set of all δ -canonical suffixes of T and the set of all δ -search paths of G, reps., under Π . We remark that any δ -canonical suffix $S \in \mathcal{CS}_{\delta}(G)$ can be recovered by its δ -certificate edge f via $\operatorname{cano}_{\delta}$, and thus the mapping $\operatorname{cano}_{\delta}$ is well-defined.

Lemma 3.2. For any $\delta \in \{-,+\}$ and any δ -canonical suffix S, its δ -canonical factoring and δ -certificate f_{δ} are unique. Consequently, the mapping $cano_{\delta}$ is a bijection between $\overline{\mathcal{E}_{\delta}^{*}} \cup \{f_{\delta}\}$ and $CS_{\delta}(G)$.

Lemma 3.3 (properties of canonical suffixes). Under any pair Π of path ordering, any suffix S in Suf(T) satisfies conditions (1) and (2) below:

- (1) S has (-)-canonical factoring if and only if S has (+)-canonical factoring.
- (2) Let S be any canonical suffix with the associated path $\pi = (f_1, \ldots, f_\ell) \in \operatorname{Path}(G)$ spelling S, and let f_{k_-}, f_{k_+} be the indexes of the (-)- and (+)-certificate k_-, k_+ in π , resp., then $1 \leq k_+ \leq k_- \leq \ell$ holds.

By Lem. 3.3, $\mathcal{CS}_{-}(G) = \mathcal{CS}_{+}(G)$ holds. Thus, we denote the set by $\mathcal{CS}(G) := \mathcal{CS}_{-}(G) = \mathcal{CS}_{+}(G)$, and simply call its members canonical suffixes of G.

Forward and backward DFSs using search paths. Recall that in our approach, we encode a target indexing array, say C, with a subset $\mathcal{CS}(G)$ of canonical suffixes under some path ordering $\Pi = (\preccurlyeq_-, \preccurlyeq_+)$ and an appropriately chosen pair $\varphi = (\mathtt{idx}, \mathtt{val})$ of mappings over $\mathcal{CS}(G)$ as the image of $\mathcal{CS}(G)$ by φ . Thus, the remaining task is to generate all index-value pairs $\varphi(S) = (\mathtt{idx}(S), \mathtt{val}(S))$ by enumerating all $S \in \mathcal{CS}(G)$ in the appropriate index order $\preccurlyeq_{\mathrm{lex}}$ or $\preccurlyeq_{\mathrm{pos}}$. To do this, we use the forward and backward DFSs using search paths of $\mathcal{SP}_{\delta}(G)$ as follows.

Consider the directed graph \mathcal{T}_{-} (resp. \mathcal{T}_{+}) obtained from $\mathcal{SP}_{-}(G)$ (resp. $\mathcal{SP}_{+}(G)$) by merging common prefixes (resp. suffixes). Then, we can easily see that (i) \mathcal{T}_{-} is connected at the root (resp. so is \mathcal{T}_{+} at the sink), (ii) \mathcal{T}_{-} is spanning over $\overline{\mathcal{E}_{-}^{\star}}$ (resp. so is \mathcal{T}_{-} over $\overline{\mathcal{E}_{+}^{\star}}$). However, the graph \mathcal{T}_{δ} may contain the same edge more than once. The next lemma states that it is not the case for $\mathcal{SP}_{\delta}(G)$. In Fig. 1, we show examples of the forward and backward search trees \mathcal{T}_{-} and \mathcal{T}_{+} .

Lemma 3.4. Let $(\mathcal{E}_{-}^{\star}, \mathcal{E}_{+}^{\star})$ be any pair of partitions of \mathcal{E} . Then, (1) the set $\mathcal{SP}_{-}(G)$ is prefix-free, and (2) the set $\mathcal{SP}_{+}(G)$ is suffix-free.

Proof. (1) Since $\mathcal{SP}_{-}(G) \subseteq (\mathcal{E}_{-}^{\star})^{*} \cdot \overline{\mathcal{E}_{-}^{\star}}$, for any distinct search paths X, Y in $\mathcal{SP}_{-}(G)$, X cannot be a proper prefix of Y. By symmetry, we can show (2). \square

Proposition 3.1. \mathcal{T}_{-} is a forward spanning tree for $\overline{\mathcal{E}_{-}^{\star}} \cup \{f_{-}\}$ rooted at root, while \mathcal{T}_{+} is a backward spanning tree for $\overline{\mathcal{E}_{+}^{\star}} \cup \{f_{+}\}$ rooted at sink. Moreover, \mathcal{T}_{-} and \mathcal{T}_{+} have at most e edges.

Proof. Since properties (i) and (ii) were proved, we show that (iii) any distinct paths of $\mathcal{SP}_{\delta}(G)$ do not share the same edge in common. Precisely speaking, \mathcal{T}_{δ} is a rooted tree whose nodes are edges of G. Note that if we embed it into G, it may form a DAG in general. However, we can show the claim (iii) from Lem. 3.4. From claim (iii), we also see that \mathcal{T}_{δ} contains at most $|\mathcal{E}| = e$ edges.

Combining the above arguments, we have the main theorem of this section.

Proposition 3.2. Assume the standard path ordering $\Pi_{\text{lex}}^{\text{pos}} = (\preccurlyeq_{\text{pos}}, \preccurlyeq_{\text{lex}})$. We can perform the following tasks in O(e) worst-case time and words of space on the ordered CDAWG G under $\Pi_{\text{lex}}^{\text{pos}}$ for T: (1) Enumerating all (-)-certificates in $\overline{\mathcal{E}_{-}^{*}} \cup \{f_{-}\}$ in the lexicographic order $\preccurlyeq_{\text{lex}}$ by the ordered ordered DFS of \mathcal{T}_{-} . (2) Enumerating all (+)-certificates in $\overline{\mathcal{E}_{+}^{*}} \cup \{f_{+}\}$ in the text order $\preccurlyeq_{\text{pos}}$ by the backward ordered DFS of \mathcal{T}_{+} .

Proof. By Prop. 3.1, we can enumerate all (-)-certificates in the lexicographic order $\preccurlyeq_{\text{lex}}$ by the standard ordered DFS of \mathcal{T}_- starting from the root of G and going downwards it iteratively following \mathcal{E}_-^* edges; When it encountered an $\overline{\mathcal{E}_-^*}$ edge, it report it and backtracks. At any node v, its outgoing edges are visited in the order of $\preccurlyeq_+^E = \preccurlyeq_{+,\text{lex}}^E$. We remark that the DFS traverses the same edge at most once because in the case that more than one edge of \mathcal{T}_δ meet at the same node on G, exactly one of them is (-)-primary. Therefore, the DFS follows only exactly one \mathcal{E}_δ^* -edge, and backtracks with all the remaining $\overline{\mathcal{E}_\delta^*}$ -edges. By symmetry, we can enumerate all (+)-certificates in the text order $\preccurlyeq_{\text{pos}}$ by the backward ordered DFS of \mathcal{T}_+ starting from the sink of G, going upwards following \mathcal{E}_+^* edges by selecting incoming edges in the order of $(\preccurlyeq_{-,\text{pos}}^E)$. In either case, the DFS over \mathcal{T}_δ traverses at most $O(|\mathcal{T}_\delta|) = O(e)$ edges.

Concerning to Prop. 3.2, from (2) of Lem. 3.3, we can put the pointer from each discovered (–)-certificate f_- to its (+)-counterpart f_+ such that $\operatorname{cano}_-(f_-) = \operatorname{cano}_+(f_+)$ in amortized O(1) time per certificate and *vice versa*.

4 Computing Run-Length BWT

Characterizations Given the BWT for a text T, the set of all irreducible ranks is given by the set $I_{BWT} := \{i \in [n] \mid BWT[i] \neq BWT[i-1]\} \subseteq [n]$. Then, we define the set QI_{BWT} of all quasi-irreducible ranks by the set $QI_{BWT} := \{ \operatorname{rnk}(S) \mid S \in \mathcal{CS}_{\delta}(G) \} \subseteq [n]$, Obviously, $|QI_{BWT}| \leqslant e$ since rnk is a bijection and $|\mathcal{CS}(G)| \leqslant e$. By assumption, we can show that $I_{BWT} \subseteq QI_{BWT}$. Consequently, QI_{BWT} satisfies the following interpolation property.

Lemma 4.1 (interpolation property). Under $(\preccurlyeq_{-}, \preccurlyeq_{+}) = (\preccurlyeq_{pos}, \preccurlyeq_{lex})$, if $i_* \notin QI_{BWT}$, $BWT[i] = BWT[i-1] \in \Sigma$ holds for $\forall i \in [n]$.

Algorithm 1: The algorithm for computing the quasi-irreducible BWT for T[1..n] from the CDAWG G for T stored in read-only memory.

```
1 Procedure RecRBWT(v);
 2 begin
                                                                      \triangleright Case: trivial suffix. T[n] = \$
         if N_+(v) = \emptyset then return ('\$',1);
 3
                                                                              \triangleright Case: non-trivial suffix
 4
              for each f = (v, X, w) \in N_+(v) in order \leqslant_{+,\text{lex}}^E compatible to \preccurlyeq_{\text{lex}} do
 5
                                                                                    \triangleright Case: (-)-primary
                    if is-primary_{-}(f) then
 6
                          RBWT' \leftarrow \mathsf{RecRBWT}(w);
 7
                                                                                  \triangleright Case: (-)-secondary
 8
                         c \leftarrow \mathtt{precsym}(f); \ \ell \leftarrow \mathtt{nleaves}(\mathtt{dst}(f)); \ RBWT' \leftarrow (c,\ell);
 9
                    RBWT \leftarrow RBWT \circ RBWT';
10
               return RBWT;
11
```

Algorithm. In Algorithm 1, we present the recursive procedure that computes the quasi-irreducible BWT for text T from an input CDAWG G for T stored in read-only memory or the self-index $G = CDAWG_{I\!I}^-(T)$ when it is invoked with v = root(G) and $RBWT = \varepsilon$. Let $F = (P_1, \ldots, P_h)$, $h \leqslant e$, be the sequence of all (-)-search path of $\mathcal{SP}_-(G)$ sorted in the lexicographic order $\preccurlyeq_{\text{lex}}$ of string labels with the index i_* of the trivial (-)-path. Let $\mathcal{I} = (I_1, \ldots, I_h)$ be the associated sequence of SA-intervals such that $I_i = [sp(P_i) \ldots ep(P_i)] \subseteq [n]$ for all $i \in [h]$. From Lem. 3.4, we can show that \mathcal{I} forms an ordered partition of [n], namely, the elements of \mathcal{I} are ordered in $\preccurlyeq_{\text{lex}}$ and any suffix falls in exactly one interval of \mathcal{I} .

Now, we give a characterization of the BWT in terms of the (-)-search paths for the canonical suffixes.

```
Lemma 4.2. Let T[1..n] \in \Sigma^n and BWT[1..n] \in \Sigma^n be the BWT for T.

(1) BWT[1..n] = BWT[I_1] \circ \cdots \circ BWT[I_h].

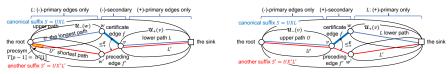
(2) For each i \in [h], the conditions (i) and (ii) below hold: (i) If i = i_*, P_i is a trivial (-)-search path, and then I_{i_*} is a singleton and BWT[I_{i_*}] = T[n] = \$'.

(ii) If i \neq i_*, P_i is a non-trivial (-)-search path with certificate f \in \overline{\mathcal{E}}_{-}^*.
```

Then, $BWT[I_1] = c^{\ell}$, where c := T[p-1], $\ell := |I_i|$, and $p = \mathsf{pos}(\mathsf{cano}_{\delta}(f))$.

Proof (sketch). Claim (1) immediately follows from the definition of \mathcal{I} . Claim (2) is obvious since trivial (-)-search path is the longest suffix T[1..] = T itself. (3) Suppose that X is a non-trivial (-)-search path with locus v. Then, X is not equal to $\mathtt{repr}_{-}(v)$. As seen in Sec. 3, it follows that X is not left-maximal in T. Therefore, there exists some $c \in \mathcal{L}$ that precedes all start positions of X in T. I_i gives the number of leaves below v. Therefore, Claim (3) is proved. \square

Case with a read-only text. Suppose that the read-only text T[1..n] is available. By Lem. 4.2, we see that all BWT-intervals $BWT[I_i]$ but $BWT[I_{i_*}]$ are equal-symbol runs with length $|I_i|$, while $BWT[I_{i_*}]$ is the singleton '\$'. Since the position p of the lex-first suffix $cano_{\delta}(f)$ in I_i can be obtained in constant time



(a) BWT: Computing a preceding symbol (b) GLPF: Computing a previous factor

Fig. 2: Computation of run-length BWT and quasi-irreducible GLPF

by $p = pos(cano_{\delta}(f))$, the preceding symbol, denoted precsym(f) := T[p-1], can be obtained in constant time in the case of a read-only text. Concatenation of two run-length encodings can be done in O(1) time by maintaining the symbols at their both ends. Hence, we can construct the RLBWT of size $r \leq e$ in O(e) worst-case time and O(e) words of space using Algorithm 1.

Extension to the case without a text. Next, we consider the case that input is the self-index version of CDAWG(T) without access to the text T.

Lemma 4.3 (computing a preceding symbol). Given $G = CDAWG_{\Pi}^{-}(T)$, the set of O(e) preceding symbols $\operatorname{precsym}(f) := T[p-1]$ of the non-trivial canonical suffix $S = \operatorname{cano}_{\delta}(f)$ for all certificate edges $f \in \overline{\mathcal{E}_{-}^{\star}}$ can be computed in O(e) worst-case time and words of space.

Proof (sketch). Let S has position p and . Since S is non-trivial, there exists the predecessor S' of S such that |S'| = |S| + 1. Furthermore, there exists another incoming edge f' such that $f' \leqslant_{-,pos}^E f$, f = (v, X, w), and f' = (v', X', w) hold. Then, S' and S meet at node v going through f' and f, and thus, they can be factorized as S' = U'X'L and S = UXL (See Fig. 2a). Since U' is the shortest in $\mathcal{U}_-(w')$, it follows from Lem. 3.1 that $\mathtt{precsym}(f) = U'[1] \in \Sigma$ can be computed by $\mathtt{fstsym\text{-shortest}}(w')$ using O(e) preprocessing and space.

Theorem 4.1. Let T[1..n] be any text over an integer alphabet Σ . Given a self-index version of CDAWG(T) without a text, Algorithm 1 constructs the RLBWT of size $r \leq e$ in O(e) worst-case time and O(e) words of space.

5 Computing Irreducible *GLPF* Arrays

Characterizations. In order to treat the PLCP and LPF arrays uniformly, we introduce their generalization, called the quasi-irreducible GLPF array for a text T parameterized by $\leq_+ \in \{\leq_{\text{lex}}, \leq_{\text{pos}}\}$ according to [7,14].

Definition 5.1. The generalized longest previous factor (GLPF) array for a text T[1..n] under \preccurlyeq_+ is the array $GLPF_{\preccurlyeq_+}[1..n] \in \mathbb{N}^n$ such that for any $p \in [n]$, $GLPF_{\preccurlyeq_+}[p] := \max(\{lcp(T_p, T_q) \mid T_q \prec_+ T_p, \ q \in [n]\} \cup \{0\})$.

Lemma 5.1. For any text T, we have $PLCP = GLPF_{\leq_{lex}}$ and $LPF = GLPF_{\leq_{pos}}$

Now, we introduce the quasi-irreducible GLPF array as a subrelation \widehat{GLPF} of GLPF indexed in \preccurlyeq_{pos} . Under $\Pi_{pos}^{pos} = (\preccurlyeq_{pos}, \preccurlyeq_{+})$, we define the quasi-irreducible GLPF array by the binary relation $\widehat{GLPF} := \{ (pos(S), val(S)) \mid$

Algorithm 2: The algorithm for computing the quasi-irreducible GLPF array for a text T from the CDAWG G for T or its self-index.

```
1 Procedure QIrrGLPF(v, QGL);
                                                         \triangleright Assume\ path\ orderings\ \Pi = (\preccurlyeq_{pos}, \preccurlyeq_{+})
 2 begin
         if N_{-}(v) = \emptyset then
 3
                                                                      ▷ Case: trivial suffix at the root
               QGL \leftarrow QGL \circ (1,0)
 4
                                                     ▷ Case: non-trivial suffix at branching node
         else
 5
               for each f = (w, X, v) in order \leq_{-,pos}^{E} compatible to \leq_{pos} do
 6
                                                                                     \triangleright Case: (+)-primary
 7
                    if is-primary<sub>+</sub>(f) then
                         \mathsf{QIrrGLPF}(w, QGL)
 8
 9
                    else
                                                                                   \triangleright Case: (+)-secondary
                         \ell \leftarrow |\mathtt{repr}_{-}(w)|; p \leftarrow n + 1 - |\mathtt{repr}_{-}(w)| - |X| - |\mathtt{repr}_{+}(v)|;
10
                         QGL \leftarrow QGL \circ (p, \ell);
                                                                                   \triangleright output : GLPF[p] = \ell
11
```

 $S \in \mathcal{CS}(G)$ $\} \subseteq [n] \times \mathbb{N}$, where $\operatorname{val}(S) = GLPF_{\preccurlyeq_+}[\operatorname{pos}(S)]$ and $QI_{GLPF} := \{ \operatorname{pos}(S) \mid S \in \mathcal{CS}(G) \}$ is the set of quasi-irreducible ranks. Since $|\mathcal{CS}(G)| \leqslant e$, GLPF has size e. We observe that GLPF[p] = 0 implies $p \in QI_{GLPF}$. Then, $GLPF_{\preccurlyeq_+}$ satisfies the interpolation property below.

Proposition 5.1 (interpolation property). For any position $p \in [n]$, if $p \notin QI_{GLPF_{\preceq_+}}$ then $GLPF_{\preceq_+}[p] = GLPF_{\preceq_+}[p-1] - 1$ holds. Consequently, PLCP and LPF satisfy the same interpolation property w.r.t. QI_{GLPF} .

Lemma 5.2 (characterization of $GLPF_{\preccurlyeq_+}$ **value).** For any pair $(p, \ell) \in [n] \times \mathbb{N}$, the conditions (a)–(c) below are equivalent (See Fig. 2b):

```
(a) (p,\ell) \in \widetilde{GLPF}.
```

- (b) $GLPF_{\preceq_+}[p] = \ell$ and $T_p[1..\ell] = T[p..p + \ell 1]$ is left-maximal in T.
- (c) For some $S \in \mathcal{CS}(G)$, $p = \mathsf{pos}(S)$. Also, if S is (+)-trivial $\ell = 0$, and if S is (+)-nontrivial, it has the form $S = \mathsf{cano}_+(f) = \mathsf{repr}_-(w) \cdot X \cdot \mathsf{repr}_+(v)$ for some (+)-certificate $f = (w, X, v) \in \overline{\mathcal{E}_+^*}$ and $\ell = |\mathsf{repr}_-(w)|$ holds.

Algorithm. In Algorithm 2, we present the recursive procedure for computing the quasi-irreducible GLPF array, \widetilde{GLPF} , of size e for a text T[1..n] from the self-index $G = CDAWG_{\Pi}^{-}(T)$ of size O(e) under a parameter pair $\Pi = (\preceq, \prec, \prec)$, when it is invoked with v = sink(G) and $QGL = \varepsilon$.

Theorem 5.1. Let T[1..n] be any text over an integer alphabet Σ . Given a self-index version of CDAWG(T) without a text, Algorithm 2 constructs the quasi-irreducible $GLPF_{\preccurlyeq_+}$ array for T of size e in O(e) worst-case time and O(e) words of space.

By a simple procedure as in [7,14], we can easily compute either the lex-parse from $GLPF_{\preceq_{\text{lex}}}$ or the LZ-parse from $GLPF_{\preceq_{\text{pos}}}$ in linear time in combined input and output sizes. Hence, the next theorem follows from Lem. 5.1 and Thm. 4.1.

Theorem 5.2. Let T[1..n] be any text over an integer alphabet Σ . The lexparse of size 2r = O(e) and the LZ-parse of size $z \leq r$ of T can be computed from a self-index version of CDAWG(T) without a text for the same text in O(e) worst-case time and words of space.

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