# Overstatement-Net-Equivalent Risk-Limiting Audit: ONEAudit 

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#### Abstract

A procedure is a risk-limiting audit (RLA) with risk limit $\alpha$ if it has probability at least $1-\alpha$ of correcting each wrong reported outcome and never alters correct outcomes. One efficient RLA method, card-level comparison (CLCA), compares human interpretation of individual ballot cards randomly selected from a trustworthy paper trail to the voting system's interpretation of the same cards (cast vote records, CVRs). CLCAs heretofore required a CVR for each cast card and a "link" identifying which CVR is for which card-which many voting systems cannot provide. This paper shows that every set of CVRs that produces the same aggregate results overstates contest margins by the same amount: they are overstatement-net-equivalent (ONE). CLCA can therefore use CVRs from the voting system for any number of cards and ONE CVRs created ad lib for the rest. In particular: - Ballot-polling RLA is algebraically equivalent to CLCA using ONE CVRs derived from the overall contest results. - CLCA can be based on batch-level results (e.g., precinct subtotals) by constructing ONE CVRs for each batch. In contrast to batch-level comparison auditing (BLCA), this avoids manually tabulating entire batches and works even when reporting batches do not correspond to physically identifiable batches of cards, when BLCA is impractical. - If the voting system can export linked CVRs for only some ballot cards, auditors can still use CLCA by constructing ONE CVRs for the rest of the cards from contest results or batch subtotals. This works for every social choice function for which there is a known RLA method, including IRV. Sample sizes for BPA and CLCA using ONE CVRs based on contest totals are comparable. With ONE CVRs from batch subtotals, sample sizes are smaller than than for BPA when batches are homogeneous, approaching those of CLCA using CVRs from the voting system, and much smaller than for BLCA: A CLCA of the 2022 presidential election in California at risk limit $5 \%$ using ONE CVRs for precinct-level results would sample approximately 70 ballots statewide, if the reported results are accurate, compared to about 26,700 for BLCA. The 2022 Georgia audit tabulated more than 231,000 cards (the expected BLCA sample size was $\approx 103,000$ cards); ONEAudit would have audited $\approx 1,300$ cards. For data from a pilot hybrid RLA in Kalamazoo, MI, in 2018, ONEAudit gives a risk of $2 \%$, substantially lower than the $3.7 \%$ measured risk for SUITE, the "hybrid" method the pilot used.


Keywords: Risk-limiting audit, BPA, card-level comparison audit, batch-level comparison audit

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## 1 Introduction: Efficient Risk-Limiting Audits

A procedure is a risk-limiting audit (RLA) with risk limit $\alpha$ if it guarantees that if the reported outcome is right, the procedure will not change it; but if the reported outcome is wrong, the chance the procedure will not correct it - the "risk" - is at most $\alpha$. "Outcome" means who or what won, not the precise vote tallies. RLA methods have been developed for many sampling designs [20,16,10,11,18,22], and to use the audit data in different ways to measure "risk" [20,16,21,17,9,8,18,22], to accommodate legal and logistical constraints and heterogeneous equipment within and across jurisdictions.
"Card" or "ballot card" means a sheet of paper; a ballot comprises one or more cards. A "cast-vote record" (CVR) is the voting system's interpretation of the votes on a particular card. A "manual-vote record" (MVR) is the auditors' reading of the votes on a card. The main approaches to RLAs are ballot-polling $R L A s$ (BPA), which examine individual randomly selected cards but do not use data from the voting system other than the totals; batch-level comparison RLAs (BLCA), which compare reported vote subtotals for randomly selected batches of cards (e.g., all cards cast in a precinct) to manual tabulations of the same batches; card-level comparison RLAs (CLCA), which compare individual CVRs to the corresponding MVRs for a random sample of cards; and hybrid audits, which combine two or more of the approaches above.

BLCA is closest to historical statutory audits, but requires larger sample sizes than other methods when outcomes are correct. BPA requires almost no data from the voting system. It is generally more efficient than BLCA, but its sample size grows approximately quadratically as the margin shrinks. The most efficient approach is CLCA, for which the sample size grows approximately linearly as the margin shrinks. But it requires the most information from the voting system: an exported CVR for every card and a way to link exported CVRs to the corresponding physical cards, without compromising voter privacy.

This paper shows that applying CLCA to any combination of CVRs provided by the voting system and CVRs created by the auditors to match batch subtotals or contest totals gives a valid RLA. When the CVRs are derived entirely from contest totals, the method is algebraically equivalent to BPA. When the CVRs are derived from batch subtotals, the method is far more efficient than BLCA and approaches the efficiency of 'pure' CLCA when the batches are sufficiently homogeneous.

Many modern voting systems can provide linked CVRs for some ballot cards (e.g., vote-by-mail) but not others (e.g., in-precinct voters). This has led to a variety of strategies:

- give up the efficiency of CLCA and use BPA
- hybrid RLAs that use different audit strategies in different strata [10,18,22,13]
- BLCA using weighted random samples [17,7,10], with batches of size 1 for cards with linked CVRs
- CLCA that rescans some or all of the cards to create linked CVRs the voting system did not originally provide $[12,5]$
- CLCA using cryptographic nonces to link CVRs to cards without compromising voter privacy [19].

Section 4 develops a simpler approach that in examples is more efficient than a hybrid audit or BLCA, works even when a BCLA is impracticable, avoids the expense of re-scanning any ballots, and does not require new or additional equipment. When reporting batches are sufficiently homogeneous, the sample size for the method approaches that of CLCA.

## 2 Testing net overstatement does not require CVRs linked to ballot cards

### 2.1 Warmup: 2-candidate plurality contest

Consider a two-candidate plurality contest, Alice v. Bob, with Alice the reported winner. We encode votes and reported votes as follows. There are $N$ cards. Let $b_{i}=1$ if card $i$ has a vote for Alice, -1 if it has a vote for Bob, and 0 otherwise. Let $c_{i}=1$ if card $i$ was counted by the voting system as a vote for Alice, $c_{i}=-1$ if it was counted as a vote for Bob, and $c_{i}=0$ otherwise. The true margin is $\sum_{i=1}^{N} b_{i}$ and the reported margin is $\sum_{i=1}^{N} c_{i}$. The overstatement of the margin on the $i$ th card is $c_{i}-b_{i} \in\{-2,-1,0,1,2\}$. It is the number of votes by which the voting system exaggerated the number of votes for Alice. Alice really won if the net overstatement of the margin, $E\left(\left\{c_{i}\right\}\right):=\sum_{i}\left(c_{i}-b_{i}\right)$, is less than the reported margin $\sum_{i} c_{i}$. Because addition is commutative and associative, if $\left\{c_{i}\right\}$ and $\left\{c_{i}^{\prime}\right\}$ are any two sets of CVRs for which $\sum_{i} c_{i}=\sum_{i} c_{i}^{\prime}$, then $E\left(\left\{c_{i}\right\}\right)=E\left(\left\{c_{i}^{\prime}\right\}\right)$ : they are overstatement net equivalent (ONE).
$E\left(\left\{c_{i}\right\}\right):=\sum_{i}\left(c_{i}-b_{i}\right)=\sum_{i} c_{i}-\sum_{i} b_{i}=\sum_{i} c_{i}^{\prime}-\sum_{i} b_{i}=\sum_{i}\left(c_{i}^{\prime}-b_{i}\right)=E\left(\left\{c_{i}^{\prime}\right\}\right)$.
Hence, if we have an RLA procedure to test whether $E\left(\left\{c_{i}\right\}\right)<\sum_{i} c_{i}$ using the "real" CVRs produced by the voting system, the same procedure can test whether the outcome is correct if it is applied to other CVRs $\left\{c_{i}^{\prime}\right\}$ provided $\sum_{i} c_{i}=\sum_{i} c_{i}^{\prime}$, even if the CVRs $\left\{c_{i}^{\prime}\right\}$ did not come from the voting system. (Audit sample sizes might be quite different.)

Thus, we can conduct a CLCA using any set $\left\{c_{i}^{\prime}\right\}$ of CVRs that reproduces the contest-level results: the net overstatement of every such set of CVRs is the same. If the system reports batch-level results, we can require that the CVRs reproduce the batch-level results as well., which might reduce audit sample sizes, especially when the batches have different political preferences. If the voting system reports CVRs for some individual ballot cards, we can conduct a CLCA that uses those CVRs, augmented by ONE CVRs for the remaining ballot cards
(derived from batch subtotals or from contest totals, by subtraction). Better agreement between the MVRs and VCRs will generally allow the audit to stop after inspecting fewer cards.

### 2.2 Numerical example

Consider a contest in which 20,000 cards were cast in all, of which 10,000 were cast by mail and have linked CVRs, with 5,000 votes for Alice, 4,000 for Bob, and 1,000 undervotes. The other 10,000 cards were cast in 10 precincts, 1,000 cards in each. Net across those 10 precincts, Alice and Bob each got 5,000 votes. In 5 precincts, Alice showed more votes than Bob; in the other 5, Bob showed more than Alice. The reported results are thus 10,000 votes for Alice, 9,000 for Bob, and 1,000 undervotes. The margin is 1,000 votes; the diluted margin (margin in votes, divided by cards cast) is $1000 / 20000=5 \%$. Consider two sets of precinct subtotals:

- 5 precincts show 900 votes for Alice and 100 for Bob; the other 5 show 900 votes for Bob and 100 for Alice.
- 5 precincts show 990 votes for Alice and 10 for Bob; the other 5 show 990 votes for Bob and 10 for Alice.

Construct ONE CVRs for the 10,000 cards cast in the 10 precincts as follows: if the precinct reports $a$ votes for Alice and $1000-a$ for Bob, the net vote for Alice is $a-(1000-a)=2 a-1000$. The "average" CVR for the precinct has $(2 a-1000) / 1000=2 a / 1000-1$ votes for Alice; that is the ONEAudit CVR for every card in the precinct. For instance, a precinct that reported 900 votes for Alice and 100 for Bob has a net margin of $900 \times 1+100 \times-1=800$ for Alice, so that precinct contributes 1,000 ONE CVRs, each with $c_{i}=(0.9) \times 1+(0.1) \times$ $(-1)=0.8$ votes for Alice.

Table 1. Overstatement-net equivalent CVRs that match batch subtotals. If a precinct of 1000 voters reported 990 votes for Alice and 10 for Bob, the net overstatement is the same as if there had been 1,000 CVRs,

| batch total | ONE CVR |
| :--- | ---: |
| 990 Alice, 10 Bob | 0.98 Alice |
| 900 Alice, 100 Bob | 0.8 Alice |
| 100 Alice, 900 Bob | -0.8 Alice ( 0.8 Bob) |
| 10 Alice, 990 Bob | -0.98 Alice (0.98 Bob) | one for each card, each showing 0.98 votes for Alice.

To audit, draw ballot cards uniformly at random, without replacement. To find the overstatement for each audited card, subtract the MVR for the card ( -1 , 0 , or 1 ) from the CVR ( $-1,0$, or 1 ) if the system provided one, or from the ONE CVR for its precinct (a number in $[-1,1]$ ) if the system did not provide a CVR. Apply a "risk-measuring function" (see, e.g., $[18,22]$ ) to the overstatements to measure the risk that the outcome is wrong based on the data collected so far;
the audit can stop without a full hand count if and when the measured risk is less than or equal to the risk limit.

The random selection can be conducted in many ways, for instance, conceptually numbering the cards from 1 to 20,000 , where cards $1-10,000$ are the ballots with CVRs, ordered in some canonical way; cards $10,001-11,000$ are the cards cast in precinct 1 , starting with the top card in the stack; cards 11,00112,000 are the cards cast in precinct 2 , starting with the top card in the stack; etc. Auditors draw random numbers between 1 and 20,000, and retrieve the corresponding card. Alternatively, if the resulting number is between 1 and 10,000, retrieve the corresponding card; but if the number is larger, draw a ballot at random from the precinct that numbered card belongs to, for instance, using the $k$-cut method [14]. That approach avoids counting into large stacks of ballots.

Table 2 summarizes expected audit sample sizes. If there had been a CVR for every card and the results were exactly correct, the sample size for a standard CLCA with risk limit $5 \%$ would be about 125 cards. A BPA at risk limit $5 \%$ would examine about 2,300 cards. A BLCA (treating individual cards as batches for those with CVRs) using sampling with probability proportional to an error bound and the Kaplan-Markov test [21] would examine about 7250 cards on average in the $900 / 100$ scenario and 5300 in the $990 / 10$ scenario. For ONEAudit, the expected sample size is about 800 cards in the $900 / 100$ scenario and 170 in the $990 / 10$ scenario. As preferences within precincts become more homogeneous, ONEAudit approaches the efficiency of CLCA.

### 2.3 The general case

We use the SHANGRLA audit framework [18] because it can be works with every social choice function for which an RLA method is known; however, ONE CVRs can be used with every extant RLA method for comparison audits. SHANRGLA reduces auditing election outcomes to multiple instances of a single problem: testing whether the mean of a finite list of bounded numbers is less than or equal to $1 / 2$. Each list results from applying a function $A$ called an assorter to the votes on the ballots; each assorter maps votes to the interval [ $0, u$ ], where the upper bound $u$ depends on the particular assorter. Different social choice functions involve different assorters and, in general, different numbers of assorters. An assertion is the claim that the average of the values the assorter takes on the true votes is greater than $1 / 2$. The contest outcome is correct if the all the SHANGRLA assertions for the contest are true.

A reported assorter margin is the amount by which that assorter applied to the reported votes exceeds $1 / 2$. For scoring rules (plurality, supermajority, multiwinner plurality, approval voting, Borda count, etc.), reported assorter margins can be computed from contest-level tallies, batch tallies, or CVRs. For auditing IRV using RAIRE [4], CVRs are generally required to construct an appropriate set of assorters and to find their margins-but the CVRs do not have to be linked to individual ballot cards.

Let $b_{i}$ denote the true votes on the $i$ th ballot card; there are $N$ cards in all. Let $c_{i}$ denote the voting system's interpretation of the $i$ th card. Suppose we

|  |  | expected <br> cards vs BPA |
| :--- | :--- | ---: | ---: | ---: |
| scenario method | CLCA |  |

Table 2. Expected RLA workloads for a two-candidate plurality contest with a 'diluted margin' of $5 \%$ at risk limit $5 \%$. In all, 20,000 cards were cast, of which 10,000 have linked CVRs; the other 10,000 cards are divided into 10 precincts with 1,000 cards each. Among cards with CVRs, 5,000 show votes for the winner, 4,000 show votes for the loser, and 1,000 have invalid votes. In the $900 / 100$ scenario, 5 precincts have 900 votes for the winner and 100 for the loser; the other 5 have 900 for the loser and 100 for the winner. In the 990/10 scenario, 5 precincts have 990 votes for the winner and 10 for the loser; the other 5 have 990 for the loser and 10 for the winner. BPA: ballot-polling audit. CLCA: card-level comparison audit (which would require re-scanning the 10,000 cards cast in precincts). BLCA: batch-level comparison audit. ONE CLCA: card-level comparison audit using the original 10,000 CVRs, augmented by 10,000 ONE CVRs derived from precinct subtotals. Column 4: sample size divided by BPA sample size. Column 5: sample size divided by CLCA sample size. Expected workload for BPA and CLCA is the same in both scenarios. Sample sizes for CLCA use the "super-simple" method [17], computed using https://www.stat.berkeley. edu/~stark/Vote/auditTools.htm (last visited 19 March 2023). Sample sizes for BPA use ALPHA [22].
have a CVR $c_{i}$ for every ballot card whose index $i$ is in $\mathcal{C}$. The cardinality of $\mathcal{C}$ is $|\mathcal{C}|$. Ballot cards not in $\mathcal{C}$ are partitioned into $G \geq 1$ disjoint groups $\left\{\mathcal{G}_{g}\right\}_{g=1}^{G}$ for which reported assorter subtotals are available. For instance $\mathcal{G}_{g}$ might comprise all ballots for which no CVR is available or all ballots cast in a particular precinct. Unadorned overbars denote the average of a quantity across all $N$ ballot cards; overbars subscripted by a set (e.g., $\mathcal{G}_{g}$ ) denote the average of a quantity across cards in that set, for instance:

$$
\begin{array}{ll}
\bar{A}^{c}:=\frac{1}{N} \sum_{i=1}^{N} A\left(c_{i}\right), \quad \bar{A}_{\mathcal{C}}^{c}:=\frac{1}{|\mathcal{C}|} \sum_{i \in \mathcal{C}} A\left(c_{i}\right), \quad \bar{A}_{\mathcal{G}_{g}}^{c}:=\frac{1}{\left|\mathcal{G}_{g}\right|} \sum_{i \in \mathcal{G}_{g}} A\left(c_{i}\right) \\
\bar{A}^{b}:=\frac{1}{N} \sum_{i=1}^{N} A\left(b_{i}\right), \quad \bar{A}_{\mathcal{C}}^{b}:=\frac{1}{|\mathcal{C}|} \sum_{i \in \mathcal{C}} A\left(b_{i}\right), \quad \bar{A}_{\mathcal{G}_{g}}^{b}:=\frac{1}{\left|\mathcal{G}_{g}\right|} \sum_{i \in \mathcal{G}_{g}} A\left(b_{i}\right) .
\end{array}
$$

The assertion is the claim $\bar{A}^{b}>1 / 2$. The reported assorter mean $\bar{A}^{c}>1 / 2$ : otherwise, according to the voting system's data, the reported outcome is wrong. Now $\bar{A}^{b}>1 / 2$ iff

$$
\begin{equation*}
\bar{A}^{c}-\bar{A}^{b}<\bar{A}^{c}-1 / 2 \tag{2}
\end{equation*}
$$

The right hand side is known before the audit starts; it is half the "reported assorter margin" $v:=2 \bar{A}^{c}-1[18]$. We assume we have a reported assorter total
$\sum_{i \in \mathcal{G}_{g}} A\left(c_{i}\right)$ from the voting system for the cards in the group $\mathcal{G}_{g}$ (e.g., reported precinct subtotals) and define the reported assorter mean for $\mathcal{G}_{g}$ :

$$
\begin{equation*}
\hat{A}_{\mathcal{G}_{g}}^{c}:=\frac{1}{\left|\mathcal{G}_{g}\right|} \sum_{i \in \mathcal{G}_{g}} A\left(c_{i}\right) \tag{3}
\end{equation*}
$$

We have

$$
\begin{equation*}
\bar{A}^{c}=\frac{\sum_{g=1}^{G}\left|\mathcal{G}_{g}\right| \hat{A}_{\mathcal{G}_{g}}^{c}+\sum_{i \in \mathcal{C}} A\left(c_{i}\right)}{N}=\frac{\sum_{g=1}^{G} \sum_{i \in \mathcal{G}_{g}} \hat{A}_{\mathcal{G}_{g}}^{c}+\sum_{i \in \mathcal{C}} A\left(c_{i}\right)}{N} \tag{4}
\end{equation*}
$$

Thus if we declare $A\left(c_{i}\right):=\hat{A}_{\mathcal{G}_{g}}^{c}$ for $i \in \mathcal{G}_{g}$, the reported assorter mean for the cards in group $\mathcal{G}_{g}$, the mean of the assorter applied to the CVRs-including these faux CVRs-will equal its reported value: using a "mean CVR" for the batch is overstatement-net-equivalent to any CVRs that give the same assorter batch subtotals. Condition 2 then can be written

$$
\begin{equation*}
\frac{1}{N} \sum_{i}\left(A\left(c_{i}\right)-A\left(b_{i}\right)\right)<v / 2 \tag{5}
\end{equation*}
$$

Following SHANGRLA [18, section 3.2], define

$$
\begin{equation*}
B\left(b_{i}\right):=\frac{u+A\left(b_{i}\right)-A\left(c_{i}\right)}{2 u-v} \in[0,2 u /(2 u-v)] \tag{6}
\end{equation*}
$$

Then $\bar{A}^{b}>1 / 2 \Longleftrightarrow \bar{B}^{b}>1 / 2$, which can be shown as follows, using the fact that $v:=2 \bar{A}^{c}-1 \leq 2 u-1<2 u$ :

$$
\begin{align*}
\bar{B}^{b} & :=\frac{1}{N} \sum_{i} \frac{u+A\left(b_{i}\right)-A\left(c_{i}\right)}{2 u-v} \\
& =\frac{u+\bar{A}^{b}-\bar{A}^{c}}{2 u-v} \\
& =\frac{u+\bar{A}^{b}-\bar{A}^{c}}{2 u-2 \bar{A}^{c}+1} \tag{7}
\end{align*}
$$

Thus if $\bar{B}^{b}>1 / 2$,

$$
\begin{align*}
\frac{u+\bar{A}^{b}-\bar{A}^{c}}{2 u-2 \bar{A}^{c}+1} & >1 / 2 \\
u+\bar{A}^{b}-\bar{A}^{c} & >u-\bar{A}^{c}+1 / 2 \\
\bar{A}^{b} & >1 / 2 \tag{8}
\end{align*}
$$

If the reported tallies are correct, i.e., if $\bar{A}^{c}=\bar{A}^{b}=(v+1) / 2$, then

$$
\begin{equation*}
\bar{B}^{b}=\frac{u}{2 u-v} \tag{9}
\end{equation*}
$$

## 3 Auditing using batch subtotals

The oldest approach to RLAs is batch-level comparison, which involves exporting batch subtotals from the voting system (e.g., for precincts or tabulators), verifying that those batch subtotals yield the reported contest results, drawing some number of batches at random (with equal probability or with probability proportional to an error bound), manually tabulating all the votes in each selected batch, comparing the manual tabulation to the reported batch subtotals, assessing whether the data give sufficiently strong evidence that the reported results are right, and expanding the sample if not $[20,16,15,21,6]$.

BLCAs have two logistical hurdles: (i) They require manually tabulating the votes on every ballot card in the batches selected for audit. (ii) When the batches of cards for which the voting system reports subtotals do not correspond to identifiable physical batches (common for vote-by-mail and vote centers), the audit has to find and retrieve every card in the audited reporting batches. Those cards may be spread across any number of physical batches, which can also make recounts prohibitively expensive [1].

Both can be avoided using CLCA with ONE CVRs. The following algorithm gives a valid RLA, but selects and compares the manual interpretation of individual cards to the implied "average" CVR of the reporting batch each card belongs to. We assume that the canvass and a compliance audit [2] have determined that the ballot manifest and physical cards are complete and trustworthy.

## Algorithm for a CLCA using ONE CVRs from batch subtotals.

1. Pick the risk limit for each contest under audit.
2. Export batch subtotals from the voting system.
3. Verify that every physical card is accounted for, ${ }^{a}$ that the physical accounting is consistent with the reported votes, and that the reported batch subtotals produce the reported winners.
4. Construct SHANGRLA assorters for every contest under audit; select a risk-measuring function for each assertion (e.g., one in [22]); set the measured risk for each assertion to 1 .
5. Calculate the reported mean assorter values for each reporting batch; these are the ONE CVRs.
6. While any measured risk is greater than its risk limit and not every card has been audited:

- Select a card at random, e.g., by selecting a batch at random with probability proportional to the size of the batch, then selecting a card uniformly at random from the batch using the $k$-cut method [14], or by selecting at random from the entire collection of cards.
- Calculate the overstatement for the selected card using the ONE CVR for the reporting batch the card belongs to.
- Update the measured risk of any assertion whose measured risk is still greater than its risk limit.
- If the measured risk for every assertion is less than or equal to its risk limit, stop and confirm the reported outcomes.

7. Report the correct contest outcomes: every card has been manually interpreted.
${ }^{a}$ For techniques to deal with missing cards, see [3,18].
This algorithm be made more efficient statistically and logistically in a variety of ways, for instance, by making an affine translation of the data so that the minimum possible value is 0 (by subtracting the minimum of the possible overstatement assorters across batches and re-scaling so that the null mean is still $1 / 2$ ) and by starting with a sample size that is expected to be large enough to confirm the contest outcome if the reported results are correct.

### 3.1 Numerical case studies

Table 3 compares expected sample sizes for BLCA to CLCA using ONE CVRs derived from the same batch subtotals, and to BPA, for two contests: the 2022 midterm Georgia Secretary of State's contest, which had a diluted margin (margin in votes divided by cards cast) of about $9.2 \%^{1}$ and the 2020 presidential election in California, which had a diluted margin of about $28.7 \%$. The Georgia contest was audited using batch-level comparisons. The Georgia SoS claims that audit was a BLCA with a risk limit of $5 \%$, but in fact the audit was not an RLA, for a variety of reasons. ${ }^{2}$ BPA and CLCA using ONE CVRs are generally much more efficient than BLCA when batches are large. CLCA with ONE CVRs is more efficient than BPA when batches are more homogenous than the contest votes as a whole, i.e., when precincts are polarized in different directions.

## 4 Auditing heterogenous voting systems

When the voting system can report linked CVRs for some but not all cards, we can augment the voting system's linked CVRs with ONE CVRs for the remaining cards, then use CLCA. The ONE CVRs can be derived from overall contest results or from reported subtotals, e.g., precinct subtotals. Finer-grained subtotals generally give smaller audit sample sizes (when the reported outcome is correct) if the smaller groups are more homogeneous than the overall election.

SUITE [10], a hybrid RLA designed for this situation, was first fielded in a pilot audit of the gubernatorial primary in Kalamazoo, MI, in 2018. The stratum with linked CVRs comprised 5,294 ballots with 5,218 reported votes in the contest; the "no-CVR" stratum comprised 22,372 ballots with 22,082 reported votes. The sample included 32 cards were drawn from the no-CVR stratum and

[^0]| Contest | total | total | actual | $\begin{array}{r} \mathrm{K}-\mathrm{M} \\ \mathrm{BL} C A \end{array}$ | ONE | Wald | BLCA/ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | turnout | batches | sample size | BLCA | $\mathrm{CLCA}$ | BP |  |
| 2020 CA U.S. Pres | 17,785,667 | 21,346 | $\approx 178,000$ | 26,700 | 70 | 72 | 381 |
| 2022 GA SoS | 3,909,983 | 12,968 | >231,000 | 103,300 | 1,380 | 700 | 75 |

Table 3. Actual and estimated expected sample sizes for various RLA methods for the 2020 U.S. presidential election in California and the 2022 Georgia Secretary of State contest, at risk limit $5 \%$. Columns 2, 3: turnout per state records. CA data from https://statewidedatabase.org/d10/g20.html (last visited 2 March 2020); the CA Statement of Vote gives slightly smaller turnout 17,785,151 (https://elections.cdn.sos.ca.gov/sov/2020-general/sov/ complete-sov.pdf, last visited 2 March 2023). GA data from (https://sos.ga.gov/ news/georgias-2022-statewide-risk-limiting-audit-confirms-results, last visited 26 February 2023). Column 4: approximate number of cards examined in the actual batch-level audits (which were not RLAs). Column 5: expected sample size for BLCA using the Kaplan-Markov risk function. Column 6: expected sample size for CLCA using ONE CVRs based on batch subtotals, for the ALPHA risk-measuring function with the truncated shrinkage estimator with parameters $c=1 / 2, d=10$, estimated from 100 Monte Carlo replications. Column 7: expected sample size for BPA using Wald's SPRT. Column 8: column 5 divided by column 6. A different risk function should reduce the GA ONE CLCA sample size to at most the BPA sample size: see section 5.1. Estimates assume that reported batch subtotals are correct.

8 from the CVR stratum. ${ }^{3}$ Table 4 summarizes the contest and audit results; auditors found no errors in the 8 CVRs, each of which yields the overstatement assorter value $u /(2 u-v)$. The new method compares the 32 cards without CVRs to ONE CVRs derived from all the votes without CVRs (not from subtotals for smaller batches) by subtracting the votes on the linked CVRs from the reported contest totals. Ignoring the fact that the sample was stratified and the difference in sampling fractions in the two strata, in 100,000 random permutations of the data, the ALPHA martingale test using a fixed alternative $0.99(2 u) /(2 u-v)$ had a mean $P$-value of 0.0201 (90th percentile 0.0321 ), about $54 \%$ of the SUITE $P$-value of 0.0374 [10]. The ONEAudit $P$-value is larger than the $P$-value of the best product supermartingale test in [13], but comparable to or smaller than the $P$-values for the other supermartingale tests. See table 5. If precinct subtotals were available to construct the ONE CVRs, the measured risk might have been lower, depending on precinct heterogeneity.

## 5 Sample sizes for contest-level ONE CLCA vs. BPA

### 5.1 Theory

Moving from tests about raw assorter values to tests about overstatements relative to ONE CVRs derived from overall contest totals is just an affine transformation: no information is gained or lost. Thus, if we audited using an affine

[^1]| Candidate CVR no-CVR | polling <br> sample |  |  |
| :--- | ---: | ---: | ---: |
| Butkovich | 6 | 66 | 0 |
| Gelineau | 56 | 462 | 1 |
| Kurland | 23 | 284 | 0 |
| Schleiger | 19 | 116 | 0 |
| Schuette | 1,349 | 4,220 | 8 |
| Whitmer | 3,765 | 16,934 | 23 |
| Non-vote | 76 | 290 | 0 |
| Total | 5,294 | 22,372 | 32 |


| Method | $P$-value | SD | 90 th <br> percentile |
| :--- | ---: | ---: | ---: |
| SUITE | 0.037 | $\mathrm{n} / \mathrm{a}$ | $\mathrm{n} / \mathrm{a}$ |
| ALPHA $P_{F}^{*}$ | 0.018 | 0.002 | 0.019 |
| ALPHA $P_{M}^{*}$ | 0.0030 .000 | 0.003 |  |
| Empirical Bernstein $P_{F}^{*}$ | 0.348 | 0.042 | 0.390 |
| Empirical Bernstein $P_{M}^{*}$ | 0.420 | 0.134 | 0.561 |
| ALPHA ONEAudit | 0.020 | 0.010 | 0.032 |

Table 5. $P$-values for the 2018 RLA pilot in Kalamazoo, MI, for different riskmeasuring functions. SUITE is a hybrid stratified approach [10]. Rows $2-5$ are from [13, Table 3]: the ALPHA and Empirical Bernstein stratumwise supermartingales combined using either Fisher's combining function $\left(P_{F}^{*}\right)$ or multiplication $\left(P_{M}^{*}\right)$. The 6th row is for ONEAudit, using the ALPHA supermartingale with fixed alternative $\eta=0.99$, a non-adaptive choice. Technically, ONEAudit should not be applied to this sample because the sample was stratified, while the risk calculation assumes the sample was a simple random sample of ballot cards: this is just a numerical illustration.
equivariant statistical test, the sample size should be the same whether the data are the original assorter values (i.e., BPA) or overstatements from ONE CVRs.

However, the statistical tests used in RLAs are not affine equivariant because they rely on a priori bounds on the assorter values. The original assorter values will generally be closer to the endpoints of $[0, u]$ than the transformed values are to the endpoints of $[0,2 u /(2 u-v)]$. To see why, suppose that there are no reported CVRs $(\mathcal{C}=\emptyset)$ and that only contest totals are reported from the system-so every cast ballot card is in $\mathcal{G}_{1}$. For a BPA, the population values from which the sample is drawn are the original assorter values $\left\{A\left(b_{i}\right)\right\}$, which for many social choice functions can take only the values $0,1 / 2$, and $u$. For instance, consider a two-candidate plurality contest, Alice v. Bob, where Alice is the reported winner. This can be audited using a single assorter that assigns the value 0 to a card with a vote for Bob, the value $u=1$ to a card with a vote for Alice, and the value $1 / 2$ to other cards. In contrast, for a comparison audit, the possible population values $\left\{B\left(b_{i}\right)\right\}$ are

$$
\left\{\frac{1-(v+1) / 2}{2-v}, 1 / 2, \frac{2-(v+1) / 2}{2-v}\right\}
$$

Unless $v=1$-i.e., unless every card was reported to have a vote for Alice the minimum value of the overstatement assorter is greater than 0 and the maximum is less than $u$. Figure 1 plots the minimum and maximum value of the overstatement assorter as a function of $v$ for $u=1$.

Fig. 1. Upper and lower bounds on the overstatement assorter as a function of the diluted margin $v$, for $u=1$.


A test that uses the prior information $x_{j} \in[0, u]$ may not be as efficient for populations for which $x_{j} \in[a, b]$ with $a>0$ and $b<u$ as it is for populations where the values 0 and $u$ actually occur. An affine transformation of the overstatement assorter values can move them back to the endpoints of the support constraint by subtracting the minimum possible value then re-scaling so that the null mean is $1 / 2$ once again, which reproduces the original assorter, $A$ :

$$
\begin{align*}
C\left(b_{i}\right) & :=\frac{1 / 2}{1 / 2-\frac{u-(v+1) / 2}{2 u-v}} \cdot\left(B\left(b_{i}\right)-\frac{u-(v+1) / 2}{2 u-v}\right) \\
& =\frac{2 u-v}{2 u-v-(2 u-(v+1))} \cdot\left(\frac{u+A\left(b_{i}\right)-(v+1) / 2}{2 u-v}-\frac{u-(v+1) / 2}{2 u-v}\right) \\
& =(2 u-v) \cdot \frac{A\left(b_{i}\right)}{2 u-v}=A\left(b_{i}\right) \tag{10}
\end{align*}
$$

### 5.2 Numerical comparison

While CLCA with ONE CVRs is algebraically equivalent to BPA, the performance of a given statistical test will be different for the two formulations. We now compare expected audit sample sizes for some common risk-measuring functions applied to CLCA with ONE CVRs derived from contest-level results and applied to the original assorter data. This is assessing the particular statistical tests, not any intrinsic difference between BPA and ONE CLCA.

Tables $7-11$ in the appendix report expected sample sizes when the reported winner received a share $\theta \in\{0.505,0.51,0.52,0.55,0.6\}$ of the reported votes, for percentages of cards that do not contain a valid vote for either candidate ranging from $10 \%$ to $75 \%$, and various values of tuning parameters in the risk-measuring functions. Table 6 gives the geometric mean of the ratios of the mean sample size for each condition to the smallest mean sample size for that condition (across
risk-measuring functions). Transforming the assorter into an overstatement assorter using the ONEAudit transformation, then testing whether the mean of the resulting population is $\leq 1 / 2$ using the ALPHA test martingale with the truncated shrinkage estimator of [22] with $d=10$ and $\eta$ between 0.505 and 0.55 performed comparably to - but slightly worse than-using ALPHA on the raw assorter values for the same $d$ and $\eta$, and within $4.8 \%$ of the overall performance of the best-performing method.

| Method | Parameters | Score | Method | Score |
| :---: | :---: | :---: | :---: | :---: |
| ALPHA | $\eta=0.505 d=10$ | 1.51 | ONEAudit | 1.52 |
|  | $\eta=0.505 d=100$ | 1.54 |  | 1.55 |
|  | $\eta=0.505 d=1000$ | 1.79 |  | 1.83 |
|  | $\eta=0.505 d=\infty$ | 3.02 |  | 3.83 |
|  | $\eta=0.51 \mathrm{~d}=10$ | 1.51 |  | 1.52 |
|  | $\eta=0.51 d=100$ | 1.53 |  | 1.55 |
|  | $\eta=0.51 d=1000$ | 1.72 |  | 1.80 |
|  | $\eta=0.51 d=\infty$ | 2.29 |  | 3.05 |
|  | $\eta=0.52 d=10$ | 1.51 |  | 1.53 |
|  | $\eta=0.52 d=100$ | 1.51 |  | 1.55 |
|  | $\eta=0.52 d=1000$ | 1.61 |  | 1.75 |
|  | $\eta=0.52 d=\infty$ | 1.84 |  | 2.32 |
|  | $\eta=0.55 d=10$ | 1.51 |  | 1.55 |
|  | $\eta=0.55 d=100$ | 1.47 |  | 1.56 |
|  | $\eta=0.55 d=1000$ | 1.44 |  | 1.62 |
|  | $\eta=0.55 d=\infty$ | 1.88 |  | 1.81 |
|  | $\eta=0.6 d=10$ | 1.50 |  | 1.59 |
|  | $\eta=0.6{ }^{\text {d }}$ d $=100$ | 1.45 |  | 1.57 |
|  | $\eta=0.6 d=1000$ | 1.51 |  | 1.52 |
|  | $\eta=0.6 d=\infty$ | 2.42 |  | 1.84 |
| SqKelly |  | 1.98 |  |  |
| a priori Kelly | $\eta=0.505$ | 2.77 |  |  |
|  | $\eta=0.51$ | 1.88 |  |  |
|  | $\eta=0.52$ | 1.60 |  |  |
|  | $\eta=0.55$ | 2.14 |  |  |
|  | $\eta=0.6$ | 3.34 |  |  |

Table 6. Geometric mean of the ratios of sample sizes to the smallest sample size for each condition described above. The smallest ratio is in bold font. In the simulations, for the hypothesis tests considered, the ONEAudit transformation entails a negligible loss in efficiency compared to ALPHA applied to the "raw" assorter.

## 6 Conclusions

Ballot-polling risk-limiting audits (BPAs) are algebraically equivalent to cardlevel comparison risk-limiting audits (CLCAs) using faux cast-vote records (CVRs) that match the overall reported results. Any set of CVRs that reproduces the reported contest tallies (more generally, reproduces reported assorter totals) has the same net overstatement of the margin: they are "overstatement-netequivalent" (ONE) to the voting system's tabulation and can be used as if the voting system had exported them.

ONE CVRs let audits use batch-level data far more efficiently than traditional batch-level comparison RLAs (BLCAs) do: create ONE CVRs for each batch, then apply CLCA as if the voting system had provided those CVRs. For real and simulated data, this saves a large mount of work compared to manually tabulating the votes on every card in the batches selected for audit, as BLCAs require. If batches are sufficiently homogeneous, the workload approaches that of "pure" CLCA using linked CVRs from the voting system. BLCAs also require locating and retrieving every card in each batch that is selected for audit. That is straightforward when reporting batches are identifiable physical batches, but not when physical batches contain a mixture of ballot cards from different reporting batches, which is common in jurisdictions that use vote centers or do not sort
vote-by-mail ballots before scanning them. In the California presidential election in 2020 and the Georgia election for Secretary of State in 2022, this approach would reduce the workload by a factor of 75 to 380 , respectively, compared to the most efficient known method for BLCA.

ONE CVRs also can obviate the need to stratify, to rescan cards, or to use "hybrid" audits when the voting system cannot export a linked CVR for every card: create ONE CVRs for the cards that lack them, then apply CLCA as if the CVRs had been provided by the voting system. The same CLCA software can be used to audit voting systems that can export a linked CVR for every card and also those that cannot. For data from a 2018 pilot audit in Kalamazoo, MI, ONE CLCA gives a measured risk much smaller than that of SUITE ( $2 \%$ versus $3.7 \%$ ), the hybrid method used in the pilot. Stratification and hybrid audits increase the complexity and opacity of audits, and rescanning substantially increases time and cost. Hence, ONEAudit may be cheaper, faster, simpler, and more transparent than previous methods. Software used to generate the tables and figures is available at https://www.github.com/pbstark/ONEAudit.

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## A Detailed simulation results for BPA versus CLCA using ONE CVRs based on contest totals



Table 7. Mean sample sizes to reject the null that the assorter mean does not exceed $1 / 2$ when the fraction of valid votes for the winner is 0.505 , for various population sample sizes, numbers of blank/invalid cards, based on 1,000 replications. The smallest sample size for each of the 12 conditions is in bold font. Some flavor of ALPHA applied to the ONEAudit transformation of the assorter values had the smallest sample size in 6 of the 12 conditions; some flavor of ALPHA applied to the raw assorter values had the smallest in 4 conditions, and some flavor of a priori Kelly applied to the raw assorter values had the smallest in 2 conditions.


Table 8. Same as table 7, but with a fraction 0.51 of the valid votes for the reported winner. Some flavor of ALPHA applied to the ONEAudit transformation of the assorter values had the smallest sample size for 6 conditions; some flavor of ALPHA applied to the raw assorter values had the smallest for 4 conditions, and some flavor of a priori Kelly applied to the raw assorter values had the smallest for 2 conditions.


Table 9. Same as table 7, but with a fraction 0.52 of the valid votes for the reported winner. Some flavor of ALPHA applied to the ONEAudit transformation of the assorter values had the smallest sample size for 4 conditions; some flavor of ALPHA applied to the raw assorter values had the smallest for 2 conditions, and some flavor of a priori Kelly applied to the raw assorter values had the smallest for 6 conditions, of which 5 used the true population mean.

|  |  |  | $N=10,000, \%$ blank |  |  |  | $N=100,000 \%$ blank |  |  |  | $N=500,000 \%$ blank |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{\theta}{0.55}$ | Method | params | 10 | 25 | 50 | 75 | 10 | 25 | 50 | 75 | 10 | 25 |  | 75 |
|  | sqKelly |  | 594 | 748 | 1,067 | 1,834 | 688 | 845 | 1,208 | 2,331 | 705 | 787 | 1,213 | 2,446 |
|  |  | $\eta=0.505$ | 2,919 | 3,426 | 4,634 | 7,180 | 3,435 | 4,144 | 6,098 | 11,800 | 3,526 | 4,167 | 6,242 | 12,311 |
| ALPHA |  | $\eta=0.505 d=10$ | 790 | 1,009 | 1,625 | 3,577 | 946 | 1,180 | 1,973 | 5,799 | 939 | 1,113 | 2,044 | 6,158 |
|  |  | $d=100$ | 790 | 1,043 | 1,694 | 3,683 | 930 | 1,187 | 2,062 | 6,012 | 935 | 1,155 | 2,117 | 6,360 |
| ONEAudit $\eta$ |  | $d=1000$ | 1,057 | 1,342 | 2,114 | 4,154 | 1,234 | 1,582 | 2,690 | 7,135 | 1,245 | 1,577 | 2,773 | 7,536 |
|  |  | $d=\infty$ | 1,800 | 2,217 | 3,315 | 5,500 | 3,284 | 4,501 | 8,283 | 20,041 | 3,767 | 5,221 | 10,930 | 34,385 |
|  |  | $\eta=0.505 d=10$ | 791 | 1,012 | 1,633 | 3,593 | 945 | 1,185 | 1,984 | 5,842 | 941 | 1,114 | 2,056 | 6,195 |
|  |  | $d=100$ | 799 | 1,054 | 1,710 | 3,706 | 942 | 1,202 | 2,083 | 6,067 | 945 | 1,170 | 2,138 | 6,417 |
|  |  | $d=1000$ | 1,108 | 1,395 | 2,174 | 4,205 | 1,308 | 1,662 | 2,792 | 7,276 | 1,317 | 1,656 | 2,869 | 7,693 |
|  |  | $d=\infty$ | 2,084 | 2,489 | 3,550 | 5,648 | 4,965 | 6,489 | 10,806 | 22,868 | 6,750 | 9,133 | 17,700 | 46,539 |
| ALPHA |  | $\eta=0.51$ | 1,659 | 1,969 | 2,809 | 4,689 | 1,831 | 2,220 | 3,256 | 6,430 | 1,856 | 2,203 | 3,322 | 6,561 |
|  |  | $\eta=0.51 \mathrm{~d}=10$ | 790 | 1,006 | 1,623 | 3,574 | 945 | 1,179 | 1,972 | 5,796 | 936 | 1,110 | 2,038 | 6,154 |
|  |  | $d=100$ | 778 | 1,024 | 1,675 | 3,668 | 916 | 1,173 | 2,043 | 5,982 | 921 | 1,135 | 2,092 | 6,339 |
| ONEAudit |  | $d=1000$ | 968 | 1,254 | 2,020 | 4,084 | 1,122 | 1,459 | 2,542 | 6,955 | 1,132 | 1,448 | 2,610 | 7,346 |
|  |  | $d=\infty$ | 1,395 | 1,799 | 2,906 | 5,236 | 1,928 | 2,710 | 5,431 | 15,712 | 2,023 | 2,826 | 6,067 | 21,572 |
|  |  | $\eta=0.51 d=10$ | 792 | 1,011 | 1,637 | 3,611 | 944 | 1,186 | 1,992 | 5,875 | 941 | 1,116 | 2,064 | 6,236 |
|  |  | $d=100$ | 796 | 1,052 | 1,710 | 3,714 | 938 | 1,199 | 2,083 | 6,102 | 941 | 1,166 | 2,140 | 6,452 |
|  |  | $d=1000$ | 1,068 | 1,352 | 2,137 | 4,186 | 1,248 | 1,601 | 2,726 | 7,224 | 1,256 | 1,593 | 2,801 | 7,641 |
|  |  | $d=\infty$ | 1,816 | 2,236 | 3,339 | 5,531 | 3,320 | 4,549 | 8,378 | 20,256 | 3,806 | 5,285 | 11,077 | 34,892 |
| ALPHA |  | $\eta=0.52$ | 952 | 1,163 | 1,673 | 2,930 | 1,045 | 1,265 | 1,858 | 3,656 | 1,047 | 1,242 | 1,869 | 3,749 |
|  |  | $\eta=0.52 d=10$ | 790 | 1,004 | 1,616 | 3,570 | 944 | 1,173 | 1,968 | 5,781 | 932 | 1,108 | 2,028 | 6,143 |
|  |  | $d=100$ | 751 | 997 | 1,646 | 3,638 | 891 | 1,144 | 2,001 | 5,921 | 896 | 1,100 | 2,047 | 6,280 |
| ONEAudit |  | $d=1000$ | 821 | 1,096 | 1,853 | 3,944 | 955 | 1,250 | 2,274 | 6,580 | 960 | 1,238 | 2,328 | 6,974 |
|  |  | $d=\infty$ | 948 | 1,283 | 2,266 | 4,721 | 1,117 | 1,548 | 3,160 | 10,491 | 1,126 | 1,562 | 3,304 | 11,965 |
|  |  | $\eta=0.52 d=10$ | 793 | 1,015 | 1,649 | 3,648 | 945 | 1,190 | 2,009 | 5,954 | 942 | 1,120 | 2,077 | 6,322 |
|  |  | $d=100$ | 786 | 1,044 | 1,712 | 3,737 | 926 | 1,191 | 2,087 | 6,150 | 930 | 1,157 | 2,144 | 6,507 |
|  |  | $d=1000$ | 983 | 1,278 | 2,060 | 4,153 | 1,147 | 1,492 | 2,602 | 7,134 | 1,153 | 1,477 | 2,676 | 7,541 |
|  |  | $d=\infty$ | 1,420 | 1,830 | 2,955 | 5,302 | 1,965 | 2,767 | 5,564 | 16,093 | 2,067 | 2,889 | 6,242 | 22,239 |
| ALPHA |  | $\eta=0.55$ | 581 | 737 | 1,054 | 1,816 | 673 | 844 | 1,186 | 2,330 | 692 | 780 | 1,189 | 2,413 |
|  |  | $\eta=0.55 d=10$ | 788 | 996 | 1,606 | 3,553 | 939 | 1,163 | 1,951 | 5,742 | 932 | 1,099 | 2,006 | 6,106 |
|  |  | $d=100$ | 696 | 926 | 1,557 | 3,566 | 826 | 1,066 | 1,887 | 5,749 | 836 | 1,020 | 1,931 | 6,099 |
| ONEAudit |  | $d=1000$ | 616 | 831 | 1,458 | 3,540 | 715 | 945 | 1,718 | 5,633 | 725 | 891 | 1,750 | 5,912 |
|  |  | $d=\infty$ | 594 | 790 | 1,381 | 3,480 | 681 | 884 | 1,581 | 5,200 | 691 | 846 | 1,589 | 5,400 |
|  |  | $\eta=0.55 d=10$ | 795 | 1,029 | 1,683 | 3,739 | 945 | 1,204 | 2,059 | 6,194 | 951 | 1,137 | 2,127 | 6,591 |
|  |  | $d=100$ | 764 | 1,023 | 1,714 | 3,803 | 903 | 1,176 | 2,092 | 6,331 | 910 | 1,129 | 2,148 | 6,705 |
|  |  | $d=1000$ | 797 | 1,079 | 1,867 | 4,056 | 925 | 1,229 | 2,287 | 6,870 | 928 | 1,215 | 2,344 | 7,275 |
|  |  | $d=\infty$ | 863 | 1,177 | 2,143 | 4,664 | 1,000 | 1,377 | 2,825 | 9,605 | 999 | 1,366 | 2,922 | 10,626 |
| apkely |  | $\eta=0.6$ | 1,044 |  | 1,557 | 2,143 | 2,795 | 3,568 | 3,811 | 5,811 | 6,072 | 5,010 | 7,595 | 11,746 |
|  |  | $\eta=0.6 \quad d=10$ | 787 | 995 | 1,585 | 3,527 | 939 | 1,155 | 1,930 | 5,690 | 940 | 1,096 | 1,974 | 6,050 |
|  |  | $d=100$ | 689 | 872 | 1,444 | 3,430 | 812 | 999 | 1,724 | 5,483 | 815 | 949 | 1,783 | 5,805 |
| ONEAudit |  | $d=1000$ | 685 | 775 | 1,149 | 2,977 | 845 | 908 | 1,334 | 4,395 | 870 | 840 | 1,347 | 4,622 |
|  |  | $d=\infty$ | 898 | 831 | 1,045 | 2,446 | 1,537 | 1,112 | 1,187 | 3,097 | 2,107 | 1,011 | 1,189 | 3,183 |
|  |  | $\eta=0.6 d=10$ | 802 | 1,057 | 1,753 | 3,905 | 951 | 1,231 | 2,153 | 6,596 | 957 | 1,170 | 2,224 | 7,056 |
|  |  | $d=100$ | 730 | 1,002 | 1,724 | 3,922 | 868 | 1,157 | 2,108 | 6,629 | 879 | 1,099 | 2,162 | 7,072 |
|  |  | $d=1000$ | 646 | 888 | 1,607 | 3,915 | 750 | 1,007 | 1,916 | 6,511 | 752 | 960 | 1,966 | 6,889 |
|  |  | $d=\infty$ | 616 | 835 | 1,514 | 3,851 | 703 | 931 | 1,746 | 6,020 | 706 | 896 | 1,772 | 6,269 |

Table 10. Same as table 7, but with a fraction 0.55 of the valid votes for the reported winner. ALPHA applied to the ONEAudit transformation never had the smallest sample size; some flavor of ALPHA applied to the raw assorter values had the smallest (or was tied for smallest) in 3 conditions, and a priori Kelly using the true population mean had the smallest (or was tied for smallest) in 10 conditions.


Table 11. Same as table 7, but with a fraction 0.6 of the valid votes for the reported winner. A priori Kelly applied to the raw assorter values using the correct population mean had the smallest sample size for all 12 conditions.


[^0]:    ${ }^{1}$ https://sos.ga.gov/news/georgias-2022-statewide-risk-limiting-audit-confirms-results, last visited 26 February 2023.
    ${ }^{2}$ https://www.stat/berkeley.edu/~stark/Preprints/cgg-rept-10.pdf, last visited 15 December 2022.

[^1]:    ${ }^{3}$ See https://github.com/kellieotto/mirla18/blob/master/code/kalamazoo_ SUITE. ipynb.

