# Data reduction for directed feedback vertex set on graphs without long induced cycles 

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#### Abstract

We study reduction rules for Directed Feedback VerTEX SET (DFVS) on instances without long cycles. A DFVS instance without cycles longer than $d$ naturally corresponds to an instance of $d$ Hitting Set, however, enumerating all cycles in an $n$-vertex graph and then kernelizing the resulting $d$-Hitting Set instance can be too costly, as already enumerating all cycles can take time $\Omega\left(n^{d}\right)$. To the best of our knowledge, the kernelization of DFVS on graphs without long cycles has not been studied in the literature, except for very restricted cases, e.g., for tournaments, in which all induced cycles are of length three. We show how to compute a kernel with at most $2^{d} k^{d}$ vertices and at most $d^{3 d} k^{d}$ induced cycles of length at most $d$ (which however, cannot be enumerated efficiently), where $k$ is the size of a minimum directed feedback vertex set. We then study classes of graphs whose underlying undirected graphs have bounded expansion or are nowhere dense; these are very general classes of sparse graphs, containing e.g. classes excluding a minor or a topological minor. We prove that for such classes without induced cycles of length greater than $d$ we can compute a kernel with $\mathcal{O}_{d}(k)$ and $\mathcal{O}_{d, \varepsilon}\left(k^{1+\varepsilon}\right)$ vertices for any $\varepsilon>0$, respectively, in time $\mathcal{O}_{d}\left(n^{\mathcal{O}(1)}\right)$ and $\mathcal{O}_{d, \varepsilon}\left(n^{\mathcal{O}(1)}\right)$, respectively. The most restricted classes we consider are strongly connected planar graphs without any (induced or non-induced) long cycles. We show that these have bounded treewidth and hence DFVS on planar graphs without cycles of length greater than $d$ can be solved in time $2^{\mathcal{O}(d)} \cdot n^{\mathcal{O}(1)}$. We finally present a new data reduction rule for general DFVS and prove that the rule together with a few standard rules subsumes all the rules applied by Bergougnoux et al. to obtain a polynomial kernel for DFVS[FVS], i.e., DFVS parameterized by the feedback vertex set number of the underlying (undirected) graph. We conclude by studying the LP-based approximation of DFVS.


## 1 Introduction

A directed feedback vertex set of a directed $n$-vertex graph $G$ is a subset $S \subseteq V(G)$ of vertices such that every directed cycle of $G$ intersects with $S$. In the Directed Feedback Vertex Set (DFVS) problem, we are given a directed graph $G$ and
an integer $k$, and the objective is to determine whether $G$ admits a directed feedback vertex set of size at most $k$. In what follows, unless stated otherwise, when we speak of a graph we always mean a directed graph, and when we speak of a cycle we mean a directed cycle.

DFVS is one of Karp's 21 NP-complete problems [Kar72]. Its NP-completeness follows easily by a reduction from Vertex Cover, which is a special case of DFVS where all induced cycles have length two. The fastest known exact algorithm for DFVS, due to Razgon [Raz07], runs in time $\mathcal{O}\left(1.9977^{n} \cdot n^{\mathcal{O}(1)}\right)$. Chen, Liu, Lu, O'Sullivan, and Razgon [CLL $\left.{ }^{+} 08\right]$ proved that the problem is fixed-parameter tractable when parameterized by solution size $k$; providing an algorithm running in time $\mathcal{O}\left(k!4^{k} k^{4} n m\right)=2^{\mathcal{O}(k \log k)} \cdot n m$, for graphs with $n$ vertices and $m$ edges. The dependence on the input size has been improved to $\mathcal{O}\left(k!4^{k} k^{5}(n+m)\right)$ by Lokshtanov, Ramanujan, and Saurabh [LRS16]. It is a major open problem in parameterized complexity whether the running time can be improved to $2^{o(k \log k)} \cdot n^{\mathcal{O}(1)}$ [LRS16]. The problem has also been studied under different parameterizations. Bonamy et al. $\left[\mathrm{BKN}^{+} 18\right]$ proved that one can solve the problem in time $2^{\mathcal{O}(t \log t)} \cdot n^{\mathcal{O}(1)}$, where $t$ denotes the treewidth of the underlying undirected graph. They also proved that this running time is tight assuming the exponential-time hypothesis (ETH). On planar graphs the running time can be improved to $2^{\mathcal{O}(t)} \cdot n^{\mathcal{O}(1)}$. A natural question is whether these results can be extended to directed width measures, e.g., whether the problem is fixedparameter tractable when parameterized by directed treewidth. Unfortunately, this is not the case. DFVS remains NP-complete even on very restricted classes of graphs such as graphs of cycle rank at most four (which in particular have bounded directed treewidth), as shown by Kreutzer and Ordyniak [KO11], and hence the problem is not even in XP when parameterized by cycle rank.

The question whether DFVS parameterized by solution size $k$ admits a polynomial kernel, i.e., an equivalent polynomial-time computable instance of size polynomial in $k$, remains one of the central open questions in the area of kernelization. Bergougnoux et al. $\left[\mathrm{BEG}^{+} 21\right]$ showed that the problem admits a kernel of size $\mathcal{O}\left(f^{4}\right)$ in general graphs and $\mathcal{O}(f)$ in graphs embeddable on a fixed surface, where $f$ denotes the size of a minimum undirected feedback vertex set in the underlying undirected graph. Note that $f$ can be arbitrarily larger than $k$. More generally, for an integer $\eta$, a subset $M \subseteq V(G)$ of vertices is called a treewidth $\eta$-modulator if $G-M$ has treewidth at most $\eta$. Lokshtanov et al. [LRS $\left.{ }^{+} 19\right]$ showed that when given a graph $G$, an integer $k$, and a treewidth $\eta$-modulator of size $\ell$, one can compute a kernel with $(k \cdot \ell)^{\mathcal{O}\left(\eta^{2}\right)}$ vertices. This result subsumes the result of Bergougnoux et al. $\left[\mathrm{BEG}^{+} 21\right]$, as the parameter $k+\ell$ is upper bounded by $\mathcal{O}(f)$ and can be arbitrarily smaller than $f$. On the other hand, unless NP $\subseteq$ coNP/poly, for $\eta \geq 2$, there cannot exist a polynomial kernel when we parameterize by the size of a treewidth- $\eta$ modulator alone, as even Vertex Cover cannot have a polynomial kernel when parameterized by the size of a treewidth-2 modulator [CLP $\left.{ }^{+} 14\right]$. Polynomial kernels for DFVS are known for several restricted graph classes, see e.g. [BJMS16, DGH $\left.{ }^{+} 10, \mathrm{FLL}^{+} 19\right]$.

From the viewpoint of approximation, the best known algorithms for DFVS are based on integer linear programs whose fractional relaxations can be solved efficiently. It was shown by Seymour [Sey95] that the integrality gap for DFVS is at most $\mathcal{O}\left(\log k^{*} \log \log k^{*}\right)$, where $k^{*}$ denotes the optimal value of a fractional directed feedback vertex set. Note that the linear programming formulation of DFVS may contain an exponential number of constraints. Even et al. $[E S S+98]$ circumvented this obstacle and provided a related combinatorial polynomial-time algorithm yielding an $\mathcal{O}\left(\log k^{*} \log \log k^{*}\right) \subseteq \mathcal{O}(\log k \log \log k)$ approximation. Assuming the Unique Games Conjecture (UGC), the problem does not admit a polynomial-time computable constant-factor approximation algorithm $\left[\mathrm{GHM}^{+}\right.$11, GL16, Sve12]. Lokshtanov et al. [LMR ${ }^{+}$21] showed how to compute a 2-approximation in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

This work was initiated after successfully participating in the PACE 2022 programming challenge [GHSS22]. In the scope of a student project at the University of Bremen, we participated in the competition and our solver ranked second in the exact track $\left[\mathrm{BDF}^{+} 22\right]$. In this paper we present our theoretical findings, whereas an empirical evaluation of the implemented rules will be presented in future work.

We first study DFVS instances without long cycles. This study is intimately linked to the study of the Hitting Set problem. Many of the known data reduction rules for DFVS are special cases of general reduction rules for Hitting SET. A hitting set in a set system $\mathcal{G}$ with ground set $V(\mathcal{G})$ and edge set $E(\mathcal{G})$, where each $S \in E(\mathcal{G})$ is a subset of $V(\mathcal{G})$, is a subset $H \subseteq V(\mathcal{G})$ such that $H \cap S \neq \varnothing$ for all $S \in E(\mathcal{G})$. Given a graph $G$, a directed feedback vertex set in $G$ corresponds one-to-one to a hitting set for the set system $\mathcal{G}$ where $V(\mathcal{G})=V(G)$ and $E(\mathcal{G})=\{V(C) \mid C$ is a cycle in $G\}$. The main difficulty in applying reduction rules designed for Hitting SEt is that we first need to efficiently convert an instance of DFVS to an instance of Hitting Set. However, in general, we want to avoid computing $\mathcal{G}$ from $G$, as $|E(\mathcal{G})|$ may be super-polynomial in the size of the vertex set, i.e., super-polynomial in $|V(G)|=|V(\mathcal{G})|$. One simple reduction rule for Hitting Set is to remove all sets $S \in E(\mathcal{G})$ such that there exists $S^{\prime} \in E(\mathcal{G})$ with $S^{\prime} \subseteq S$. Instances of Hitting Set that do not contain such pairs of sets are called vertex induced. The remaining minimal sets in the corresponding DFVS instance are the induced cycles of $G$. It follows that in a DFVS instance it suffices to hit all induced cycles. Unfortunately, it is NPcomplete to detect if a vertex or an edge lies on an induced cycle [FKMP95] even on planar graphs, implying that it is not easy to exploit this property for DFVS directly. Overcoming this obstacle requires designing data reduction rules based on sufficient conditions guaranteeing that a vertex or an edge does not lie on an induced cycle and can therefore be safely removed.

An instance of DFVS without cycles of length greater than $d$ naturally corresponds to an instance of $d$-Hitting Set. As shown in [AK10], $d$-Hitting SET admits a kernel with $k+(2 d-1) k^{d-1}$ vertices, which can be efficiently computed when the $d$-Hitting Set instance is explicitly given as input. This is known to be near optimal, as $d$-Hitting Set does not admit a kernel of
size $\mathcal{O}\left(k^{d-\varepsilon}\right)$ unless the polynomial hierarchy collapses [DVM14]; note that here size refers to the total size of the instance and not to the number of vertices. The question of whether there exists a kernel for $d$-Hitting Set with fewer elements is considered to be one of the most important open problems in kernelization $\left[\mathrm{BFG}^{+} 11, \mathrm{DGH}^{+} 10, \mathrm{FLL}^{+} 23, \mathrm{FLSZ19}, \mathrm{YWC17}\right]$. However, even in this restricted case we cannot efficiently generate the $d$-Hitting Set instance from the DFVS instance, as even testing if a vertex lies on an induced cycle of length at most $d$ is W[1]-hard [HH06] when parameterized by $d$. We hence have to avoid computing a Hitting Set instance explicitly but must rather work on the implicit graph representation of a DFVS instance. To the best of our knowledge the kernelization of DFVS on graphs without long cycles has not been studied in the literature, except for some very restricted cases, e.g., on tournaments in which all induced cycles are of length three $\left[\mathrm{BFG}^{+} 11, \mathrm{DGH}^{+} 10, \mathrm{FLL}^{+} 19\right]$.

We show that after applying the standard reduction rules we can compute in polynomial time a superset $W$ of the vertices that lie on induced cycles of length at most $d$ and which is of size at most $2^{d} k^{d}$. As it suffices to hit all induced cycles, $G[W]$ is an equivalent instance. Up to a factor $k$ and constants depending only on $d$ this matches the best bounds we know for the kernelization of $d$ Hitting Set. Potentially in the kernelized instance on $2^{d} k^{d}$ vertices we could have $\left(2^{d} k^{d}\right)^{d}=2^{d^{2}} k^{d^{2}}$ induced cycles. Based on the classical sunflower lemma, we prove however, that kernelized instances contain at most $d^{3 d} k^{d}$ induced cycles of length at most $d$, for any fixed $d \geq 2$. In light of the major open question whether DFVS admits a polynomial kernel, we pose as a question whether it admits a kernel of size $\mathcal{O}_{d}\left(k^{\mathcal{O}(1)}\right)$ computable in time $\mathcal{O}_{d}\left(n^{\mathcal{O}(1)}\right)$ on instances without induced cycles of length greater than $d$.

We then turn our attention to restricted graph classes for which we can efficiently test whether a vertex lies on an induced cycle of length at most $d$, e.g., by efficient algorithms for first-order model-checking [DMS23, GKS17]. We study classes of graphs whose underlying undirected graphs have bounded expansion or are nowhere dense. These are very general classes of sparse graphs [NDM08, NdM11], including, e.g., all classes that exclude a minor or a topological minor, such as planar graphs. We show that DFVS on classes of bounded expansion admits a kernel with $\mathcal{O}_{d}(k)$ vertices, and a kernel with $\mathcal{O}_{d, \varepsilon}\left(k^{1+\varepsilon}\right)$ vertices, for any $\varepsilon>0$, on nowhere dense classes, respectively, computable in time $\mathcal{O}_{d}\left(n^{\mathcal{O}(1)}\right)$ and $\mathcal{O}_{d, \varepsilon}\left(n^{\mathcal{O}(1)}\right)$, respectively. This answers our above question for very general classes of sparse graph positively. Our method is based on the approach of $\left[\mathrm{DDF}^{+} 16, \mathrm{EGK}^{+} 17\right]$ for the kernelization of the Distance- $r$ Dominating SET problem on bounded expansion and nowhere dense classes.

We conclude our study of restricted graph classes by observing that a strongly connected planar graph without any long (induced or non-induced) cycles has bounded treewidth. We observe that after the application of the reduction rules, weak components are equal to strong components. Hence, the DAG of strong components in fact is a tree. Then, if each strong component has bounded treewidth, we can combine the tree decompositions of the strong components with the tree
of strong components to derive that the whole graph after application of the rules has has bounded treewidth and solve it efficiently by the algorithm of Bonamy et al. [BKN $\left.{ }^{+} 18\right]$.

We proceed by designing a new data reduction rule that provides a sufficient condition for a vertex or edge to lie on an induced cycle. The new rule conveniently generalizes many of the complicated rules presented by Bergougnoux et al. $\left[\mathrm{BEG}^{+} 21\right]$ to establish a kernel of size $\mathcal{O}\left(f^{4}\right)$, where $f$ is the size of a minimum feedback vertex set for the underlying undirected graph. In addition to being simpler, our rule does not require the initial computation of a feedback vertex set for the underlying undirected graph.

Finally, we study the LP-based approximation of DFVS. As previously mentioned, we can formulate an integer linear program (ILP) that is equivalent to DFVS. In the natural formulation, which we call the cycles $I L P$, we introduce a binary variable $d_{v}$ for every $v \in V(G)$ where $d_{v}=1$ means that $v$ is part of a solution. The goal is to minimize the number of variables set to 1 , given that all induced cycles are hit. Note that this formulation can have an exponential number of constraints. Instead, we study an equivalent ILP of polynomial size, the order $I L P$, which is based on the fact that a directed graph is acyclic if and only if there is a topological order on its vertex set. We prove that the optimal solution to the LP-relaxation of the order ILP is at most 3 times smaller than the optimal solution to the LP-relaxation of the cycles ILP. This makes LPbased approximation approaches directly accessible and avoids the specialized combinatorial algorithm of Even et al. [ESS $\left.{ }^{+} 98\right]$.

## 2 Preliminaries

A graph $G$ consists of a (non-empty) vertex set $V(G)$ and edge set $E(G) \subseteq$ $V(G) \times V(G)$. For vertices $u, v \in V(G)$ we write $u v$ for the edge directed from $u$ to $v$. An edge $v v$ is called a loop. We denote the in- and out-neighborhood of $v \in V(G)$ by $N_{G}^{-}(v)=\{u \mid u v \in E(G)\}$ and $N_{G}^{+}(v)=\{u \mid v u \in E(G)\}$, respectively. The neighborhood of $v$ is denoted by $N_{G}(v)=N_{G}^{-}(v) \cup N_{G}^{+}(v)$. A cycle $C$ in a graph $G$ is a sequence of vertices $v_{1} v_{2} \ldots v_{\ell+1}$ such that $v_{1}=v_{\ell+1}$, $v_{i} \neq v_{j}$ for all $i \neq j \leq \ell$, and $v_{i} v_{i+1} \in E(G)$ for all $i \leq \ell$. We denote the set of vertices that appear in $C$ by $V(C)=\left\{v_{1}, \ldots, v_{\ell}\right\}$. We denote by $\ell$ the length of $C$. A $u$ - v-path is a sequence of vertices $v_{1} v_{2} \ldots v_{\ell+1}$ of pairwise distinct vertices such that $v_{1}=u$ and $v_{\ell+1}=v$. Likewise, we define $V(P)=\left\{v_{1}, \ldots, v_{\ell+1}\right\}$ and call $\ell$ the length of the path, that is, the number of edges of $P$. For a $u$ - $v$-path $P$ and a $v$-w-path $Q$ we write $P Q$ for the $u$-w-walk obtained by concatenating $P$ and $Q$ (removing the repetition of $v$ in the middle). Recall that a $u$-w-walk in a graph $G$ implies the existence of a $u$-w-path in $G$ using a subset of the vertices and edges of the walk. By a slight abuse of notation, we sometimes use $P Q$ to denote the $u$-w-path. For a set $S \subseteq V(G)$, we write $N_{G}^{d+}[S]$ to denote the $d$-out-neighborhood of $S$ in $G$ and $N_{G}^{d-}[S]$ to denote the $d$-in-neighborhood of $S$ in $G$. That is, $N_{G}^{d+}[S]$ contains all vertices of $G$ that are reachable from some vertex in $S$ via a path of length at most $d \geq 0$, and $N_{G}^{d-}[S]$ contains all vertices
of $G$ that can reach some vertex in $S$ via a path of length at most $d$. Note that since paths of length zero are allowed, we have $S \subseteq N_{G}^{d+}[S] \cap N_{G}^{d-}[S]$. We write $N_{G}^{d+}[v]$ and $N_{G}^{d-}[v]$ whenever $S=\{v\}$.

For a vertex subset $U \subseteq V(G)$, we denote by $G[U]$ the graph induced by $U$, that is the graph obtained from $G$ where we only keep the vertices in $U$ and the edges incident on them. We write $G-U$ for the graph $G[V(G) \backslash U]$ and for a singleton vertex set $\{v\}$ we write $G-v$ instead of $G-\{v\}$. For an edge $u v$ we write $G+u v$ and $G-u v$ for the graph obtained by adding or removing the edge $u v$, respectively. A cycle $C$ is an induced cycle if the graph $G[V(C)]$ is isomorphic to a cycle and a path $P$ is an induced path if the graph $G[V(P)]$ induces a path. We call a $u$ - $v$-path almost induced if $P$ is an induced path in $G-v u$. Slightly abusing notation, when $u, v$ are distinct vertices on an induced cycle $C$, we will say that $C$ decomposes into an induced $u$ - $v$-path and an induced $v$-u-path, even though this is not true if $u v \in E(G)$, in which case the $u$-v-path is only almost induced.

We call a set $S \subseteq V(G)$ a directed feedback vertex set, dfvs for short, if $G-S$ does not contain any (directed) cycles. An input of the Directed Feedback Vertex Set problem consists of a graph $G$ and a positive integer $k$. The goal is to determine whether $G$ admits a dfvs of size at most $k$. We will constantly make use of the following simple lemma.

Lemma 2.1. Let $G$ be a graph without induced cycles of length greater than $d$ and let $k$ be a positive integer. Then, we can compute in polynomial time either a $d f v s$ of size at most $d k$ or decide that there does not exist a dfvs of size at most $k$ in $G$.

Proof. By greedily packing induced (pairwise) vertex-disjoint cycles we can find at most $k$ cycles (each consisting of at most $d$ vertices), otherwise we can conclude that no solution of size at most $k$ exists. Then, all other (non-packed) induced cycles intersect one of the packed cycles since $G$ does not contain induced cycles of length greater than $d$.

Observe that every cycle contains an induced cycle. Hence, even though we cannot decide efficiently whether a vertex lies on an induced cycle we can efficiently pack induced pairwise vertex-disjoint cycles as needed.

A hypergraph (also called a set system) $\mathcal{G}$ consists of a (non-empty) vertex set $V(\mathcal{G})$ and hyperedge set $E(\mathcal{G})=\left\{S_{1}, \ldots, S_{m}\right\}$, where $S_{i} \subseteq V(\mathcal{G})$ for all $i \leq m$. For a vertex subset $U \subseteq V(\mathcal{G})$, we denote by $\mathcal{G}[U]$ the hypergraph $\mathcal{G}$ induced by $U$, that is, the hypergraph with vertex set $U$ and hyperedge set $\left\{S_{i} \in E(G) \mid\right.$ $\left.S_{i} \subseteq U\right\}$. Note that we only keep those hyperedges that are fully contained in $U$. A hitting set of a hypergraph $\mathcal{G}$ is a set $H \subseteq V(\mathcal{G})$ such that $H \cap S_{i} \neq \varnothing$ for all $i \leq m$. In other words, $H$ contains at least one vertex from every hyperedge. The input of a Hitting Set instance consists of a hypergraph $\mathcal{G}$ and a positive integer $k$, where $E(\mathcal{G})$ explicitly enumerates all sets. The goal is to determine whether $\mathcal{G}$ admits a hitting set of size at most $k$. Given a graph $G$, a directed feedback vertex set in $G$ corresponds one-to-one to a hitting set for the set
system $\mathcal{G}$ where $V(\mathcal{G})=V(G)$ and $E(\mathcal{G})=\{V(C) \mid C$ is an induced cycle in $G\}$. In the following, with any graph $G$ we associate the corresponding hypergraph $\mathcal{G}$. We call $\mathcal{G}$ vertex induced if there are no two sets $S, S^{\prime} \in E(\mathcal{G})$ with $S^{\prime} \subseteq S$.

### 2.1 Standard reduction rules

We present standard reduction rules from the literature, of which many are instantiations of standard rules for Hitting Set. We also list and attribute special cases that may be more efficiently computable. These rules will be used in the later sections.

The first rule is presented, e.g., as Rule 1 in [FWY09].
Rule DFVS 1. If $v \in V(G)$ lies on a loop then add $v$ to the solution, remove $v$ from $G$, and decrease the parameter by one.

The next rule is based on a folklore rule for Hitting Set, which in the literature is often attributed to [Wei98]. If there are two vertices $u, v \in V(\mathcal{G})$ such that $u$ appears in every hyperedge in which $v$ appears then remove $v$. We say that $u$ dominates $v$. Removing an element from the universe of $\mathcal{G}$ almost corresponds to the following operation in $G$. For $v \in V(G)$, we write $G \ominus v$ for the graph obtained by connecting all in-neighbors of $v$ with all out-neighbors of $v$ and then removing $v$, or simply removing $v$ if it has no in- or out-neighbors. We say that $G \ominus v$ is obtained from $G$ by shortcutting $v$. Shortcutting may introduce new cycles that cannot be recovered in $G$ by re-inserting $v$. However, shortcutting cannot introduce new induced cycles (that did not exist in $G$ ). To the best of our knowledge, this rule was not studied before in full generality in the literature, hence we provide a proof of safeness.

Rule DFVS 2. If there are distinct vertices $u, v \in V(G)$ such that $u$ appears on every cycle on which $v$ appears then shortcut $v$ in $G$.

For the proof of the safeness of the rule we make use of the fact that it suffices to hit all induced cycles.

Lemma 2.2. Rule DFVS 2 is safe and can be implemented in time $\mathcal{O}\left(n^{2}(n+m)\right)$.
Proof. Let $S$ be a dfvs of $G$. We may assume, without loss of generality, that $S$ does not contain $v$, as we could replace it by $u$. That is, $(S \backslash\{v\}) \cup\{u\}$ is also a dfvs of $G$.

Let $C^{\prime}$ be an induced cycle in $G \ominus v$. If $C^{\prime}$ is not affected by the shortcutting of $v$, that is, if $C^{\prime}$ does not contain an in-neighbor $x$ and an out-neighbor $y$ of $v$, or it does contain such vertices $x$ and $y$ but the edge $x y$ was already present in $E(G)$, then $C^{\prime}$ is also an induced cycle of $G$, hence is hit by $S$ in $G$ and $G \ominus v$. This implies that $S$ is also a dfvs in $G \ominus v$.

Assume $C^{\prime}$ contains an in-neighbor $x$ and an out-neighbor $y$ of $v$ and the edge $x y$ is a newly introduced shortcut edge $x y$ (in $G \ominus v$ ). First, observe that this edge is the unique shortcut edge on $C^{\prime}$. Assume otherwise that $C^{\prime}$ contains
at least two shortcut edges $x y$ and $x^{\prime} y^{\prime}$, say, $C^{\prime}=x y P x^{\prime} y^{\prime} Q x$. By construction, $G \ominus v$ also contains the edges $x y^{\prime}$ and $x^{\prime} y$. Then $x y^{\prime} Q x$ is a cycle whose vertex set is a subset of the vertex set of $C^{\prime}$, contradicting the fact that $C^{\prime}$ is an induced cycle. Thus $V(C)=V\left(C^{\prime}\right) \cup\{v\}$ induces a unique cycle in $G$. As $C$ is induced in $G$ and $S$ does not contain $v$ by assumption, $C$ and (consequently) $C^{\prime}$ are hit by $S$. Hence, $S$ is also a dfvs in $G \ominus v$.

Conversely, let $S^{\prime}$ be a dfvs of $G \ominus v$. If $C$ is an induced cycle of $G$ then $V(C) \backslash\{v\}$ is the vertex set of a cycle in $G \ominus v$, and is therefore hit by $S^{\prime}$. Note that $V(C) \backslash\{v\}$ is the vertex set of a cycle in $G \ominus v$ regardless of whether $C$ contains $v$, or an in-neighbor and an out-neighbor of $v$, or $C$ is unaffected by the shortcutting of $v$. Hence, $S^{\prime}$ intersects with $V(C) \backslash\{v\}$ in $G \ominus v$ and is also a dfvs in $G$.

We show how the rule can be implemented efficiently. The condition that every cycle containing $v$ also contains $u$ is equivalent to the condition that $v$ does not lie on a cycle of $G-u$. For each fixed pair of vertices $u, v$ we can hence test (by a simple breadth-first search in $G-u$ starting from $v$ ) whether $v$ lies on a cycle in $G-u$. If it does not, we may remove $v$. Testing this for all pairs of vertices takes time $\mathcal{O}\left(n^{2}(n+m)\right)$.

We mention several special cases of Rule 2 that can be implemented more efficiently. The first special case is presented, e.g., as Rule 1 in $\left[\mathrm{BEG}^{+} 21\right]$ and Rule 3 in [FWY09].

Rule DFVS 2.1 If $v \in V(G)$ has no in- or no out-neighbor then $v$ can be removed from the graph $G$.

A second special case occurs when $u$ is the only in-neighbor of $v$ or when $v$ is the only out-neighbor of $u$. The corresponding rule is presented as Rule 3 in [ $\mathrm{BEG}^{+} 21$ ] and Rule 4 in [FWY09].

Rule DFVS 2.2 If a vertex $v \in V(G)$ has only one in-neighbor or one outneighbor then shortcut $v$ in $G$.

A final special case is presented as Rule 5 in [FWY09].
Rule DFVS 2.3 If $v$ does not lie on two cycles that are vertex-disjoint except for $v$ then shortcut $v$.

To see why Rule DFVS 2.3 is a special case of Rule 2 apply Menger's theorem to find a vertex $u \neq v$ that hits all cycles on which $v$ lies. Here, and in all future applications of Menger's theorem, to find a set of vertices that intersects with all cycles containing a vertex $v$ we construct the following graph $G^{\prime}$. We make a "copy" $v^{\prime}$ of $v$ then delete all outgoing edges of $v$ and make them outgoing edges of $v^{\prime}$ instead, i.e., the out-neighbors of $v$ become out-neighbors of $v^{\prime}$ instead. We complete the construction by making $v^{\prime}$ the unique out-neighbor of $v$. Then, the cycles containing $v$ in $G$ correspond one-to-one to the $v^{\prime}-v$-paths in $G^{\prime}$. By Menger's theorem, the size of the minimum vertex cut for $v^{\prime}$ and $v$ (the minimum
number of vertices, distinct from $v^{\prime}$ and $v$, whose removal disconnects $v^{\prime}$ and $v$ ) is equal to the maximum number of pairwise internally vertex-disjoint paths from $v^{\prime}$ to $v$. By finding a minimum vertex cut (using a flow algorithm) we find a set of vertices intersecting all cycles in $G$ that contain $v$. In particular, the vertices of a minimum cut intersect all cycles in $G$ that are pairwise vertex-disjoint except for $v$. Going back to the rule, a vertex $u \neq v$ that hits all cycles on which $v$ lies must exist if there exists no two cycles that are vertex disjoint except for $v$. In other words, there exist no two internally vertex-disjoint paths from $v^{\prime}$ to $v$ and removing $u$ disconnects $v^{\prime}$ and $v$. This vertex $u$ in fact dominates $v$.

The following sunflower-like rule was presented (in a weaker form) as Rule 3 in $\left[\mathrm{BEG}^{+} 21\right]$ and the special case of $u=v$ as Rule 6 in [FWY09].

Rule DFVS 3. Let $u, v \in V(G)$ and let $\mathcal{M}$ be a set of internally vertex-disjoint $u$-v-paths. Denote by $M$ the set of internal vertices of the paths in $\mathcal{M}$ (excluding $u$ and $v$ ). Let $\ell$ be a lower bound for the size of a minimum dfvs in $G-M$. If $|\mathcal{M}|>k-\ell$ insert the edge $u v$.

Observe that $\ell=0$ (realized by the empty set) is always a trivial lower bound for $G-M$. This yields the following special case: whenever two vertices $u$ and $v$ are connected by more than $k$ internally vertex-disjoint paths, we may insert the edge $u v$. A better bound for $\ell$ can be obtained using the $\mathcal{O}(\log k \log \log k)$ approximation algorithms of Even et al. [ESS $\left.{ }^{+} 98\right]$ or by the LP-based lower bound presented in Section 7.

Finally, we state the following conditional rule. In Section 6.1 we present multiple special cases of Rule 4 that are efficiently implementable.

Rule DFVS 4. If possible in polynomial time, remove all vertices and edges that do not lie on induced cycles.

From now on we assume that all rules (except for Rule DFVS 4, since this is not possible in general) have been applied exhaustively, i.e., in order and reiterated after any successful application of any rule. We slightly abuse notation and use $G$ to denote the resulting graph.

## 3 DFVS in graphs without long induced cycles

We begin our study of DFVS in graphs without induced cycles of length greater than $d$. In the following we assume that all graphs have no induced cycles of length greater than $d$. Unfortunately, it is NP-complete to determine if a vertex lies on an induced cycle [FKMP95]. In fact, this is even W[1]-hard when parameterized by $d$ [HH06]. By Lemma 2.1 we can approximate a small dfvs $S$. As a first rule we can delete all vertices not in $N_{G}^{d+}[S] \cap N_{G}^{d-}[S]$. It would be even better to delete all vertices that do not lie on an induced path of length at most $d$ between two vertices $u, v \in S$ (making a copy of $u$ when dealing with the case $u=v$ ). Since $G-S$ is acyclic, one could hope that this is possible in time $\mathcal{O}_{d}\left(n^{\mathcal{O}(1)}\right)$, however, as we show (in Lemma 8.1) even this is not possible. The

Directed Chordless ( $s, v, t$ )-Path problem asks, given a graph $G$, vertices $s, v, t$, and integer $d$, whether there exists an induced $s$ - $t$-path in $G$ of length at most $d$ containing $v$. The W[1]-hardness of the problem on general (directed and undirected) graphs was proved in [HH06]. We show hardness on directed acyclic graphs via a reduction from Grid Tiling. To not disturb the flow of the paper we postpone the proof to Section 8.

We hence have to come up with "new" reduction rules that are efficiently implementable. We start with a high-level description of our strategy as well as the obstacles that we need to overcome. Given a reduced graph $G$ (on which none of the reduction rules is applicable), we first compute a dfvs $S$ of size at most $d k$ as guaranteed by Lemma 2.1. Since we assume that $G$ has no induced cycles of length greater than $d$, all vertices of $G$ that are at distance $d+1$ or more from every vertex in $S$ can be discarded as they cannot belong to induced cycles of length at most $d$ that intersect with $S$. Hence, in what follows, we let $G=G\left[N_{G}^{d+}[S] \cap N_{G}^{d-}[S]\right]$ (which can be easily computed in polynomial time by standard breadth-first searches). The vertex set of $G=G\left[N_{G}^{d+}[S] \cap N_{G}^{d-}[S]\right]$ is partitioned into $S$ and $R=V(G) \backslash S$, where $|S| \leq d k$ and every vertex in $R$ is at distance at most $d$ to/from some vertex in $S$. Note that we would like to check for each $w \in R$ whether there exists an induced path of length at most $d$ from some $u \in S$ to $w$ and back. However, this is not possible due to Lemma 8.1, since it implies that we cannot efficiently iterate through the vertices of $R$ one by one and decide if they belong to some induced path. Our solution consists of adopting a "relaxed approach". That is, let $I_{u}^{d} \subseteq V(G)$ denote the set of all vertices that belong to some induced cycle of length at most $d$ that also includes $u \in S$. We shall compute, for each vertex $u \in S$, a set $W_{u}^{d} \supseteq I_{u}^{d}$. In other words, we compute a superset, which we call $W_{u}^{d}$, of the vertices that share an induced cycle of length at most $d$ with $u$. We call $W_{u}^{d}$ the set of $d$-weakly relevant vertices for $u$. Most crucially, we show that each $W_{u}^{d}$ can be computed efficiently and will be of bounded size. We let $W_{S}^{d}=\bigcup_{u \in S} W_{u}^{d}$ and we call $W_{S}^{d}$ the set of $d$ weakly relevant vertices for $S$. It is not hard to see that $G\left[S \cup W_{S}^{d}\right]$ is indeed an equivalent instance (to $G$ ) as it includes all vertices that participate in induced cycles of length at most $d$.

We describe the construction of $W_{u}^{d}$ for a single vertex. That is, we fix a non-reducible directed graph $G$, an integer $k \geq 2$, a constant $d \geq 2$, a dfvs $S$ of size at most $d k$, and a vertex $u \in S$. We first construct a graph $H_{u}^{d}$ as follows:

- We begin by setting $H_{u}^{d}=G\left[N_{G}^{d+}[u]\right]$.
- Then, we add a new vertex $v$ to $H_{u}^{d}$ and make all the in-neighbors of $u$ become in-neighbors of $v$ instead, i.e, $u$ will only have out-neighbors and $v$ will only have in-neighbors.
- Next, we delete all vertices in $H_{u}^{d}$ that do not belong to some directed path from $u$ to $v$ of length at most $d$.

Note that $H_{u}^{d}$ can be computed in polynomial time. Moreover, there exists an induced cycle of length at most $d$ containing $u$ in $G$ if and only if there exists an induced $u$ to $v$ path of length at most $d$ in $H_{u}^{d}$. By a slight abuse of

```
\(\underline{\text { Algorithm } 1 \text { Algorithm for computing weakly relevant vertices for } u \in S}\)
    procedure WeaklyRelevant \((G, S, u, d)\)
        return \(\operatorname{Recurse}\left(H_{u, v}^{d}, u, v,\{ \}, d\right) \quad \triangleright \operatorname{Returns} W_{u}^{d}\)
    end procedure
    procedure \(\operatorname{Recurse}(H, x, z, W, d)\)
        if \(|V(H) \backslash\{x, z\}| \leq k\) or \(d==2\) then
            return \(W \cup(V(H) \backslash\{x, z\})\)
        end if
        \(Y \leftarrow \operatorname{VertexSeparator}(H, x, z) \quad \triangleright\) Recall that \(|Y| \leq k\) and \(x, z \notin Y\)
        \(W \leftarrow W \cup Y\)
        for \(y \in Y\) do
            \(W \leftarrow W \cup \operatorname{Recurse}\left(H_{x, y}^{d-1}, x, y, W, d-1\right) \cup \operatorname{Recurse}\left(H_{y, z}^{d-1}, y, z, W, d-1\right)\)
        end for
        return \(W\)
    end procedure
```

notation, we also denote the graph $H_{u}^{d}$ by $H_{u, v}^{d}$ to emphasize the source and sink vertices. We call a directed graph $k$-nice whenever any two vertices $x, z$ are either connected by the directed edge $x z$ or by a set of at most $k$ pairwise internally vertex-disjoint (directed) paths. In particular, either $x z$ is an edge or there exists a set $Y$ (disjoint from $\{x, z\}$ ) of at most $k$ vertices that hits every directed path from $x$ to $z$. Observe that $H_{u}^{d}$ is indeed $k$-nice (since Rule 3 has been exhaustively applied on $G$ ). Given a $k$-nice graph $H_{u}^{d}$, two vertices $x, z \in V\left(H_{u}^{d}\right)$, and $2 \leq d^{\prime}<d$, we let $H_{x, z}^{d^{\prime}}$ denote the ( $k$-nice) graph obtained from $H_{u}^{d}$ by deleting all incoming edges of $x$, deleting all outgoing edges of $z$, and deleting all vertices that do not belong to a path of length at most $d^{\prime}$ from $x$ to $z$. We are now ready to compute $W_{u}^{d}$, for $u \in S$, recursively as described in Algorithm 1. Recall that since Rule DFVS 3 is not applicable in $G$, there does not exist $k$ internally vertex-disjoint (directed) paths between any two vertices of $G$ (and any $H_{x, z}^{d^{\prime}}$ resulting from the recursive calls). Hence, whenever we compute (via a flow algorithm) a set $Y$ separating two non-adjacent vertices we know that $Y$ will be of size at most $k$.

Lemma 3.1. For $u \in S$, every induced cycle $C_{u}$ of length at most dincluding $u$ only includes vertices that are d-weakly relevant for $u$, i.e., $V\left(C_{u}\right) \subseteq W_{u}^{d}$.

Proof. We prove, by induction on $2 \leq \ell \leq d$, that every vertex of every induced $u$ - $v$-path $P$ of length at most $\ell$ in $H_{u}^{\ell}$ is contained in $W_{u}^{\ell}$. Recall that every induced cycle of length at most $d$ including $u$ in $G$ corresponds to an induced path of length at most $d$ in $H_{u}^{d}$. Hence, all vertices of such induced cycles belong to $W_{u}^{d}$, proving the statement of the lemma.

The claim is true for $\ell=2$; the only (induced) $u-v$-paths of length $\ell=2$ in $H_{u, v}^{2}$ involve at most $k$ distinct vertices by Rule 3 ; otherwise $u$ belongs to $k+12$-cycles that pairwise intersect at $u$ and $u$ would be removed by Rule 1 . These at most $k$ vertices belong to $W_{u}^{2}$. Now assume the claim is true for some
$\ell>2$. We prove it for $\ell+1$. As $u v \notin E(G)$, there exists an $u$ - $v$-separator $Y$ of size at most $k$. In particular, there is $y \in Y \cap V(P)$, say $y$ is the $j$-th vertex on $P$ when walking from $u, 1 \leq j \leq \ell$. Then $P=P_{1} P_{2}$, where $P_{1}$ has length $1 \leq j \leq \ell$ and $P_{2}$ has length $\ell+1-j \leq \ell$. By the induction hypothesis, the vertices of $P_{1}$ are contained in $\operatorname{RECuRSE}\left(H_{u, y}^{\ell}, u, y, W, \ell\right)$ and the vertices of $P_{2}$ are contained in $\operatorname{Recurse}\left(H_{y, v}^{\ell}, y, v, W, \ell\right)$. By construction, $W_{u}^{\ell+1}$ contains the vertices of both of these sets, as needed to conclude the proof.

Lemma 3.1 immediately implies the safeness of the following rule.
Rule DFVS 5. If a vertex $w \notin S$ is not d-weakly relevant for some vertex $u \in S$ then remove $w$ from $G$.

It remains to prove that the rule can be efficiently implemented and that its application leads to a small kernel.

Lemma 3.2. For $u \in S$ and $2<\ell \leq d$ we have $\left|W_{u}^{\ell}\right| \leq k\left(2\left|W_{u}^{\ell-1}\right|+1\right) \leq$ $2^{\ell-1} k^{\ell-1}$ (assuming $k \geq 1$ ).

Proof. The claim follows by induction. For $\ell=2$ we have $\left|W_{u}^{2}\right| \leq k$. By the recursive definition of $W_{u}^{\ell+1}$ we have $\left|W_{u}^{\ell+1}\right| \leq k\left(2\left|W_{u}^{\ell-1}\right|+1\right) \leq\left(2^{\ell} k^{\ell}-2 k\right) /(4 k-2)$ $\leq\left(2^{\ell} k^{\ell}\right) /(4 k-2) \leq\left(2^{\ell} k^{\ell}\right) /(2 k) \leq 2^{\ell-1} k^{\ell-1}$.

Lemma 3.3. Rule DFVS 5 is safe and, if $2^{d} k^{d} \leq n^{\mathcal{O}(1)}$, it can be implemented in polynomial time.

Proof. For each $u \in S$ we simply compute the set $W_{u}^{d}$ by applying Algorithm 1. Each run of the algorithm requires $2^{d-1} k^{d-1} n^{\mathcal{O}(1)}$ time in the worst case. Since $|S| \leq k d$, the total running time is $2^{d} k^{d} n^{\mathcal{O}(1)}$ in the worst case. Hence, if $2^{d} k^{d} \leq$ $n^{\mathcal{O}(1)}$ we have $2^{d} k^{d} n^{\mathcal{O}(1)} \leq n^{\mathcal{O}(1)}$, as needed.

Theorem 3.1. DFVS parameterized by solution size $k$ and restricted to graphs without induced cycles of length greater than $d$ admits a kernel with $2^{d} k^{d}$ vertices computable in polynomial time.

Proof. Either $2^{d} k^{d}>n$, in which case we are done. Otherwise, the rule is efficiently applicable and yields a kernel of the claimed size.

Finally, we further study the structure of kernelized instances and count how many induced cycles we can find. Our key tool is the classical sunflower lemma. A sunflower with $\ell$ petals and a core $Y$ is a collection of sets $S_{1}, \ldots, S_{\ell} \in E(\mathcal{G})$ such that $S_{i} \cap S_{j}=Y$ for all $i \neq j \leq \ell$. The sets $S_{i} \backslash Y$ are called petals and we require none of them to be empty (while the core $Y$ may be empty). Erdös and Rado [ER60] proved in their famous sunflower lemma that every hypergraph with edges of size at most $d$ with at least $\operatorname{sun}(d, k)=d!k^{d}$ edges contains a sunflower with at least $k+1$ petals. Kernelization for $d$-Hitting Set based on the sunflower lemma yields a kernel with at most $\mathcal{O}\left(d!k^{d}\right)$ sets on hypergraphs with hyperedges of size at most $d$, see e.g. [FK15, VB14]. We can prove the following lemma.

Lemma 3.4. Kernelized instances of DFVS contain at most $d^{3 d} k^{d}$ induced cycles of length at most d.

Proof. We consider the hypergraph $\mathcal{G}$ of induced cycles. We prove that $\mathcal{G}$ does not contain a sunflower with more than $(d / 2)^{2} k$ petals (each petal of size at most $d$ ). By the sunflower lemma, $\mathcal{G}$ contains at most $\operatorname{sun}\left(d,(d / 2)^{2} k\right)=d!\left((d / 2)^{2} k\right)^{d}=$ $d!(d / 2)^{2 d} k^{d} \leq d^{3 d} k^{d}$ hyperedges (of size at most $d$ ), as claimed.

A sunflower in $\mathcal{G}$ corresponds to a set of induced cycles in $G$ of length at most $d$ that share a common core. If the core has the elements $\left\{v_{1}, \ldots, v_{c}\right\}$, then each petal $S$ of the sunflower completes the vertices $v_{1}, \ldots, v_{c}$ to a cycle $C_{S}$. Observe that when two vertices $v_{i}, v_{j}$ are connected by an edge, say $v_{i} v_{j} \in E(G)$, then they appear in that order on every cycle $C_{S}$, and we have $v_{j} v_{i} \notin E(G)$ as this would contradict the fact that the cycles are induced. Hence, the vertices $v_{1}, \ldots, v_{c}$ can be partitioned into maximal path segments $P_{1}, \ldots, P_{t}$ such that the vertices of each $P_{i}$ are connected consecutively as a path and such that there are no edges between $P_{i}$ and $P_{j}$ for $i \neq j$. Each of the cycles $C_{S}$ connects the path segments in some order using the petal vertices (the case $t=1$ is the simplest so we consider the case $t>1$ ). Note that if the path segments $P_{i}$ and $P_{j}$ are connected in that order, then the connection is via the last vertex of $P_{i}$ and the first vertex of $P_{j}$.

Observe that $t \leq d / 2$, as each cycle has length at most $d$ and for every two consecutive path segments there must be at least one petal vertex connecting the two. Now, if there is a sunflower with more than $(d / 2)^{2} k$ petals, then one of the possible $(d / 2)^{2}$ pairs of path segments must be connected by more than $k$ paths. Since the connection is always between the last vertex $v$ of the first segment and the first vertex $w$ of the second segment, there are more than $k$ disjoint paths connecting $v$ and $w$. As Rule 3 can no longer be applied, there is a direct edge between $v$ and $w$, contradicting the fact that the cycles of the sunflower are induced.

Of course for small values of $d$ we can prove better bounds, however, they can improve the bounds of Lemma 3.4 only up to the constants depending on $d$. A special case of Rule 3 for 2-cycles is the well-known high-degree rule for Vertex Cover: if a vertex $v$ belongs to more than $k$ distinct 2 -cycles (assuming no duplicate edges) then add $v$ to the solution and decrement the parameter by one. We immediately derive the following bound on the number of 2-cycles.

Lemma 3.5. Kernelized instances of DFVS contain at most $k^{2}$ cycles of length 2.
Lemma 3.6. Kernelized instances of DFVS contain at most $k^{3}$ induced cycles of length 3.

Proof. Assume there are more than $k^{3}$ induced cycles of length 3. In any dfvs of size at most $k$, there must exist a vertex $v_{1}$ that hits a $1 / k$ fraction of these cycles, i.e., $v_{1}$ must intersect with more than $k^{2}$ of the induced cycles of length 3 . We fix such a $v_{1}$ and consider all induced 3 -cycles containing $v_{1}$.

For every in-neighbor $v_{2}$ of $v_{1}$, i.e., for every edge $v_{2} v_{1} \in E(G)$, we can have at most $k$ (distinct) vertices $v_{3}$ such that $v_{1} v_{3}, v_{3} v_{2} \in E(G)$; as otherwise, by Rule 3, $v_{1} v_{2} \in E(G)$ and the cycles are not induced. Similarly, for every outneighbor $v_{2}$ of $v_{1}$, i.e., for every edge $v_{1} v_{2} \in E(G)$, we can have at most $k$ (distinct) vertices $v_{3}$ such that $v_{2} v_{3}, v_{3} v_{1} \in E(G)$; as otherwise, again by Rule 3 , $v_{2} v_{1} \in E(G)$ and the cycles are not induced.

Since $v_{1}$ does not lie on $k+1$ cycles that pairwise intersect only at $v_{1}$ after the application of Rule 3, by Menger's theorem there is a set of at most $k$ vertices (different from $v_{1}$ ) that hits all cycles containing $v_{1}$. Hence, at most $k$ (in or out) neighbors of $v_{1}$ hit all induced 3 -cycles containing $v_{1}$. Combined with the fact that each of those neighbors can belong to at most $k$ induced 3 -cycles containing $v_{1}$, this implies that $v_{1}$ belongs to at most $k^{2}$ induced cycles of length 3 , contradicting the fact that $v_{1}$ must hit more than $k^{2}$ induced 3 -cycles for a dfvs of size $k$ to hit more than $k^{3}$ induced 3 -cycles.

In fact the bounds of Lemma 3.4 are optimal up to factors depending only on $d$. Consider for example the graph on vertices $v_{i, j}$ for $1 \leq i \leq d, 1 \leq j \leq k$. Connect $v_{i, j}$ with $v_{i+1, \ell}, 1 \leq i \leq d, 1 \leq j, \ell \leq k$, where we compute $i+1$ modulo $d$. This graph on $d k$ vertices has a dfvs of size $k$. None of the presented reduction rules is applicable. Finally, it has $k^{d}$ cycles.

## 4 Nowhere dense classes without long induced cycles

We now improve the general kernel construction for DFVS on graphs without induced cycles of length greater than $d$ by further restricting the class of (the underlying undirected) graphs. We obtain a kernel with $\mathcal{O}_{d}(k)$ vertices on classes with bounded expansion and $\mathcal{O}_{d, \varepsilon}\left(k^{1+\varepsilon}\right)$ vertices, for any $\varepsilon>0$, on nowhere dense classes of graphs (when we say $G$ belongs to a class $\mathscr{C}$ of graphs we in fact mean that the underlying undirected graph belongs to $\mathscr{C})$. We present the proof for nowhere dense classes since it subsumes the bounded expansion case. To keep the presentation clean we omit the details for the latter case since the required modifications are negligible. We refer the reader to [NDM08, NdM11] for formal definitions of bounded expansion and nowhere dense classes of graphs. We only need the following properties, which will also motivate our additional reduction rule. Recall that every class of bounded expansion is also nowhere dense. For every nowhere dense class of graphs $\mathscr{C}$ there exists a positive integer $t>0$ such that $K_{t, t}$ (the complete biparite graph with $t$ vertices in each part) is not a subgraph of any $G \in \mathscr{C}$.

Let us fix an approximate solution $S$ as described in Lemma 2.1. We build a projection closure around our approximate solution $S$. This is possible in nowhere dense classes as stated in the next lemma.

Let $X \subseteq V(G)$ and let $u \in V(G) \backslash X$. The undirected d-projection of $u$ onto $X$ is defined as the set $\Pi_{d}(u, X)$ of all vertices $w \in X$ for which there exists an undirected path $P$ in $G$ that starts in $u$, ends in $w$, has length at most $d$, and whose internal vertices do not belong to $X$.

Lemma 4.1 ( $\left.\left[\mathbf{E G K}^{+} \mathbf{1 7}\right]\right)$. Let $\mathscr{C}$ be a nowhere dense class of graphs. There exists a polynomial time algorithm that given a graph $G \in \mathscr{C}, d, \varepsilon>0$ and $X \subseteq V(G)$, computes the $d$-projection-closure of $X$, denoted by $X^{\circ}$, with the following properties:

1. $X \subseteq X^{\circ}$,
2. $\left|X^{\circ}\right| \leq \kappa_{d, \varepsilon} \cdot|X|^{1+\varepsilon}$ for a constant $\kappa_{d, \varepsilon}$ depending only on $d$ and $\varepsilon$,
3. $\left|\Pi_{d}\left(u, X^{\circ}\right)\right| \leq \kappa_{d, \varepsilon} \cdot|X|^{\varepsilon}$ for each $u \in V(G) \backslash X^{\circ}$, and
4. $\left|\left\{\Pi_{d}(u, X): u \in V(G) \backslash X^{\circ}\right\}\right| \leq \kappa_{d, \varepsilon} \cdot|X|^{1+\varepsilon}$.

We need the following strengthening for $\ell$-tuples [PST18]. For a set $X \subseteq V(G)$ and an $\ell$-tuple $\bar{x}$ of vertices we call the tuple $\left(N\left[\bar{x}_{1}\right] \cap X, \ldots, N\left[\bar{x}_{\ell}\right] \cap X\right)$ the undirected projection of $\bar{x}$ onto $X$. We say that $\bar{X}=\left(X_{1}, \ldots, X_{\ell}\right)$ is realized as a projection if there is a tuple $\bar{x}$ whose projection is equal to $\bar{X}$.

Lemma 4.2 ( [PST18]). Let $\mathscr{C}$ be a nowhere dense class of graphs and let $\ell$ be a natural number. Let $G \in \mathscr{C}$ and $X \subseteq V(G)$. Then, for every $\varepsilon>0$ there exists a constant $\tau_{\ell, \varepsilon}$ such that there are at most $\tau_{\ell, \varepsilon} \cdot|X|^{\ell+\varepsilon}$ different realized undirected projections of $\ell$-tuples.

Let $X \subseteq V(G)$ and let $x, y \in X$. Let $P=u_{1}, \ldots, u_{\ell}$ be an almost induced $x-y$ path with $|V(P)|=\ell \leq d$ and let $u_{i} \in V(P)$. Then, the $X$-path-projection profile of $(P, u)$ is the tuple $\left(i, N^{-}\left(u_{1}\right) \cap X, N^{+}\left(u_{1}\right) \cap X, \ldots, N^{-}\left(u_{\ell}\right) \cap X, N^{+}\left(u_{\ell}\right) \cap X\right)$. The $X$-path-projection profile of vertex $u$ is the set of all $X$-path-projection profiles $(P, u)$, where $P$ is any almost induced $x$ - $y$-paths on at most $d$ vertices and $x, y \in X$ are any two vertices in $X$. Two vertices $u, v$ are equivalent over $X$ if they have the same $X$-path-projection profiles.

Lemma 4.3. Let $\mathscr{C}$ be a nowhere dense class of graphs and let $t>0$ be some fixed positive integer such that $K_{t, t} \nsubseteq G$, for all $G \in \mathscr{C}$. Let $G \in \mathscr{C}$ and $X \subseteq V(G)$. Then, for every $\varepsilon>0$, there exists a constant $\chi_{d, t, \varepsilon}$ such that the number of $X$-path-projection profiles for $u \in V(G) \backslash X$ is bounded by $\chi_{d, t, \varepsilon}$. $|X|^{d+\varepsilon}$.

Proof. We have $d$ choices for the number $i$. By Lemma 4.2 we have at most $\tau_{\ell, \varepsilon}|X|^{\ell+\varepsilon}$ different undirected projections of $\ell$-tuples. If the undirected projection of a single vertex $v$ within an $\ell$-tuple has size smaller than $t(|N[v] \cap X| \leq t-1)$, then even though we can have many vertices with the same undirected projection, there are at most $2^{t-1}$ possible ways of orienting this undirected projection to obtain a directed projection; orienting an undirected projection $N[v] \cap X$ yields a directed projection $\left(N^{-}[v] \cap X, N^{+}[v] \cap X\right)$. Otherwise, when $|N[v] \cap X| \geq t$, we can have at most $t-1$ other vertices with the same undirected projection; this follows from the fact that $G$ does not contain $K_{t, t}$ as a subgraph. Consequently, there are at most $t-1$ possible orientations of $N[v] \cap X$ whenever $|N[v] \cap X| \geq t$. Putting it all together, we know that any undirected projection (of any size) can be oriented in at most $2^{t-1}$ different ways. Summing over all possible choices of $\ell \leq d$, we get at most $d^{2} \cdot 2^{d t} \cdot \tau_{d, \varepsilon} \cdot|X|^{d+\varepsilon} X$-path-projection profiles. To conclude the proof, we define $\chi_{d, t, \varepsilon}$ as $d^{2} \cdot 2^{d t} \cdot \tau_{d, \varepsilon}$.

We now state a new reduction rule, which depends on a constant $c$ that we fix later.
Rule DFVS 6. Assume we can find in polynomial time sets $B, X \subseteq V(G)$ such that

1. the $d+1$-neighborhoods in $G-X$ of distinct vertices from $B$ are disjoint,
2. every induced cycle using a vertex of $N_{G}^{d}[B]$ also uses a vertex of $X$,
3. vertices in $B$ are pairwise equivalent over $X$, i.e., they have the same $X$ -path-projection profile (in particular, if one vertex of $B$ lies on an $x-y$-path of length $\ell \leq d$, then all vertices of $B$ do as well), and
4. $|B|>c+d+1$ and $|X| \leq c$.

Then, choose an arbitrary vertex of $B$ and delete it from $G$.
Lemma 4.4. Rule 6 is safe.
Proof. Denote by $G^{\prime}$ the graph obtained after one application of the rule with sets $B, X \subseteq V(G)$, where $|B|>c+d+1$ and $|X| \leq c$. Let $u \in B$ be the deleted vertex. Since the rule only removes a vertex it is clear that every dfvs $S$ of $G$ is also a dfvs of $G^{\prime}$. It remains to show that for every dfvs $S^{\prime}$ of $G^{\prime}$ there exists a dfvs of $G$ that is not larger than $S^{\prime}$.

Let $C_{u}$ be an induced cycle in $G$ going through $u$ such that $\left|V\left(C_{u}\right)\right| \leq d$. All other induced cycles (that do not contain $u$ ) are also induced cycles in $G^{\prime}$ and are hence hit by $S^{\prime}$. By assumption, every induced cycle including a vertex of $N_{G}^{d}[B]$, in particular $u$, also includes at least one vertex of $X$. Pick $x, y \in X$ (possibly $x=y$ ) such that $C_{u}$ includes $x, u, y$ in that order and such that no other vertices of $X$ appear in between $x$ and $y$ (in $C_{u}$ ).

Fix the $x$-u-path $P_{x u}$ and the $u$ - $y$-path $P_{u y}$ that are subpaths of $C_{u}$. Let $P_{u}=$ $P_{x u} P_{u y}$. Since $C_{u}$ is an induced cycle, if $C_{u}$ contains vertices of $X \backslash\{x, y\}$ then these other vertices do not appear in the $X$-path-projection profile of $\left(P_{u}, u\right)$.

Since all $v \in B$ have the same $X$-path-projection profile, for each $v \neq u$ there are paths $P_{x v}$ from $x$ to $v$ and $P_{v y}$ from $v$ to $y$ such that $P_{v}=P_{x v} P_{v y}$ and $\left(P_{v}, v\right)$ has the same $X$-path-projection profile as $\left(P_{u}, u\right)$. Since the $d+1$-neighborhoods in $G-X$ of all vertices of $B$ are pairwise disjoint, all of these paths are pairwise vertex disjoint except for $x$ and $y$ and at most $d$ vertices $v \in B \backslash\{u\}$ can have other vertices of $V\left(C_{u}\right) \backslash X$ that are in the 1-neighborhood of $P_{v}$. Since all these pairs $\left(P_{v}, v\right), v \neq u$, have the same $X$-path-projection profile as $\left(P_{u}, u\right)$, just like $\left(P_{u}, u\right)$, vertices in $\left(V\left(C_{u}\right) \cap X\right) \backslash\{x, y\}$ do not appear in the $X$-path-projection profile of $\left(P_{v}, v\right)$. Because there are at least $c+d+1$ vertices in $B \backslash\{u\}$, we get $c+1$ induced $x$ - $y$-paths that are pairwise vertex disjoint except for $x$ and $y$ in $G^{\prime}$ and that are not adjacent to vertices of $V\left(C_{u}\right) \backslash\{x, y\}$. Hence, for every $v \in B$ we get an induced cycle $C_{v}$ on at most $d$ vertices by replacing $P_{u}$ by $P_{v}$.

Assuming that $S^{\prime}$ is not a dfvs in $G$, the vertices in $V\left(C_{u}\right) \backslash V\left(P_{u}\right)$ are not hit by $S^{\prime}$. Then, all the (at least $c+1$ ) cycles $C_{v}$ are hit on the paths $P_{v}$, i.e., $S^{\prime} \cap V\left(C_{v}\right) \subseteq V\left(P_{v}\right)$. All vertices of $S^{\prime}$ that hit the cycles $C_{v}$ on $P_{v}$, call those vertices $Y$, lie in the $d$-neighborhood of $v$ in $G-X$. Hence, by assumption, they do not hit any cycles that do not also go through $X$. Then, $\left(S^{\prime} \backslash Y\right) \cup X$ is a dfvs of $G$ of size at most $\left|S^{\prime}\right|$; as $|Y| \geq c+1$ and $|X| \leq c$. This completes the proof.

Lemma 4.5. Given a graph $G, X \subseteq V(G)$ and $u \in V(G)$, we can test in time $\mathcal{O}_{d, \varepsilon}\left(n^{\mathcal{O}(1)}\right)$ whether every induced cycle (on at most $d$ vertices) using a vertex of $N_{G}^{d}[u]$ also uses a vertex of $X$. Furthermore, we can decide in the same time bound if two vertices $u, v$ have the same $X$-path-projection profile.

Proof. We apply the efficient first-order model checking algorithm of [GKS17]. To do so, we first mark the set $X$ using a unary predicate to make it accessible to first-order logic. Both properties are easily expressible by a first-order formula whose length depends only on $d$.

Nowhere dense classes of graphs are uniformly quasi-wide [NdM11]. A class of graphs is uniformly quasi-wide if for every $q$ there exists a constant $s$ and a function $N(m)$ such that the following holds. If $G \in \mathscr{C}$ and $A \subseteq V(G)$ has size at least $N(m)$, then there exists a set $Y \subseteq V(G)$ of size at most $s$ and a set $B \subseteq A \backslash Y$ of size at least $m$ such that all distinct vertices of $B$ have disjoint $q$-neighborhoods in $G-Y$. The best bounds for the function $N$ are given in [KRS19, PST18]. It is important that the function is polynomial, that is, $N(m)=m^{t}$ for some constant $t$.

Theorem 4.1. Rule 6 can be efficiently applied until the reduced graph has $\mathcal{O}_{d, \varepsilon}\left(k^{1+\varepsilon}\right)$ vertices.

Proof. Let $N(m)$ and $s$ be the function and constant witnessing that $\mathscr{C}$ is uniformly quasi-wide for parameter $q=2 d$. Assume $N(m)=m^{t}$. Recall that $S$ is an approximate solution of size at most $d k$. Let $\delta>0$ be a constant that we determine later. We build the projection closure $S^{\circ} \supseteq S$ for parameters $2 d$ and $\delta$, which by Lemma 4.1 is of size at most $\kappa_{2 d, \delta} \cdot(d k)^{1+\delta}$ and such that the $2 d$ projection of each $v \in V(G) \backslash S^{\circ}$ has size at most $\kappa_{2 d, \delta} \cdot(d k)^{\delta}$. Let $\chi_{2 d, t, \delta}$ be the constant from Lemma 4.3. Define $c$ for the application of Rule 6 as $\kappa_{2 d, \delta} \cdot(d k)^{\delta}+s$.

Assume $|V(G)|>\kappa_{2 d, \delta}(d k)^{1+\delta}+\kappa_{2 d, \delta} \cdot N\left(\chi_{d, t, \delta} \cdot c^{\delta} \cdot(c+d+2)\right)(d k)^{1+\delta}$. We show that we can efficiently apply Rule 6.

First, there are at least $\kappa_{2 d, \delta} \cdot N\left(\chi_{d, t, \delta} \cdot(c+d+2)\right)(d k)^{1+\delta}$ vertices in $V(G) \backslash S^{\circ}$. Moreover, every induced cycle using a vertex $u \in V(G) \backslash S^{\circ}$ uses a vertex of $\Pi_{2 d}(u) \subseteq S^{\circ}$. This is true because $G-S$ is acyclic and all paths from $u$ to $S$ must use a vertex of $\Pi_{2 d}(u)$. In fact, this is true for every vertex in the $d$ neighborhood of $u$ in $G-\Pi_{2 d}(u)$; this is why we consider parameter $2 d$ instead of $d$.

Since there are at most $\kappa_{2 d, \delta} \cdot(d k)^{1+\delta}$ different projection classes there is at least one class $A$ with at least $N\left(\chi_{d, t, \delta} \cdot c^{\delta} \cdot(c+d+2)\right)$ vertices. We denote the set of projection vertices by $\Pi \subseteq X$. We apply uniform quasi-wideness to $A$ to find a set $Y$ of size at most $s$ and a set $B^{\prime} \subseteq A \backslash Y$ containing at least $\chi_{d, t, \delta} \cdot c^{\delta} \cdot(c+d+2)$ vertices that have pairwise disjoint $2 d$-neighborhoods in $G-Y$. Let $X=\Pi \cup Y$. Note that $X$ has size at most $\kappa_{2 d, \delta} \cdot(d k)^{\delta}+s=c$.

By Lemma 4.3 there are at most $\chi_{d, t, \delta} \cdot c^{\delta}$ many different $X$-path-projection profiles, hence, we find a set $B \subseteq B^{\prime}$ of size greater than $c+d+1$ of vertices that all have the same $X$-path-projection profile. Hence, all assumptions to apply Rule 6
are satisfied and we can still carry out the rule to decrease the size of $V(G)$. As all lemmas can be applied efficiently, Rule 6 with the given sets $B$ and $X$ can also be applied efficiently.

It remains to define the constant $\delta$. We need $N\left(\chi_{d, t, \delta} \cdot c^{\delta} \cdot(c+d+2)\right)(d k)^{1+\delta} \in$ $\mathcal{O}_{d, \varepsilon}\left(k^{1+\varepsilon}\right)$. By assumption we have $N(m)=m^{t}$, hence, $N\left(\chi_{d, t, \delta} \cdot c^{\delta} \cdot(c+d+2)\right)$ $(d k)^{1+\delta} \leq\left(\chi_{d, t, \delta} \cdot c^{\delta} \cdot(c+d+2)\right)^{t}(d k)^{1+\delta} \leq\left(\chi_{d, t, \delta} \cdot\left(\kappa_{2 d, \delta} \cdot(d k)^{\delta}+s\right)^{\delta} \cdot\left(\left(\kappa_{2 d, \delta} \cdot\right.\right.\right.$ $\left.\left.\left.(d k)^{\delta}+s\right)+d+2\right)\right)^{t}(d k)^{1+\delta}$. It hence suffice to define $\delta$ such that $\left(\delta+\delta^{2}\right) t+1 \leq \varepsilon$.

## 5 DFVS in planar graphs without long cycles

One may wonder whether the stronger assumption that a graph does not contain long cycles, induced or non-induced, leads to even more efficient algorithms. We show that this is indeed the case when considering planar graphs. We show that strongly connected planar graphs without cycles of length $d$ have treewidth $\mathcal{O}(d)$. We observe in the next section (Observation 2) that after the application of Rule DFVS 7, weak components are equal to strong components. Hence, the DAG of strong components in fact is a tree. Then, if each strong component has bounded treewidth, we can combine the tree decompositions of the strong components with the tree of strong components to derive that the whole graph after application of Rule DFVS 7 has bounded treewidth. We can then use the algorithm of Bonamy et al. $\left[\mathrm{BKN}^{+} 18\right]$ to solve the instance in time $2^{\mathcal{O}(d)} \cdot n^{\mathcal{O}(1)}$.

Lemma 5.1. Let $G$ be a strongly connected graph and let $u, v \in V(G)$. Let $P$ be an undirected path between $u$ and $v$ in the underlying undirected graph. If $G$ does not have cycles of length greater than $d$, then it contains a (directed) u-v-path $Q$ such that every vertex of $Q$ is at distance at most $d$ from some vertex of $P$.
Proof. Assume $P=v_{1} \ldots v_{\ell}$. For each $v_{i} v_{i+1}$ fix a cycle $C_{i}$ of length at most $d$ containing both $v_{i}$ and $v_{i+1}$. Such a cycle must exist as $G$ is strongly connected and does not contain cycles longer than $d$. We can now appropriately stitch subpaths from these cycles to find a $u$ - $v$-path $Q$ in $G$. It is immediate that every vertex of $Q$ is at distance at most $d$ from some vertex of $P$.

Theorem 5.1. Let $G$ be a strongly connected planar graph without cycles of length greater than $d$. Then, $G$ has treewidth at most $30 d$.
Proof. As proved in [RST94], every planar graph of treewidth at least $6 t$ contains a grid of order $t$ as a minor. Assume towards a contradiction that $G$ has treewidth greater than $30 d$. Then it contains a grid of order $5 d$ as a minor. We fix four vertices $v_{1}, v_{2}, v_{3}, v_{4}$ from the four corner branch sets of the inner sub-grid minor of order $3 d$ and undirected paths $P_{i,(i \bmod 4)+1}$ leading from $v_{i}$ to $v_{(i \bmod 4)+1}$ using only vertices of branch sets of the boundary of the grid minor model. By Lemma 5.1 and because $G$ is planar we find $v_{i}-v_{(i \bmod 4)+1}$-paths that use only vertices inside the regions defined by the boundary of the grid minor of order $5 d$ and the boundary of the central sub-grid minor of order $d$. By gluing the paths we find a closed walk which contains a cycle that fully encloses the central grid minor of order $d$. This cycle has length at least $4 d$, contradicting the fact that $G$ has no cycles longer than $d$.

## 6 Long induced cycles

Since in general we cannot efficiently implement Rule 4, in this section we provide sufficient conditions to decide whether a vertex or an edge lies on an induced cycle, and thereby approximate the effect of the rule. Let us first define a bit more notation.

It will be convenient to partition the edge set of $G$ into red and blue edges. We say that an edge $u v \in E(G)$ is red if $v u \notin E(G)$, and say that $u v$ is blue if $v u \in E(G)$, i.e., if $u v u$ is a cycle in $G$. By $R(G) \subseteq E(G)$, we denote the set of all red edges of $G$, and by $B(G) \subseteq E(G)$ we denote the set of all blue edges of $G$. We define the blue neighborhood of a vertex $v \in V(G)$ as $B_{G}(v)=N_{G}^{+}(v) \cap N_{G}^{-}(v)$, i.e., the set of neighbors of $v$ that are connected by blue edges. Similarly, let $R_{G}(v)=N_{G}(v) \backslash B_{G}(v)$, that is, the set of neighbors of $v$ that are connected by red edges. If $G$ is clear from the context, we drop the subscript $G$ in all of the above definitions. The degree (resp. in-degree, out-degree, blue-degree) of a vertex $v$ is defined as $|N(v)|$ (resp. $\left|N^{-}(v)\right|,\left|N^{+}(v)\right|,|B(v)|$ ). Observe that a minimum dfvs $S$ of a graph $G$ with only blue edges is the same as a minimum vertex cover in the underlying undirected graph, as $S$ hits all edges (every blue edge is a cycle of length two), i.e., $G-S$ is edgeless.

Observe that no graph $G$ has an induced cycle containing both a blue and a red edge, as any blue edge $u v$ already implies a cycle of length two. Hence, as every blue edge lies on an induced cycle by definition, it suffices to check whether red edges lie on induced cycles or not.

Observation 1 If an edge $u v \in R(G)$ does not lie on an induced cycle in $G[R(G)]$ then uv can be removed from $G$.

We will present rules that are special cases of Rule 4 but can be implemented efficiently.

Rule DFVS 7. Let $u v \in R(G)$ and let $G^{\prime}$ be the graph induced by $R(G)$ where all vertices $z$ with $u z \in E(G)$ or $z v \in E(G)$ have been removed. If there does not exist a $v$-u-path in $G^{\prime}$ then remove uv from $G$.

Lemma 6.1. Rule DFVS 7 is safe and can be implemented in time $\mathcal{O}(m(n+m))$.
Proof. By Observation 1, if $u v$ does not lie on an induced cycle it may be removed from $G$. Assume $u v$ lies on an induced cycle $C=u P$, where $P$ is an induced $v$ to $u$ path. Then $P$ cannot contain a vertex $z$ with $u z \in E(G)$ or $z v \in E(G)$. Assume otherwise, say, $P=Q z R$. Then $u z R$ or $v Q z v$, respectively, are subcycles of $C$, contradicting the fact that $C$ is an induced cycle. Hence, if there does not exist a $v$ - $u$-path in $G^{\prime}$ then $u v$ does not lie on an induced cycle and may be removed.

For any edge $u v$ we can construct $G^{\prime}$ from $G$ in time $\mathcal{O}(n+m)$ and perform a depth-first search. Iterating through all edges leads to a running time of $\mathcal{O}(m(n+m))$.

Special cases of Rule DFVS 7 that can be checked even more efficiently are the following.

Rule DFVS 7.1 If $u v \in R(G)$ and $u$ has no incoming red edge or $v$ has no outgoing red edge then remove uv.

Rule DFVS 7.2 If $v w \in R(G)$ and if for all $u \in N^{-}(v)$ we also have $u w \in$ $R(G)$ then remove $v w$. If $u v \in R(G)$ and if for all $w \in N^{+}(v)$ we also have $u w \in R(G)$ then remove uv.

Observe that, in particular, after the application of Rule DFVS 7 every edge of $G$ lies on a cycle. After the applications of these rules it makes sense to recompute strong components. The following observation shows that it now suffices to apply a regular component search (for weak components).

Observation 2 After the exhaustive application of Rule DFVS 7 every weak component is strongly connected.

Proof. Let $u_{1} \ldots u_{t}$ be a path in the undirected underlying graph of $G$. Each of the edges $u_{i} u_{i+1}$ lies on a directed cycle $C_{i}$ of $G$. Then in $C_{1} \ldots C_{t-1}$ we find a directed $u_{1}-u_{t}$-path; we first find a directed walk by appropriately gluing parts of the cycles and then find a path contained in the walk.

We now formulate a modified depth-first search rule. We say that a cycle $C=$ $v_{1}, \ldots, v_{\ell}$ for $\ell \geq 4$ is induced on an initial segment of length $i$ (for some $3 \leq i<\ell$ ) if $v_{1}, \ldots v_{i}$ induce a path in $G[V(C)]$ and only $v_{i}$ can have out-neighbors among $v_{i+1}, \ldots, v_{\ell-1}$ and only $v_{1}$ can have in-neighbors among $v_{i+1}, \ldots, v_{\ell-1}$. Note that this definition depends on the vertex that we distinguish as $v_{1}$ on the cycle. Note also that by definition every cycle of length three is induced on an initial segment of length three. We say that an edge $u v \in R(G)$ lies on a cycle that is induced on an initial segment of length $i$ if there exists a cycle $C=v_{1}, \ldots, v_{\ell}$ that is induced on an initial segment of length $i$ such that $\left(v_{1}, v_{2}\right)=(u, v)$.

Lemma 6.2. If $u v \in R(G)$ does not lie on a cycle that is induced on an initial segment of length $i$ for some $i \geq 3$, then uv does not lie on an induced cycle of $G$. Furthermore, we can test this property in time $\mathcal{O}\left(n^{i-2}(n+m)\right)$.

Proof. The first statement is immediate from the fact that an induced cycle of length $\ell$ is induced on an initial segment of length $\ell-1$ (independent of the choice of initial vertex $v_{1}$ ). Moreover, if a cycle is induced on an initial segment of length $i$, then it is induced on an initial segment of length $j$ for every $3 \leq j \leq i$.

For the running time we consider an algorithm that non-deterministically guesses vertices $v_{3}, \ldots, v_{i}$ such that $v_{1}, \ldots, v_{i}$ is an induced path in $G$. We remove all vertices that are out-neighbors of one of the $v_{j}$ for $1 \leq j \leq i-1$ and all vertices that are in-neighbors of one of the $v_{j}$ for $2 \leq j \leq i$ and carry out a regular depth-first search from $v_{i}$. If we find $v_{1}$ in this search, say by visiting the vertices $v_{i+1}, \ldots, v_{\ell}=v_{1}$ we return the cycle $v_{1}, \ldots, v_{\ell}$, which is induced on the initial segment of length $i$ by construction. Otherwise, we return that $u v$
does not lie on such a cycle. A deterministic version of the algorithm iterates through all possible sets $v_{3}, \ldots, v_{i}$ in time $\mathcal{O}\left(n^{i-2}\right)$. For each set, the algorithm constructs the graph with deleted vertices and carries out a depth-first search in time $\mathcal{O}(n+m)$.

We will prove later that testing for containment in cycles that are induced on an initial segment of length three subsumes many non-trivial reduction rules of Bergougnoux et al. $\left[\mathrm{BEG}^{+} 21\right]$. Hence, we state the following reduction rule, though rules with larger $i$ may be interesting to consider.

Rule DFVS 8. If an edge $u v \in R(G)$ does not lie on a cycle that is induced on an initial segment of length three then remove uv from $G$.

Lemma 6.3. Rule 8 is safe and can be implemented in time $\mathcal{O}(n m(n+m))$.
Proof. The safeness of the rule is immediate by Observation 1 and Lemma 6.2. The running time is obtained by iterating over all edges of $G$.

The following corollary shows that Rule 8 generalizes the two previous reduction rules, i.e., Rule DFVS 7.1 and Rule DFVS 7.2.

Corollary 6.1. After exhaustive application of Rule 8 every edge lies on a cycle that is induced on an initial segment of length three. In particular, for every $v w \in R(G)$ there is an incoming red edge $u v \in R(G)$ and an out-going red edge $w z \in R(G)$. Furthermore, if $u v \in R(G)$ is an incoming edge of $v$, then there is some uw $\notin E(G)$ and if $w z \in R(G)$ is an out-going edge of $w$, then there is some $v z \notin E(G)$. Finally, every weak component is strongly connected.

### 6.1 Analysis of kernel size

In this section we prove that Rule 1 , Rule 2 , Rule 3 and Rule 8 lead to a kernel of size $\mathcal{O}\left(f^{4}\right)$, where $f$ is the size of a minimum feedback vertex set in the underlying undirected graph. In fact, we prove the stronger bound of $\mathcal{O}\left(f^{3} k\right)$. Our analysis is based on the analysis of Bergnouxnoux et al. $\left[\mathrm{BEG}^{+} 21\right]$. Essentially, we prove that all complicated rules of Bergnouxnoux et al. are subsumed by Rule 8. In the following, fix an undirected feedback vertex set $F$ (which does not have to be computed and in particular may be assumed to be minimum). We prove that the rules lead to a kernel of size $\mathcal{O}\left(|F|{ }^{3} k\right)$. We (almost) follow the terminology of Bergnouxnoux et al. For the sake of clarity, when it suffices to apply a special case of one of the above rules, we refer to the special case.

Let $B$ be the set of blue vertices, i.e., those vertices that are incident to at least one blue edge. By Lemma 3.5, we may assume that there are at most $k^{2}$ blue edges, hence, $|B| \leq 2 k^{2}$ (otherwise we have a negative instance). Note that the standard argument applicable for Vertex Cover that we may assume that there are at most $k^{2}$ blue vertices cannot be applied here, since there can be additional red edges. Let $A=V(G) \backslash(F \cup B)$.

An ordered pair $(u, v)$ of (not necessarily distinct) vertices of $F$ is called a potential edge of $F$. If $(u, v) \notin E(G)$, then it is a non-edge of $F$. If $u=v$, then $(u, v)$ is called a loop. A vertex $w \in V(G) \backslash(F \cup B)=A$ directly contributes to a potential edge $(u, v)$ if $(u, w) \in E(G)$ and $(w, v) \in E(G)$. Note that unlike Bergougnoux et al., no vertex of $A$ can directly contribute to a loop (the case $u=v$ ), these vertices are incident with a blue edge and have already been collected in $B$. The following lemma follows from the fact that Rule 3 is no longer applicable.
Lemma 6.4. For every non-edge $(u, v)$ of $F$ there are at most $k$ vertices that directly contribute to $(u, v)$. Consequently, there are at most $|F|(|F|-1) k$ vertices in $A$ that directly contribute to a non-edge of $F$.

We follow the approach of Bergnouxnoux et al. to bound the number of vertices in $A$. Denote by $A_{0}, A_{1}, A_{2}$ and $A_{\geq 3}$ the sets of vertices of $A$ that have a total degree $0,1,2$ and at least 3 , respectively, in $G-(F \cup B)$.
Lemma 6.5. Every vertex in $A_{0} \cup A_{1}$ directly contributes to a non-edge of $F$.
Proof. First observe that because Rule DFVS 2.2 cannot be applied, every vertex has at least 2 in- and 2 out-neighbors, hence, every $v \in A_{0} \cup A_{1}$ has an inneighbor and an out-neighbor in $F$. Assume that $w$ does not directly contribute to a loop or a non-edge. Then for every pair of in- and out-edges $u w \in E(G)$ and $w v \in E(G)$ we have $u v \in E(G)(u \neq v$ as otherwise $w \in B)$. If $w$ has no in-neighbor from $A$ (then it can have at most one out-neighbor in $A$ ), then all edges $w v$ with $v \in F$ have been removed by Rule 7. Then $w$ has at most one out-neighbor and is removed either by Rule DFVS 2.1 or Rule DFVS 2.2. Analogously, if $w$ has no out-neighbor in $A$ (then it can have at most one inneighbor from $A$ ), then all edges $u w$ with $u \in F$ have been removed by Rule 7 . Then $w$ has at most one in-neighbor and is removed either by Rule DFVS 2.1 or Rule DFVS 2.2.

Corollary 6.2. $\left|A_{0} \cup A_{1}\right| \leq|F|(|F|-1) k$ and $\left|A_{\geq 3}\right| \leq|F|(|F|-1) k-2$.
Proof. The underlying undirected graph of $G[A]$ induces an undirected forest and the number of vertices of degree at least 3 in an undirected forest is at most equal to the number of leaves minus two; the number of leaves is $\left|A_{1}\right|$. Hence, we have $\left|A_{\geq 3}\right| \leq\left|A_{1}\right|-2 \leq\left|A_{0} \cup A_{1}\right|-2 \leq|F|(|F|-1) k-2$, where the latter inequality is a consequence of the previous two lemmas (Lemma 6.4 and Lemma 6.5).

It remains to bound the size of $A_{2}$. Following the terminology of Bergnouxnoux et al., we call a vertex $w \in A_{2}$ a sink vertex or a source vertex if the two neighbors of $w$ in $A$ are both in-neighbors or out-neighbors, respectively. Otherwise we call $w$ a balanced vertex. As Rule DFVS 2.2 cannot be applied, every vertex of $A_{2}$ has at least 2 distinct neighbors in $F$.

Let $P=\left(w_{1}, \ldots, w_{r}\right)$ be an inclusion-wise maximal directed path in $G[A]$ whose internal vertices are in $A_{2}$. We call $P$ a path segment in $A$. We call $P$ an outer path segment if at least one of its endpoints is not in $A_{2}$ and an inner
path segment, otherwise. Note that path segments are directed paths, which, by maximality, can never start or end with a balanced vertex. Moreover, every internal vertex of a path segment must be a balanced vertex.

We first bound number of outer path segments, that is, the number of path segments with at least one endpoint in $A_{1} \cup A_{3}$.

Lemma 6.6. The number of outer path segments is at most $4(|F|(|F|-1) k)$.
Proof. Let $H$ be the undirected graph obtained from the undirected graph underlying $G[A]$ by contracting all edges that are incident to at least one vertex of degree 2. Then every outer path segment runs between the endpoint of an edge in $H$ and an endpoint or inner vertex of the contracted path in a unique direction, as $A$ does not contain blue edges. Hence, the number of outer path segments in $A$ is bounded by twice the number of edges of $H$. As $H$ is a forest without vertices of degree two, its number of edges is equal to the number of leaves plus the number of non-leaves minus one. As shown in Corollary 6.2 both of these numbers are bounded from above by $|F|(|F|-1) k$.

We say that a path segment $P=\left(w_{1}, \ldots, w_{r}\right)$ contributes to a potential edge $(u, v)$ of $F$ if there are $i$ and $j, 1 \leq i \leq j \leq r$, such that $u w_{i} \in E(G)$ and $w_{j} v \in E(G)$. We say that $P$ contributes to a loop on $u \in F$ if there are $i$ and $j$, $1 \leq i \leq j \leq r$, such that $u w_{i} \in E(G)$ and $w_{j} u \in E(G)$.

Lemma 6.7. Every inner path segment contributes to a non-edge or a loop of $F$.
Proof. Assume towards a contradiction that a non-trivial $P=\left(w_{1}, \ldots, w_{r}\right)$ does not contribute to a non-edge or a loop of $F$. First assume that $P$ contains at least the inner vertex $w_{2}$. Note that $w_{1}$ is a source and $w_{r}$ is a $\operatorname{sink}$ in $A$, that is, there is no in-neighbor of $w_{1}$ in $A$ and no out-neighbor of $w_{r}$ in $A$. On the other hand, $w_{2}$ is balanced and has only the in-neighbor $w_{1}$ and out-neighbor $w_{3}$ in $A$. As Rule DFVS 2.3 cannot be applied anymore and $w_{2}$ is not incident with a blue edge, $w_{2}$ has at least one in-neighbor $u \in F$. Consider an edge $u w_{2} \in E(G)$, where $u \in F$. Every cycle containing $u w_{2}$ must use some $w_{j} \in V(P), j \geq 2$, and some $v \in F$ with $w_{j} v \in E(G)$. As $P$ does not contribute to a loop we have $u \neq v$. Hence, $u v \in E(G)$, as $P$ does not contribute to a non-edge of $F$. But then the edge $u w_{2}$ would have been deleted by the application of Rule 7. As $u$ was an arbitrary in-neighbor from $F$ of $w_{2}$, then $w_{2}$ has only the in-neighbor $w_{1}$ and out-neighbor $w_{3}$, hence, should have been shortcutted by Rule DFVS 2.2.

Now assume that $P$ does not contain an inner vertex, that is, $P=\left(w_{1}, w_{2}\right)$. As $w_{1}$ is a source and $w_{2}$ is a sink in $A$, the only in-neighbors for $w_{1}$ are from $F$ and the only out-neighbors of $w_{2}$ are from $F$. Hence all cycles using $w_{1} w_{2}$ have the form $w_{1}, w_{2}, v, \ldots, u, w_{1}$, where $u, v \in F$. Consider an arbitrary such cycle. Then $u \neq v$ as $P$ does not contribute to a loop of $F$ and $u v \in E(G)$ as $P$ does not contribute to a non-edge of $F$. However, this edge $u v$ shows that the cycle is not induced on an initial segment of length 3 , and consequently, $w_{1} w_{2}$ should have been removed by Rule 8 .

Lemma 6.8. There are at most $3|F|(|F|-1) k$ inner path segments.

Proof. As shown in Lemma 6.7, every inner path segment contributes to a nonedge or a loop of $F$. As every inner path segment has a source in $A_{2}$ as its starting and a sink in $A_{2}$ as its ending vertex and all inner vertices are balanced vertices, it can intersect with at most two other inner path segments (at their endpoints). In any set $X$ of inner path segments we can hence find $|X| / 3$ independent inner path segments. If there are more than $3|F|(|F|-1) k$ inner path segments that contribute to a non-edge or a loop, then we find $|F|(|F|-1) k$ many independent ones. Then some pair must be connected by more than $k$ disjoint paths. Then this pair is connected by an edge by Rule 3 and the path segment does not contribute to it.

Corollary 6.3. Overall there are at most $\mathcal{O}\left(|F|^{2} k\right)$ path segments.
Observe that all path segments are induced paths in $A$. Let $u \in F$ and let $P=\left(w_{1}, \ldots, w_{r}\right)$ be an induced directed path in $A$ such that $w_{1}, \ldots, w_{r-1}$ are balanced in $A$. If $u w_{1} \in E(G)$ and $u w_{r} \in E(G)$ and for every $1<i<r$ we have $u w_{i} \notin E(G)$, then we call $P$ an out-segment for $u$. We say that an out-segment for $u$ denoted by $P$ contributes to a potential edge or loop $(u, v)$ in $F$ if there is an index $1 \leq i<r$ such that $w_{i} v \in E(G)$ for some $v \in F$.

Lemma 6.9. Every out-segment for $u$ contributes to a non-edge or loop of $F$.
Proof. Assume $P$ is an out-segment for $u$ that does not contribute to a non-edge or loop. Because $w_{1}$ is balanced in $A$, its only out-neighbor in $A$ is $w_{2}$. Hence, every cycle $C$ using the edge $u w_{1}$ must contain the vertex $w_{r}$ or a vertex $v \in F$ with $w_{i} v \in E(G)$ for some $i<r$. Note that $v \neq u$ because $P$ does not contribute to a loop. Because $P$ does not contribute to a non-edge of $F$ in the latter case there is an edge $u v \in E(G)$. This however, in either case, implies that the edge $u w_{1}$ is deleted by Rule 7 .

Lemma 6.10. For each $u \in F$ there are at most $|F| k$ out-segments for $u$.
Proof. According to Lemma 6.9, every out-segment for $u$ contributes to a nonedge or loop of $F$. Note that the contribution for $P=\left(w_{1}, \ldots, w_{r}\right)$ is from the (balanced) vertices $w_{1}$ and $w_{i}$, for some $i<r$. Hence, when two out-segments contribute to the same non-edge $(u, v)$ and intersect in $w_{r}$, the contributing paths still give rise to two internally vertex disjoint $u-v$-paths. Hence, there can be at most $k$ out-segments contributing to a non-edge $(u, v)$. Finally, there are at most $|F|$ choices for $v$.

We can now bound the size of $A_{2}$.
Lemma 6.11. $\left|A_{2}\right| \in \mathcal{O}\left(|F|^{3} k\right)$.
Proof. Note that every balanced and source vertex of $A_{2}$ is an out-neighbor of at least one $u \in F$ and lies on at least one path segment. Fix some $u \in F$. Every out-neighbor of $u$ on a path segment can either be associated to the path segment itself (if $u$ has only one out-neighbor on the whole path segment) or
to an out-segment. Therefore, the number of out-neighbors of $u$ in $A_{2}$ is at most the number of path segments plus the number of out-segments for $u$. Both numbers are bounded by $\mathcal{O}\left(|F|^{2} k\right)$ by Corollary 6.3 and Lemma 6.10. Hence, the total number of out-neighbors of vertices of $F$ in $A_{2}$, which is equal to the total number of balanced and source vertices in $A_{2}$, is bounded by $\mathcal{O}\left(|F|{ }^{3} k\right)$. Combined with the fact that the number of sink vertices in $A_{2}$ is also bounded by $\mathcal{O}\left(|F|^{2} k\right)$, we get the claimed bound of $\left|A_{2}\right| \in \mathcal{O}\left(|F|^{3} k\right)$.

Theorem 6.1. After the exhaustive application of Rule 1, Rule 2, Rule 3 and Rule 8 we obtain a kernel with $\mathcal{O}\left(|F|{ }^{3} k\right)$ vertices.

## 7 LP-based approximation

We can derive the following cycles $I L P$ for DFVS naturally from the Hitting SET formulation. Given a DFVS instance $G$, we introduce a binary variable $d_{v}$ for every $v \in V(G)$ where $d_{v}=1$ means that $v$ is part of the solution. The goal is to minimize the number of variables set to 1 , given that all induced cycles are hit.

$$
\begin{aligned}
& \min \sum_{v \in V(G)} d_{v} \\
& \text { s.t. } \sum_{v \in V(C)} d_{v} \geq 1 \text { for all induced cycles } C \text { in } G \\
& \qquad d_{v} \in\{0,1\} \text { for all } v \in V(G)
\end{aligned}
$$

Note that this formulation can have an exponential number of contraints. In the following, we assume that our instances contain no loops. We work with the following equivalent order ILP of polynomial size which uses the fact that a graph is acyclic if and only if there is a topological order on its vertex set. To be more precise, we order the vertices linearly, minimizing the number of vertices having an incident edge pointing in the incorrect direction. We introduce a binary variable $x_{u v}$ for all distinct $u, v \in V(G)$ where $x_{u v}=1$ indicates that $u$ is smaller than $v$ in the order. Furthermore, we introduce a binary variable $y_{v}$ for every $v \in V(G)$ with the same meaning as the $d_{v}$ in the cycles ILP.

$$
\begin{aligned}
\min \sum_{v \in V(G)} y_{v} & \\
\text { s.t. } x_{u v}+x_{v u} & =1 \text { for all distinct } u, v \in V(G) \\
x_{u v}+x_{v w}-x_{u w} & \geq 1 \text { for all distinct } u, v, w \in V(G) \\
x_{u v}+y_{u}+y_{v} & \geq 1 \text { for all } u v \in E(G) \\
x_{u v}, y_{v} & \in\{0,1\} \text { for all } u, v \in V(G)
\end{aligned}
$$

The first two constraints (ensuring anti-symmetry and transitivity) yield a linear order on $V(G)$, and the third constraint ensures that at least one endpoint of any edge pointing in the incorrect direction must be part of the solution. We prove that the relaxation of the order ILP is within a constant factor of the all cycles ILP.

Theorem 7.1. The optimal solution of the order ILP relaxation is at most 3 times smaller than the optimal solution of the cycles ILP relaxation.

Proof. Fix an optimal solution (assignment of variables) of the order ILP relaxation. Let $C=v_{1} \ldots v_{\ell}$ be an induced cycle in $G$. We show that $\sum_{1 \leq i \leq \ell} y_{v_{i}} \geq 1 / 3$. First assume that all $x_{v_{i} v_{i+1}}=1$ for $1 \leq i<\ell$. Then by transitivity we have $x_{v_{1} v_{\ell}}=1$ and by anti-symmetry $x_{v_{\ell} v_{1}}=0$. Then by the third constraint $y_{v_{1}}+y_{v_{\ell}} \geq 1$ and we are done.

Hence assume $x_{v_{i} v_{i+1}}=1-\varepsilon_{i}$ for some $\varepsilon_{i} \geq 0$, where at least one $\varepsilon_{i}>0$. For every $j \geq i+1$ we have $x_{v_{i} v_{j}} \geq 1-\sum_{i \leq i<q} \varepsilon_{q}$ by transitivity. Hence, $x_{v_{0} v_{\ell}} \geq 1-\sum_{1 \leq q<\ell} \varepsilon_{q}$. Then $x_{v_{\ell} v_{1}} \leq \sum_{1 \leq q<\ell} \varepsilon_{q}$ and $y_{v_{1}}+y_{v_{\ell}} \geq 1-\sum_{1 \leq q<\ell} \varepsilon_{q}$.

If $\sum_{1 \leq q<\ell} \varepsilon_{q} \leq 2 / 3$, then $y_{v_{1}}+y_{v_{\ell}} \geq 1 / 3$ and $C$ collects weight at least $1 / 3$. Otherwise we have $\sum_{1 \leq q<\ell} \varepsilon_{q}>2 / 3$. Hence, we can write $\sum_{1 \leq i<\ell} x_{v_{i} v_{i+1}}=$ $\ell-\sum_{1 \leq i<\ell} \varepsilon_{i} \geq \sum_{1 \leq i<\ell}\left(1-y_{v_{i}}-y_{v_{i+1}}\right)=\ell-y_{v_{1}}-y_{v_{\ell}}-2 \sum_{1<i<\ell} y_{i}$. Plugging in the inequality we obtain $2 / 3<\sum_{1 \leq i<\ell} \varepsilon_{i} \leq y_{v_{1}}+y_{v_{\ell}}+2 \sum_{2 \leq i<\ell-1} y_{v_{i}}$. Hence, $C$ collects more than weight $1 / 3$.

Denote the optimal solution value for the cycles ILP relaxation by $h^{*}$, the optimal solution value for the order ILP relaxation by $x^{*}$, and the optimal ILP solution value by $k$. Then $\mathcal{O}(k /(\log k \log \log k)) \leq h^{*} \leq 3 x^{*} \leq 3 k$, where the first inequality follows from [Sey95].

Corollary 7.1. We can approximate in polynomial time the cycles ILP relaxation up to factor 3 .

## 8 Hardness of Directed Chordless Path

The Directed Chordless $(s, v, t)$-Path problem asks, given a graph $G$, vertices $s, v, t$, and integer $d$, whether there exists an induced $s$-t-path in $G$ of length at most $d$ containing $v$. The $\mathrm{W}[1]$-hardness of the problem on general (directed and undirected) graphs was proved in [HH06]. We show hardness on directed acyclic graphs via a reduction from Grid Tiling.

An instance of Grid Tiling consists of an even integer $k$, an integer $n$, and a collection $\mathcal{S}$ of $k^{2}$ nonempty sets $S_{i, j} \subseteq[n] \times[n]$, where $1 \leq i, j \leq k$. The goal is to decide whether there exists, for each $1 \leq i, j \leq k$, a pair $s_{i . j} \in S_{i, j}$ such that:

$$
\text { - If } s_{i, j}=(a, b) \text { and } s_{i+1, j}=\left(a^{\prime}, b^{\prime}\right) \text {, then } a=a^{\prime}
$$

- If $s_{i, j}=(a, b)$ and $s_{i, j+1}=\left(a^{\prime}, b^{\prime}\right)$, then $b=b^{\prime}$.

In other words, if $(i, j)$ and $\left(i^{\prime}, j^{\prime}\right)$ are adjacent in the first or second coordinate, then $s_{i, j}$ and $s_{i^{\prime}, j^{\prime}}$ agree in the first or second coordinate, respectively. We visualize $S_{i, j}$ to be in a "cell" at row $i$ and column $j$ of a "matrix". Observe that the constraints ensure that the first coordinate of the solution is the same in each column and the second coordinate is the same in each row.

Lemma 8.1. The Directed Chordless $(s, v, t)$-Path problem parameterized by the length $d$ of a path is W[1]-hard even when restricted to directed acyclic graphs.

Proof. Given an instance of Grid Tiling, we construct an instance of Directed Chordless $(s, v, t)$-Path as follows. We first construct a directed acyclic graph $G$. For each $S_{i, j} \in \mathcal{S}, 1 \leq i, j \leq k$, we add a new set of vertices $V_{i, j}$ to $V(G)$. $V_{i, j}$ contains one vertex $v_{a, b}^{i, j}$ for each pair $s_{i, j}=(a, b) \in S_{i, j}$. Then we add two new sets of vertices $X=\left\{x_{1}, \ldots, x_{k}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{k}\right\}$. We partition $X$ into $X_{\text {odd }}=\left\{x_{j} \mid j\right.$ is odd $\}$ and $X_{\text {even }}=\left\{x_{j} \mid j\right.$ is even $\}$. Similarly, we partition $Y$ into $Y_{o d d}=\left\{y_{j} \mid j\right.$ is odd $\}$ and $Y_{\text {even }}=\left\{y_{j} \mid j\right.$ is even $\}$. We now describe the edges in $G$ :

- For every vertex $x_{j} \in X_{o d d}$, we add the edges $x_{j} v_{a, b}^{1, j}$, for all $a, b$. In other words, every vertex in $V_{1, j}$ is made an out-neighbor of $x_{j}, j \in\{1,3, \ldots, k-1\}$.
- For every vertex $x_{j} \in X_{\text {even }}$, we add the edges $v_{a, b}^{1, j} x_{j}$, for all $a, b$. In other words, every vertex in $V_{1, j}$ is made an in-neighbor of $x_{j}, j \in\{2,4, \ldots, k\}$.
- For every vertex $y_{j} \in Y_{o d d}$, we add the edges $v_{a, b}^{k, j} y_{j}$, for all $a, b$. In other words, every vertex in $V_{k, j}$ is made an in-neighbor of $y_{j}, j \in\{1,3, \ldots, k-1\}$.
- For every vertex $y_{j} \in Y_{\text {even }}$, we add the edges $y_{j} v_{a, b}^{k, j}$, for all $a, b$. In other words, every vertex in $V_{k, j}$ is made an out-neighbor of $y_{j}, j \in\{2,4, \ldots, k\}$.
- We add the edges $\left\{y_{1} y_{2}, y_{3} y_{4}, \ldots, y_{k-1} y_{k}\right\} \cup\left\{x_{2} x_{3}, x_{4} x_{5}, \ldots, x_{k-2} x_{k-1}\right\}$.
- For odd $j \in\{1,3, \ldots, k-1\}$ and $i \in[k-1]$, if there exists $v_{a, b}^{i, j} \in V_{i, j}$ and $v_{a^{\prime}, b^{\prime}}^{i+1, j} \in V_{i+1, j}$ such that $a=a^{\prime}$ then add the edge $v_{a, b}^{i, j} v_{a^{\prime}, b^{\prime}}^{i+1, j}$.
- For even $j \in\{2,4, \ldots, k\}$ and $i \in[k] \backslash\{1\}$, if there exists $v_{a, b}^{i, j} \in V_{i, j}$ and $v_{a^{\prime}, b^{\prime}}^{i-1, j} \in V_{i-1, j}$ such that $a=a^{\prime}$ then add the edge $v_{a, b}^{i, j} v_{a^{\prime}, b^{\prime}}^{i-1, j}$.
- For $j=1, j^{\prime} \in\{2,3,4, \ldots, k\}$, and $i \in\{1,2,3, \ldots, k\}$, if there exists $v_{a, b}^{i, 1} \in$ $V_{i, 1}$ and $v_{a^{\prime}, b^{\prime}}^{i, j^{\prime}} \in V_{i, j^{\prime}}$ such that $b \neq b^{\prime}$ then add the edge $v_{a, b}^{i, 1} v_{a^{\prime}, b^{\prime}}^{i, j^{\prime}}$.

To complete the construction of the Directed Chordless $(s, v, t)$-Path instance, we choose $s=x_{1}, v=y_{1}, t=x_{k}$, and $d=k(k+1)+k-1=k^{2}+2 k-1$.

Observe that $G$ is acyclic since the edges in odd columns are all directed "downwards" and the edges in even columns are all directed "upwards". Moreover, the edges connecting vertices in $X$ or $Y$ are all directed "rightwards". Similarly, the edges connecting vertices in the first column to vertices in later columns are all directed "rightwards". Hence, following the layout of the construction, we can topologically order the vertices of $G$ such that for every directed edge $u v$ from vertex $u$ to vertex $v, u$ comes before $v$ in the ordering.

We also note that any $(s, v, t)$-path in $G$ must contain exactly one vertex from each $V_{i, j}$ as well as all vertices in $X \cup Y$; for a total of $k^{2}+2 k$ vertices. Hence, any $(s, v, t)$-path in $G$ will have length exactly $d$. In addition, $P$ must visit $X, Y$, and the "cells" of the matrix in a unique order. That is, $P$ must start with $x_{1}$ then visit all the cells of the first column to reach $y_{1}$. After $y_{1}$ the only out-neighbor is $y_{2}$. From $y_{2}, P$ then proceeds upwards along the second column to reach $x_{2}$. This zig-zag behavior continues until $P$ reaches $x_{k}$. In other words, $P$ consists of the ordered vertices $s=x_{1}, \ldots, v=y_{1}, y_{2}, \ldots, x_{2}, x_{3}, \ldots, y_{3}, y_{4}, \ldots, x_{4}, x_{5}$, $\ldots, y_{k-1}, y_{k}, \ldots, t=x_{k}$.

Assume that we have a yes-instance of Directed Chordless $(s, v, t)$-Path and let $P$ be an induced $(s, v, t)$-path in $G$. The vertices of $V(P) \backslash(X \cup Y)$ correspond one-to-one to pairs in $\mathcal{S}$. We claim that those pairs form a valid solution for the Grid Tiling instance. Assume otherwise. Then, either $P$ contains two consecutive vertices in the same column that do not agree on the first coordinate or $P$ contains two vertices in the same row that do not agree on the second coordinate. The former case is not possible by construction; we only add edges between consecutive vertices in the same column whenever they agree on the first coordinate. For the latter case, we claim that it would contradict the fact that $P$ is induced. To see why, let $i$ be a row containing two vertices $v_{2}, v_{3}$ that do not agree on the second coordinate. Without loss of generality, we assume that neither of these two vertices belongs to the first column. Let $j_{1}$ and $j_{2}$, $j_{1}<j_{2}$, denote their respective columns. Recall that $P$ must include one vertex $v_{1}$ from $V_{i .1}$ and this vertex cannot agree with both $v_{2}, v_{3}$ on the second coordinate. Hence, by construction, $G$ either contains the edge $v_{1} v_{2}$ or the edge $v_{1} v_{3}$, which contradicts the fact that $P$ is induced.

Using almost identical arguments, it can be shown that whenever we have a yes-instance of Grid Tiling we can immediately construct an induced ( $s, v, t$ )path in $G$ of length exactly $d$, as needed.

By connecting the bottom right vertex with the top left vertex with an edge we obtain the following corollaries of (the proof of) Lemma 8.1.

Corollary 8.1. It is W[1]-hard to decide if a vertex lies on an induced cycle of length at most d even on graphs that become acyclic after the deletion of a single edge.

Corollary 8.2. It is W/1]-hard to decide if a graph contains an induced cycle of length at least $d$ even on graphs that become acyclic after the deletion of a single edge.

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