# Approximating the Revenue Maximization Problem with Sharp Demands 

Vittorio Bilo ${ }^{1}$, Michele Flammini ${ }^{2}$, and Gianpiero Monaco ${ }^{2}$<br>${ }^{1}$ Department of Mathematics and Physics "Ennio De Giorgi", University of Salento Provinciale Lecce-Arnesano, P.O. Box 193, 73100 Lecce, Italy<br>vittorio.bilo@unisalento.it<br>${ }^{2}$ Department of Information Engineering Computer Science and Mathematics, University of L'Aquila, Via Vetoio, Coppito, 67100 L'Aquila, Italy<br>\{flammini, gianpiero.monaco\}@di.univaq.it


#### Abstract

We consider the revenue maximization problem with sharp multi-demand, in which $m$ indivisible items have to be sold to $n$ potential buyers. Each buyer $i$ is interested in getting exactly $d_{i}$ items, and each item $j$ gives a benefit $v_{i j}$ to buyer $i$. We distinguish between unrelated and related valuations. In the former case, the benefit $v_{i j}$ is completely arbitrary, while, in the latter, each item $j$ has a quality $q_{j}$, each buyer $i$ has a value $v_{i}$ and the benefit $v_{i j}$ is defined as the product $v_{i} q_{j}$. The problem asks to determine a price for each item and an allocation of bundles of items to buyers with the aim of maximizing the total revenue, that is, the sum of the prices of all the sold items. The allocation must be envy-free, that is, each buyer must be happy with her assigned bundle and cannot improve her utility. We first prove that, for related valuations, the problem cannot be approximated to a factor $O\left(m^{1-\epsilon}\right)$, for any $\epsilon>0$, unless $P=N P$ and that such result is asymptotically tight. In fact we provide a simple $m$-approximation algorithm even for unrelated valuations. We then focus on an interesting subclass of "proper" instances, that do not contain buyers a priori known not being able to receive any item. For such instances, we design an interesting 2 -approximation algorithm and show that no $(2-\epsilon)$-approximation is possible for any $0<\epsilon \leq 1$, unless $P=N P$. We observe that it is possible to efficiently check if an instance is proper, and if discarding useless buyers is allowed, an instance can be made proper in polynomial time, without worsening the value of its optimal solution.


## 1 Introduction

A major decisional process in many business activities concerns whom to sell products (or services) to and at what price, with the goal of maximizing the total revenue. On the other hand, consumers would like to buy at the best possible prices and experience fair sale criteria.

In this work, we address such a problem from a computational point of view, considering a two-sided market in which the supply side consists of $m$ indivisible items and the demand one is populated by $n$ potential buyers (in the following
also called consumers or customers), where each buyer $i$ has a demand $d_{i}$ (the number of items that $i$ requests) and valuations $v_{i j}$ representing the benefit $i$ gets when owing item $j$. As several papers on this topic (see for instance [12|22|17|7|15), we assume that, by means of market research or interaction with the consumers, the seller knows each customer's valuation for each item.

The seller sets up a price $p_{j}$ for each item $j$ and assigns (i.e., sells) bundle of items to buyers with the aim of maximizing her revenue, that is the sum of the prices of all the sold items. When a consumer is assigned (i.e., buys) a set of items, her utility is the difference between the total valuation of the items she gets (valuations being additive) and the purchase price.

The sets of the sold items, the purchasing customers and their purchase prices are completely determined by the allocation of bundles of items to customers unilaterally decided by the seller. Nevertheless, we require such an allocation to meet two basic fairness constraints: (i) each customer $i$ is allocated at most one bundle not exceeding her demand $d_{i}$ and providing her a non-negative utility, otherwise she would not buy the bundle; (ii), the allocation must be envy-free [30, i.e., each customer $i$ does not prefer any subset of $d_{i}$ items different from the bundle she is assigned.

The envy-freeness notion adopted in this paper is the typical one of pricing problems. Anyway, in the literature there also exist weaker forms usually applied in fair division settings (see for instance [16]) where, basically, no buyer wants to switch her allocation with that of another buyer, without combining different bundles. Notice that in our scenario a trivial envy-free solution always exists that lets $p_{j}=\infty$ for each item $j$ and does not assign any item to any buyer.

Many papers (see the Related Work section for a detailed reference list) considered the unit demand case in which $d_{i}=1$ for each consumer $i$. Arguably, the multi-demand case, where $d_{i} \geq 1$ for each consumer $i$, is more general and finds much more applicability. To this aim, we can identify two main multi-demand schemes. The first one is the relaxed multi-demand model, where each buyer $i$ requests at most $d_{i} \geq 1$ items, and the second one is the sharp multi-demand model, where each buyer $i$ requests exactly $d_{i} \geq 1$ items and, therefore, a bundle of size less than $d_{i}$ has no value for buyer $i$.

For relaxed multi-demand models, a standard technique can reduce the problem to the unit demand case in the following way: each buyer $i$ with demand $d_{i}$ is replaced by $d_{i}$ copies of buyer $i$, each requesting a single item. However, such a trick does not apply to the sharp demand model. Moreover, as also pointed out in [7], the sharp multi-demand model exhibits a property that unit demand and relaxed multi-demand ones do not posses. In fact, while in the latter model any envy-free pricing is such that the price $p_{j}$ is always at most the value of $v_{i j}$, in the sharp demand model, a buyer $i$ may pay an item $j$ more than her own valuation for that item, i.e., $p_{j}>v_{i j}$ and compensate her loss with profits from the other items she gets (see section 3.1 of [7]). Such a property, also called overpricing, clearly adds an extra challenge to find an optimal revenue.

The sharp demand model is quite natural in several settings. Consider, for instance, a scenario in which a public organization has the need of buying a fixed
quantity of items in order to reach a specific purpose (i.e. locations for offices, cars for services, bandwidth, storage, or whatever else), where each item might have a different valuation for the organization because of its size, reliability, position, etc. Yet, suppose a user wants to store on a remote server a file of a given size $s$ and there is a memory storage vendor that sells slots of fixed size $c$, where each cell might have different features depending on the server location and speed and then yielding different valuations for the user. In this case, a number of items smaller than $\left\lceil\frac{s}{c}\right\rceil$ has no value for the user. Similar scenarios also apply to cloud computing. In [7], the authors used the following applications for the sharp multi-demand model. In TV (or radio) advertising [23], advertisers may request different lengths of advertising slots for their ads programs. In banner (or newspaper) advertising, advertisers may request different sizes or areas for their displayed ads, which may be decomposed into a number of base units. Also, consider a scenario in which advertisers choose to display their advertisement using medias (video, audio, animation) 4|24 that would usually need a fixed number of positions, while text ads would need only one position each. An example of formulation sponsored search using sharp multi-demands can be found in 13. Other results concerning the sharp multi-demand model in the Bayesian setting can be found in 11.

Related Work. Pricing problems have been intensively studied in the literature, see e.g., $26|27| 21|20| 25|1| 18$ just to cite a few, both in the case in which the consumers' preferences are unknown (mechanism design [29|5]) and in the case of full information that we consider in this paper. In fact, our interest here is in maximizing the seller's profit assuming that consumers' preferences are gathered through market research or conjoint analysis [12|22|17|7|15]. From an algorithmic point of view, [17] is the first paper dealing with the problem of computing the envy-free pricing of maximum revenue. The authors considered the unit demand case for which they gave an $O(\log n)$-approximation algorithm and showed that computing an optimal envy-free pricing is APX-hard. Briest [2] showed that, under reasonable complexity assumptions, the revenue maximization problem in the unit demand model cannot be approximated within $O\left(\log ^{\varepsilon} n\right)$ for some $\varepsilon>0$. The subcase in which every buyer positively evaluates at most two items has been studied in [6]. The authors proved that the problem is solvable in polynomial time and it becomes NP-hard if some buyer gets interested in at least three items.

For the multi-demand model, Chen et. al. [8] gave an $O(\log D)$ approximation algorithm when there is a metric space behind all items, where $D$ is the maximum demand, and Briest [2] showed that the problem is hard to approximate within a ratio of $O\left(n^{\varepsilon}\right)$ for some $\varepsilon>0$.

To the best of our knowledge, [7] is the first paper explicitly dealing with the sharp multi-demand model. The authors considered a particular valuation scheme (also used in 1428 for keywords advertising scenarios) where each item $j$ has a parameter $q_{j}$ measuring the quality of the item and each buyer $i$ has a value $v_{i}$ representing the benefit that $i$ gets when owing an item of unit quality. Thus, the benefit that $i$ obtains from item $j$ is given by $v_{i} q_{j}$. For such a problem,
the authors proved that computing the envy-free pricing of maximum revenue is NP-hard. Moreover, they showed that if the demand of each buyer is bounded by a constant, the problem becomes solvable in polynomial time. We remark that this valuation scheme is a special case of the one in which the valuations $v_{i j}$ are completely arbitrary and given as an input of the problem. Throughout the paper, we will refer to the former scheme as to related valuations and to the latter as to unrelated valuations. Recently [10] considered the sharp multidemand model with the additional constraint in which items are arranged as a sequence and buyers want items that are consecutive in the sequence.

Finally [15] studied the pricing problem in the case in which buyers have a budget, but no demand constraints. The authors considered a special case of related valuations in which all qualities are equal to 1 (i.e., $q_{j}=1$ for each item $j$ ). They proved that the problem is still NP-hard and provided a 2-approximation algorithm. Such algorithm assigns the same price to all the sold items.

Many of the papers listed above deal with the case of limited supply. Another stream of research considers unlimited supply, that is, the scenario in which each item $j$ exists in $e_{j}$ copies and it is explicitly allowed that $e_{j}=\infty$. The limited supply setting seems generally more difficult than the unlimited supply one. In this paper we consider the limited supply setting. Interesting results for unlimited supply can be found in 1793 .
Our Contribution. We consider the revenue maximization problem with sharp multi-demand and limited supply. We first prove that, for related valuations, the problem cannot be approximated to a factor $O\left(m^{1-\epsilon}\right)$, for any $\epsilon>0$, unless P $=\mathrm{NP}$ and that such result is asymptotically tight. In fact we provide a simple $m$-approximation algorithm even for unrelated valuations.

Our inapproximability proof relies on the presence of some buyers not being able to receive any bundle of items in any envy-free outcome. Thus, it becomes natural to ask oneself what happens for instances of the problem, that we call proper, where no such pathological buyers exist. For proper instances, we design an interesting 2-approximation algorithm and show that the problem cannot be approximated to a factor $2-\epsilon$ for any $0<\epsilon \leq 1$ unless $\mathrm{P}=$ NP. Therefore, also in this subcase, our results are tight. We remark that it is possible to efficiently decide whether an instance is proper. Moreover, if discarding useless buyers is allowed, an instance can be made proper in polynomial time, without worsening the value of its optimal solution.

## 2 Model and Preliminaries

In the Revenue Maximization Problem with Sharp Multi-Demands (RMPSD) investigated in this paper, we are given a market made up of a set $M=$ $\{1,2, \ldots, m\}$ of items and a set $N=\{1,2, \ldots, n\}$ of buyers. Each item $j \in M$ has unit supply (i.e., only one available copy). We consider both unrelated and related valuations. In the former each buyers $i$ has valuations $v_{i j}$ representing the benefit $i$ gets when owing item $j$. In the latter each item is characterized by a quality (or desirability) $q_{j}>0$, while each buyer $i \in N$ has a value $v_{i}>0$,
measuring the benefit that she gets when receiving a unit of quality, thus, the valuation that buyer $i$ has for item $j$ is $v_{i j}=v_{i} q_{j}$. We notice that related is a special case of unrelated valuations. Throughout the paper, when not explicitly indicated, we refer to related valuations. Finally each buyer $i$ has a demand $d_{i} \in \mathbb{Z}^{+}$, which specifies the exact number of items she wants to get. In the following we assume items and bidders ordered in non-increasing order, that is, $v_{i} \geq v_{i^{\prime}}$ for $i<i^{\prime}$ and $q_{j} \geq q_{j^{\prime}}$ for $j<j^{\prime}$.

An allocation vector is an $n$-tuple $\mathbf{X}=\left(X_{1}, \ldots, X_{n}\right)$, where $X_{i} \subseteq M$, with $\left|X_{i}\right| \in\left\{0, d_{i}\right\}, \sum_{i \in N}\left|X_{i}\right| \leq m$ and $X_{i} \cap X_{i^{\prime}}=\emptyset$ for each $i \neq i^{\prime} \in N$, is the set of items sold to buyer $i$. A price vector is an $m$-tuple $\mathbf{p}=\left(p_{1}, \ldots, p_{m}\right)$, where $p_{j}>0$ is the price of item $j$. An outcome of the market is a pair $(\mathbf{X}, \mathbf{p})$.

Given an outcome ( $\mathbf{X}, \mathbf{p}$ ), we denote with $u_{i j}(\mathbf{p})=v_{i j}-p_{j}$ the utility that buyer $i$ gets when she is sold item $j$ and with $u_{i}(\mathbf{X}, \mathbf{p})=\sum_{j \in X_{i}} u_{i j}(\mathbf{p})$ the overall utility of buyer $i$ in $(\mathbf{X}, \mathbf{p})$. When the outcome (or the price vector) is clear from the context, we simply write $u_{i}$ and $u_{i j}$. An outcome $(\mathbf{X}, \mathbf{p})$ is feasible if $u_{i} \geq 0$ for each $i \in N$.

We denote with $M(\mathbf{X})=\bigcup_{i \in N} X_{i}$ the set of items sold to some buyer according to the allocation vector $\mathbf{X}$. We say that a buyer $i$ is a winner if $X_{i} \neq \emptyset$ and we denote with $W(\mathbf{X})$ the set of all the winners in $\mathbf{X}$. For an item $j \in M(\mathbf{X})$, we denote with $b_{\mathbf{X}}(j)$ the buyer $i \in W(\mathbf{X})$ such that $j \in X_{i}$, while, for an item $j \notin M(\mathbf{X})$, we define $b_{\mathbf{X}}(j)=0$. Moreover, for a winner $i \in W(\mathbf{X})$, we denote with $f_{\mathbf{X}}(i)=\min \left\{j \in M: j \in X_{i}\right\}$ the best-quality item in $X_{i}$. Also in this case, when the allocation vector is clear from the context, we simply write $b(j)$ and $f(i)$. Finally, we denote with $\beta(\mathbf{X})=\max \{i \in N: i \in W(\mathbf{X})\}$ the maximum index of a winner in $\mathbf{X}$. An allocation vector $\mathbf{X}$ is monotone if $\min _{j \in X_{i}}\left\{q_{j}\right\} \geq q_{f\left(i^{\prime}\right)}$ for each $i, i^{\prime} \in W(\mathbf{X})$ with $v_{i}>v_{i^{\prime}}$, that is, all the items of $i$ are of quality greater of equal to the one of all the items of $i^{\prime}$.

Definition 1. A feasible outcome ( $\mathbf{X}, \mathbf{p}$ ) is an envy-free outcome if, for each buyer $i \in N, u_{i} \geq \sum_{j \in T} u_{i j}$ for each $T \subseteq M$ of cardinality $d_{i}$.

Notice that, by definition, an outcome ( $\mathbf{X}, \mathbf{p}$ ) is envy-free if and only if the following three conditions holds:

1. $u_{i} \geq 0$ for each $i \in N$,
2. $u_{i j} \geq u_{i j^{\prime}}$ for each $i \in W(\mathbf{X}), j \in X_{i}$ and $j^{\prime} \notin X_{i}$,
3. $\sum_{j \in T} u_{i j} \leq 0$ for each $i \notin W(\mathbf{X})$ and $T \subseteq M$ of cardinality $d_{i}$.

Note also that, as already remarked, envy-free solutions always exist, since the outcome ( $\mathbf{X}, \mathbf{p}$ ) such that $X_{i}=\emptyset$ for each $i \in N$ and $p_{j}=\infty$ for each $j \in M$ is envy-free. Moreover, deciding whether an outcome is envy-free can be done in polynomial time.

By the definition of envy-freeness, if $i \in W(X)$ is a winner, then all the buyers $i^{\prime}$ with $v_{i^{\prime}}>v_{i}$ and $d_{i^{\prime}} \leq d_{i}$ must be winners as well, otherwise $i^{\prime}$ would envy a subset of the bundle assigned to $i$. This motivates the following definition, which restricts to instances not containing buyers not being a priori able to receive any item (useless buyers).

Definition 2. An instance $I$ is proper if, for each buyer $i \in N$, it holds $d_{i}+$ $\sum_{i^{\prime} \mid v_{i^{\prime}}>v_{i}, d_{i^{\prime}} \leq d_{i}} d_{i^{\prime}} \leq m$.

The (market) revenue generated by an outcome ( $\mathbf{X}, \mathbf{p}$ ) is defined as $\operatorname{rev}(\mathbf{X}, \mathbf{p})=\sum_{j \in M(\mathbf{X})} p_{j}$. RMPSD asks for the determination of an envy-free outcome of maximum revenue. We observe that it is possible to efficiently check if an instance is proper, and if discarding useless buyers is allowed, an instance can be made proper in polynomial time, without worsening the value of its optimal solution. An instance of the RMPSD problem can be modeled as a triple $(\mathbf{V}, \mathbf{D}, \mathbf{Q})$, where $\mathbf{V}=\left(v_{1}, \ldots, v_{n}\right)$ and $\mathbf{D}=\left(d_{1}, \ldots, d_{n}\right)$ are the vectors of buyers' values and demands, while $\mathbf{Q}=\left(q_{1}, \ldots, q_{m}\right)$ is the vector of item qualities. We conclude this section with three lemmas describing some properties that need to be satisfied by any envy-free outcome.

Lemma 1 ([7]). If an outcome ( $\mathbf{X}, \mathbf{p}$ ) is envy-free, then $\mathbf{X}$ is monotone.
Proof. Let ( $\mathbf{X}, \mathbf{p}$ ) be an envy-free outcome and assume, for the sake of contradiction, that $\mathbf{X}$ is not monotone, i.e., that there exist two buyers $i, i^{\prime} \in W(\mathbf{X})$ with $v_{i}>v_{i}^{\prime}$ and two items $j \in X_{i}$ and $j^{\prime} \in X_{i^{\prime}}$ such that $q_{j}<q_{j^{\prime}}$. By the envyfreeness of ( $\mathbf{X}, \mathbf{p}$ ), it holds $u_{i j} \geq u_{i j^{\prime}}$ which implies $p_{j}-p_{j^{\prime}} \leq v_{i}\left(q_{j}-q_{j^{\prime}}\right)$ and $u_{i^{\prime} j^{\prime}} \geq u_{i^{\prime} j}$ which implies $p_{j}-p_{j^{\prime}} \geq v_{i^{\prime}}\left(q_{j}-q_{j^{\prime}}\right)$. By dividing both inequalities by $q_{j}-q_{j^{\prime}}<0$, we get $v_{i^{\prime}} \geq \frac{p_{j}-p_{j^{\prime}}}{q_{j}-q_{j^{\prime}}}$ and $v_{i} \leq \frac{p_{j}-p_{j^{\prime}}}{q_{j}-q_{j^{\prime}}}$ which implies $v_{i} \leq v_{i^{\prime}}$, a contradiction.

Given an outcome $(\mathbf{X}, \mathbf{p})$, an item $j \in X_{i}$ is overpriced if $u_{i j}<0$.
Lemma 2 ([7]). Let $(\mathbf{X}, \mathbf{p})$ be an envy-free outcome. For each overpriced item $j^{\prime} \in M(\mathbf{X})$, it holds $b\left(j^{\prime}\right)=\beta(\mathbf{X})$.

Proof. Let ( $\mathbf{X}, \mathbf{p}$ ) be an envy-free outcome and assume, for the sake of contradiction, that there exists an overpriced item $j \in X_{i}$ with $i<\beta(\mathbf{X})$. Hence, $u_{i j}=$ $v_{i j}-p_{j}<0$. Since $\beta(\mathbf{X}) \in W(\mathbf{X})$ and $(\mathbf{X}, \mathbf{p})$ is feasible, it holds $u_{\beta(\mathbf{X})} \geq 0$ which implies that there exists an item $j^{\prime} \in X_{\beta(\mathbf{X})}$ such that $u_{\beta(\mathbf{X}) j^{\prime}}=v_{\beta(\mathbf{X}) j^{\prime}}-p_{j^{\prime}} \geq 0$. Moreover, by the envy-freeness of ( $\mathbf{X}, \mathbf{p}$ ), it also holds $u_{i j} \geq u_{i j^{\prime}}$. By using $v_{i} \geq v_{\beta(\mathbf{X})}$, we get $u_{i j} \geq u_{i j^{\prime}}=v_{i} q_{j^{\prime}}-p_{j^{\prime}} \geq v_{\beta(\mathbf{X})} q_{j^{\prime}}-p_{j^{\prime}}=u_{\beta(\mathbf{X}) j^{\prime}} \geq 0$, which contradicts the assumption that $u_{i j}<0$.

The following lemma establishes that, if a buyer $i$ is not a winner, then the total number of items assigned to buyers with valuation strictly smaller than $v_{i}$ is less than $d_{i}$.

Lemma 3. Let $(\mathbf{X}, \mathbf{p})$ be an envy-free outcome. For each buyer $i$ such that $i \notin$ $W(\mathbf{X})$, it holds $d_{i}>\sum_{k>i: k \in W(\mathbf{X})} d_{k}$.

Proof. Let $(\mathbf{X}, \mathbf{p})$ be an envy-free outcome and let $i, i^{\prime}$ be two buyers such that $v_{i}>v_{i^{\prime}}, i \notin W(\mathbf{X})$ and $i^{\prime} \in W(\mathbf{X})$. Assume, for the sake of contradiction, that $d_{i} \leq \sum_{k>i: k \in W(\mathbf{X})} d_{k}$. This implies that there exists $T \subseteq M$, of cardinality $d_{i}$, such that all items $j \in T$ are assigned to buyers with values of at most $v_{i}$ and
at least one item $j^{\prime} \in T$ is assigned to buyer $i^{\prime}$. Moreover, since $u_{k} \geq 0$ for each $k \in W(\mathbf{X})$ by the feasibility of $(\mathbf{X}, \mathbf{p})$, there exists one such $T$ for which $u_{i^{\prime} j^{\prime}}+\sum_{j \in T \backslash\left\{j^{\prime}\right\}} u_{b(j) j} \geq 0$. Hence, we obtain

$$
\sum_{j \in T} u_{i j}=u_{i j^{\prime}}+\sum_{j \in T \backslash\left\{j^{\prime}\right\}} u_{i j}>u_{i^{\prime} j^{\prime}}+\sum_{j \in T \backslash\left\{j^{\prime}\right\}} u_{b(j) j} \geq 0
$$

where the strict inequality follows from the fact that $v_{i}>v_{i^{\prime}}$ and $v_{i} \geq v_{b(j)}$ for each $j \in T \backslash\left\{j^{\prime}\right\}$. Thus, since there exists a set of items $T$ of cardinality $d_{i}$ such that $\sum_{j \in T} u_{i j}>0$, it follows that $(\mathbf{X}, \mathbf{p})$ is not envy-free, a contradiction.

## 3 A Pricing Scheme for Monotone Allocation Vectors

Since we are interested only in envy-free outcomes, by Lemma 1 in the following we will implicitly assume that any considered allocation vector is monotone.

We call pricing scheme a function which, given an allocation vector $\mathbf{X}$, returns a price vector. In this section, we propose a pricing scheme for allocation vectors which will be at the basis of our approximability and inapproximability results. For the sake of readability, in describing the following pricing function, given $\mathbf{X}$, we assume a re-ordering of the buyers in such a way that all the winners appear first, still in non-increasing order of $v_{i}$.
For an allocation vector $\mathbf{X}$, define the price vector $\widetilde{\mathbf{p}}$ such that, for each $j \in M$,

$$
\widetilde{p}_{j}= \begin{cases}\infty & \text { if } b(j)=0 \\ v_{b(j)} q_{j}-\sum_{k=b(j)+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) & \text { otherwise }\end{cases}
$$

Quite interestingly, such a scheme resembles one presented [19. Next lemma shows that $\widetilde{\mathbf{p}}$ is indeed a price vector.

Lemma 4. For each $j \in M$, it holds $\widetilde{p}_{j}>0$.
Proof. Clearly, the claim holds for each $j$ such that $b(j) \in\{0, \beta(\mathbf{X})\}$. For each $j$ such that $0<b(j)<\beta(\mathbf{X})$, it holds

$$
\begin{aligned}
\widetilde{p}_{j} & =v_{b(j)} q_{j}-\sum_{k=b(j)+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& =v_{b(j)}\left(q_{j}-q_{f(b(j)+1)}\right)+\sum_{k=b(j)+1}^{\beta(\mathbf{X})-1}\left(\left(q_{f(k)}-q_{f(k+1)}\right) v_{k}\right)+v_{\beta(\mathbf{X})} q_{f(\beta(\mathbf{X}))} \\
& >0
\end{aligned}
$$

where the inequality holds since $v_{\beta(\mathbf{X})} q_{f(\beta(\mathbf{X}))}>0$ and all the other terms are non-negative since $\mathbf{X}$ is monotone.

We continue by showing the following important property, closely related to the notion of envy-freeness, possessed by the outcome ( $\mathbf{X}, \widetilde{\mathbf{p}}$ ) for each allocation vector $\mathbf{X}$.

Lemma 5. For each allocation vector $\mathbf{X}$, the outcome ( $\mathbf{X}, \widetilde{\mathbf{p}})$ is feasible and, for each winner $i \in W(\mathbf{X})$, $u_{i} \geq \sum_{j \in T} u_{i j}$ for each $T \subseteq M$ of cardinality $d_{i}$. Thus, the allocation is envy-free for the subset of the winners buyers.

Proof. Given an allocation vector $\mathbf{X}$, consider a winner $i \in W(\mathbf{X})$. If $i$ is the only winner in $W(\mathbf{X})$, it immediately follows that $u_{i} \geq \sum_{j \in T} u_{i j}$ for each $T \subseteq M$ of cardinality $d_{i}$ since items not assigned to $i$ have infinite price. We prove this claim for the case in which $|W(\mathbf{X})|>1$ by showing that, for each $j, j^{\prime} \in M$ such that $j \in X_{i}$ and $j^{\prime} \notin X_{i}$, it holds $u_{i j} \geq u_{i j^{\prime}}$.

To this aim, consider an item $j^{\prime}$ such that $0<b\left(j^{\prime}\right)<i=b(j)$ (whenever it exists). It holds

$$
\begin{aligned}
u_{i j}-u_{i j^{\prime}}= & v_{i} q_{j}-\widetilde{p}_{j}-v_{i} q_{j^{\prime}}+\widetilde{p}_{j^{\prime}} \\
= & v_{i} q_{j}-v_{b(j)} q_{j}+\sum_{k=b(j)+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& -v_{i} q_{j^{\prime}}+v_{b\left(j^{\prime}\right)} q_{j^{\prime}}-\sum_{k=b\left(j^{\prime}\right)+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
= & v_{b\left(j^{\prime}\right)} q_{j^{\prime}}-v_{i} q_{j^{\prime}}+\sum_{k=i+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right)-\sum_{k=b\left(j^{\prime}\right)+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
= & \left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}}-\sum_{k=b\left(j^{\prime}\right)+1}^{i}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
\geq & \left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}}-\sum_{k=b\left(j^{\prime}\right)+1}^{i}\left(\left(v_{k-1}-v_{k}\right) q_{j^{\prime}}\right) \\
= & \left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}}-\left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}} \\
= & 0
\end{aligned}
$$

where the second equality comes from $i=b(j)$ and the inequality follows from the monotonicity of $\mathbf{X}$.

Now consider an item $j^{\prime}$ such that $b\left(j^{\prime}\right)>i=b(j)$ (whenever it exists). Similarly as above, it holds

$$
\begin{aligned}
u_{i j}-u_{i j^{\prime}} & =v_{i} q_{j}-\widetilde{p}_{j}-v_{i} q_{j^{\prime}}+\widetilde{p}_{j^{\prime}} \\
& =v_{b\left(j^{\prime}\right)} q_{j^{\prime}}-v_{i} q_{j^{\prime}}+\sum_{k=i+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right)-\sum_{k=b\left(j^{\prime}\right)+1}^{\beta(\mathbf{X})}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& =\left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}}+\sum_{k=i+1}^{b\left(j^{\prime}\right)}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& \geq\left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}}+\sum_{k=i+1}^{b\left(j^{\prime}\right)}\left(\left(v_{k-1}-v_{k}\right) q_{j^{\prime}}\right) \\
& =\left(v_{b\left(j^{\prime}\right)}-v_{i}\right) q_{j^{\prime}}+\left(v_{i}-v_{b\left(j^{\prime}\right)}\right) q_{j^{\prime}} \\
& =0
\end{aligned}
$$

where the inequality follows from the monotonicity of $\mathbf{X}$ and the fact that $q_{j^{\prime}} \leq$ $q_{f\left(b\left(j^{\prime}\right)\right)}$ by the definition of $f_{\mathbf{X}}$.

Finally, for any item $j^{\prime}$ with $b_{j^{\prime}}=0$, for which it holds $\widetilde{p}_{j^{\prime}}=\infty, u_{i j} \geq u_{i j^{\prime}}$ trivially holds.

Thus, in order to conclude the proof, we are just left to show that $u_{i} \geq 0$ for each $i \in W(\mathbf{X})$. To this aim, note that, for each $j^{\prime} \in X_{\beta(\mathbf{X})}$, it holds $u_{\beta(\mathbf{X}) j^{\prime}}=0$ by definition of $\widetilde{\mathbf{p}}$, which yields $u_{\beta(\mathbf{X})}=0$. Let $j^{\prime}$ be any item belonging to $X_{\beta(\mathbf{X})}$. Since, as we have shown, for each buyer $i \in W(\mathbf{X})$ and item $j \in X_{i}$, it holds $u_{i j} \geq u_{i j^{\prime}}$, it follows that $u_{i}=\sum_{j \in X_{i}}\left(v_{i} q_{j}-\widetilde{p}_{j}\right) \geq d_{i}\left(v_{i} q_{j^{\prime}}-\widetilde{p}_{j^{\prime}}\right) \geq$ $d_{i}\left(v_{\beta(\mathbf{X})} q_{j^{\prime}}-\tilde{p}_{j^{\prime}}\right)=d_{i} u_{\beta(\mathbf{X}) j^{\prime}}=0$ and this concludes the proof.

## 4 Results for Generic Instances

In this section, we show that it is hard to approximate the RMPSD to a factor $O\left(m^{1-\epsilon}\right)$ for any $\epsilon>0$, even when considering related valuations, whereas a simple $m$-approximation algorithm can be designed for unrelated valuations.

### 4.1 Inapproximability Result

For an integer $k>0$, we denote with $[k]$ the set $\{1, \ldots, k\}$. Recall that an instance of the Partition problem is made up of $k$ strictly positive numbers $q_{1}, \ldots, q_{k}$ such that $\sum_{i \in[k]} q_{i}=Q$, where $Q>0$ is an even number. It is well-known that deciding whether there exists a subset $J \subset[k]$ such that $\sum_{i \in J} q_{i}=Q / 2$ is an NPcomplete problem. The inapproximability result that we derive in this subsection is obtained through a reduction from a specialization of the Partition problem, that we call Constrained Partition problem, which we define in the following.

An instance of the Constrained Partition problem is made up of an even number $k$ of non-negative numbers $q_{1}, \ldots, q_{k}$ such that $\sum_{i \in[k]} q_{i}=Q$, where $Q$ is
an even number and $\frac{3}{2} \min _{i \in[k]}\left\{q_{i}\right\} \geq \max _{i \in[k]}\left\{q_{i}\right\}$. In this case, we are asked to decide whether there exists a subset $J \subset[k]$, with $|J|=k / 2$, such that $\sum_{i \in J} q_{i}=Q / 2$.

Lemma 6. The Constrained Partition problem is NP-complete.
Proof. Let $I=\left\{q_{1}, \ldots, q_{k}\right\}$ be an instance of the Partition problem and denote with $q_{\text {min }}=\min _{i \in[k]}\left\{q_{i}\right\}$ and $q_{\max }=\max _{i \in[k]}\left\{q_{i}\right\}$. We construct an instance $I^{\prime}=\left\{q_{1}^{\prime}, \ldots, q_{k^{\prime}}^{\prime}\right\}$ of the Constrained Partition problem as follows: set $k^{\prime}=2 k$, then, for each $i \in[k]$, set $q_{i}^{\prime}=q_{i}+2 q_{\max }$, while, for each $k+1 \leq i \leq k^{\prime}$, set $q_{i}^{\prime}=2 q_{\max }$. It is easy to see that, by construction, it holds that $k^{\prime}$ is an even number, $\frac{3}{2} \min _{i \in\left[k^{\prime}\right]}\left\{q_{i}^{\prime}\right\} \geq 3 q_{\text {max }}=\max _{i \in\left[k^{\prime}\right]}\left\{q_{i}^{\prime}\right\}$ and that $\sum_{i \in\left[k^{\prime}\right]} q_{i}^{\prime}=$ $\sum_{i \in[k]} q_{i}+2 k^{\prime} q_{\max }=Q+2 k^{\prime} q_{\max }$ is an even number, so that $I^{\prime}$ is a valid instance of the Constrained Partition problem.

In order to show the claim, we have to prove that there exists a positive answer to $I$ if and only if there exists a positive answer to $I^{\prime}$.

To this aim, let $J \subset[k]$, with $\sum_{i \in J} q_{i}=Q / 2$, be a positive answer to $I$. Let $J^{\prime} \subseteq\left\{k+1, \ldots, k^{\prime}\right\}$, with $\left|J^{\prime}\right|=k-|J|$, be any set of $k-|J|$ numbers of value $2 q_{\max }$. Note that, by the definition of $k^{\prime}$ and the fact that $|J|<k, J^{\prime} \neq \emptyset$. We claim that the set $J \cup J^{\prime}$ is a positive answer to $I^{\prime}$. In fact, it holds $\left|J \cup J^{\prime}\right|=k$ and $\sum_{i \in J \cup J^{\prime}} q_{i}^{\prime}=\sum_{i \in J}\left(q_{i}+2 q_{\max }\right)+2 q_{\max }(k-|J|)=Q / 2+k^{\prime} q_{\max }$.

Now, let $J^{\prime} \subset\left[k^{\prime}\right]$, with $\sum_{i \in J^{\prime}} q_{i}^{\prime}=Q / 2+k^{\prime} q_{\max }$, be a positive answer to $I^{\prime}$. Note that, since $k^{\prime}=2 k$, it holds $\sum_{i=k+1}^{k^{\prime}} q_{i}^{\prime}=k^{\prime} q_{\max }$. Hence, since $Q>0$, there must exist at least one index $i \in J^{\prime}$ such that $i \in[k]$. Let $J=\left\{i \in J^{\prime}: i \in\right.$ $[k]\} \neq \emptyset$ be the set of all such indexes. We claim that $J$ is a positive answer to $I$. In fact, it holds $\sum_{i \in J} q_{i}=\sum_{i \in J^{\prime}} q_{i}^{\prime}-k^{\prime} q_{\max }=Q / 2$.

We can now proceed to show our first inapproximability result, by means of the following reduction. Given an integer $k \geq 3$, consider an instance $I$ of the Constrained Partition problemwith $2(k-1)$ numbers $q_{1}, \ldots, q_{2(k-1)}$ such that $\sum_{i=1}^{2(k-1)} q_{i}=Q$ and define $q_{\text {min }}=\min _{i \in[2(k-1)]}\left\{q_{i}\right\}$. Remember that, by definition, $Q$ is even and it holds $\frac{3}{2} q_{\min } \geq \max _{i \in[2(k-1)]}\left\{q_{i}\right\}$. Note that, this last property, together with $Q \geq 2(k-1) q_{m i n}$, implies that $q_{j} \leq \frac{3 Q}{4(k-1)}<\frac{Q}{2}$ for each $j \in[2(k-1)]$ since $k \geq 3$.

For any $\epsilon>0$, define $\alpha=\left\lceil\frac{2}{\epsilon}\right\rceil+1$ and $\lambda=k^{\alpha}$. Note that, by definition, $\lambda \geq k^{2}$. We create an instance $I^{\prime}$ of the RMPSD as follows. There are $n=5$ buyers and $m=\lambda+k-1$ items divided into four groups: $k$ items of quality $Q$, one item of quality $Q / 2,2(k-1)$ items of qualities $q_{i}$, with $i \in[2(k-1)]$, inherited from $I$, and $\lambda-2 k$ items of quality $\bar{q}:=\frac{q_{\text {min }}}{100}>0$. The five buyers are such that $v_{1}=2$ and $d_{1}=k, v_{2}=1+\frac{1}{\lambda} \frac{Q-2 k \bar{q}+k Q(\lambda+1) / 2}{Q k+Q-2 k \bar{q}+\lambda \bar{q}}$ and $d_{2}=\lambda, v_{3}=1+\frac{1}{\lambda}$ and $d_{3}=k, v_{4}=1+\frac{1}{\lambda} \frac{Q-k \bar{q}}{Q+(\lambda-2 k) \bar{q}}$ and $d_{4}=\lambda-k, v_{5}=1$ and $d_{5}=\lambda-2 k$.

Note that it holds $v_{i}>v_{i+1}$ for each $i \in[4]$. In fact, $v_{4}>1=v_{5}$, since $\lambda>2 k$ and $Q \geq 2(k-1) q_{\min }=200(k-1) \bar{q}>k \bar{q}$ for $k \geq 2$. Moreover, $v_{4}<1+\frac{1}{\lambda}$, since $\lambda>k$ implies $Q-k \bar{q}<Q+(\lambda-2 k) \bar{q}$. Finally, $v_{2}>1+\frac{1}{\lambda}$, since
$\lambda>2=\frac{k Q}{k(Q-Q / 2)}>\frac{k Q}{k Q-2 \bar{q}}$ implies $Q-2 k \bar{q}+\frac{k Q(\lambda+1)}{2}>Q k+Q-2 k \bar{q}+\lambda \bar{q}$ and $v_{2}<2=v_{1}$, since $\lambda>\frac{k}{2}+1$ implies $Q-2 k \bar{q}+\frac{k Q(\lambda+1)}{2}<\lambda(Q k+Q-2 k \bar{q}+\lambda \bar{q})$.

Our aim is to show that, if there exists a positive answer to $I$, then there exists an envy-free outcome for $I^{\prime}$ of revenue at least $(\lambda-2 k) \bar{q}$, while, if a positive answer to $I$ does not exists, then no envy-free outcome of revenue greater than $6(k+3)(k-1) q_{\text {min }}$ can exist for $I^{\prime}$.
Lemma 7. If there exists a positive answer to $I$, then there exists an envy-free outcome for $I^{\prime}$ of revenue greater than $(\lambda-2 k) \bar{q}$.
Proof. Consider the allocation vector $\mathbf{X}$ such that $X_{1}$ is made up of $k$ items of quality $Q, X_{3}$ contains the item of quality $Q / 2$ plus the $k-1$ items forming a positive answer to $I, X_{5}$ is made up of the $\lambda-2 k$ items of quality $\bar{q}$ and $X_{2}=X_{4}=\emptyset$. Note that $\mathbf{X}$ is monotone. We show that the outcome $(\mathbf{X}, \widetilde{\mathbf{p}})$ is envy-free.

According to the price vector $\widetilde{\mathbf{p}}$, it holds $\widetilde{p}_{j}=\frac{(3 \lambda+1) Q-2 \bar{q}}{2 \lambda}$ for each $j \in X_{1}$, $\widetilde{p}_{j}=\frac{(\lambda+1) q_{j}-\bar{q}}{\lambda}$ for each $j \in X_{3}$ and $\widetilde{p}_{j}=\bar{q}$ for each $j \in X_{5}$.

Because of Lemma 5, in order to show that ( $\mathbf{X}, \widetilde{\mathbf{p}})$ is envy-free, we only need to prove that, for each buyer $i \notin W(\mathbf{X})$ and $T \subseteq M$ with $|T|=d_{i}$, it holds $\sum_{j \in T} u_{i j} \leq 0$. Note that the buyers not belonging to $W(\mathbf{X})$ are buyers 2 and 4 .

For buyer 2 , since there are exactly $\lambda$ items having a non-infinite price, it follows that $T=X_{1} \cup X_{3} \cup X_{5}$ is the only set of items of cardinality $d_{2}$ which can give buyer 2 a non-negative utility. It holds

$$
\begin{aligned}
& \sum_{j \in T}\left(v_{2} q_{j}-\widetilde{p}_{j}\right) \\
&=\left(1+\frac{1}{\lambda} \frac{Q-2 k \bar{q}+\frac{k Q}{2}(\lambda+1)}{Q k+Q-2 k \bar{q}+\lambda \bar{q}}\right)(k Q+Q+(\lambda-2 k) \bar{q}) \\
&=-\frac{k((3 \lambda+1) Q-2 \bar{q})}{2 \lambda}-\frac{(\lambda+1) Q-k \bar{q}}{\lambda}-(\lambda-2 k) \bar{q} \\
&=
\end{aligned}
$$

For buyer 4 , for each pair of items $\left(j, j^{\prime}\right)$ with $j \in X_{1}$ and $j^{\prime} \in X_{3}$, it holds $u_{4 j}<u_{4 j^{\prime}}$, while, for each pair of items $\left(j^{\prime}, j^{\prime \prime}\right)$ with $j^{\prime} \in X_{3}$ and $j^{\prime \prime} \in X_{5}$, it holds $u_{4 j^{\prime}}<u_{4 j^{\prime \prime}}$. In fact, we have

$$
\begin{aligned}
u_{4 j^{\prime}}-u_{4 j} & =v_{4} q_{j}-v_{4} Q-q_{j}\left(1+\frac{1}{\lambda}\right)+\frac{Q}{2}\left(3+\frac{1}{\lambda}\right) \\
& >\frac{1}{\lambda}\left(\frac{Q}{2}-q_{j}\right) \\
& \geq 0
\end{aligned}
$$

where the first inequality follows from $1<v_{4}<3 / 2$ and the second one follows from $q_{j} \leq Q / 2$ for each $j \in X_{3}$; and

$$
\begin{aligned}
u_{4 j^{\prime \prime}}-u_{4 j^{\prime}} & =v_{4} \bar{q}-\bar{q}-v_{4} q_{j}+q_{j}+\frac{q_{j}}{\lambda}-\frac{\bar{q}}{\lambda} \\
& =\left(q_{j}-\bar{q}\right)\left(1+\frac{1}{\lambda}-v_{4}\right) \\
& >0
\end{aligned}
$$

where the inequality follows from $v_{4}<1+1 / \lambda$ and $q_{j}>\bar{q}$ for each $j \in X_{3}$.
Hence, the set of items of cardinality $d_{4}$ which gives the highest utility to buyer 4 is $T=X_{3} \cup X_{5}$. It holds

$$
\begin{aligned}
& \sum_{j \in T}\left(v_{4} q_{j}-\widetilde{p}_{j}\right) \\
= & \left(1+\frac{1}{\lambda} \frac{Q-k \bar{q}}{Q+\bar{q}(\lambda-2 k)}\right)(Q+(\lambda-2 k) \bar{q})-\frac{(\lambda+1) Q-k \bar{q}}{\lambda}-(\lambda-2 k) \bar{q} \\
= & 0 .
\end{aligned}
$$

Thus, we can conclude that the outcome $(\mathbf{X}, \widetilde{\mathbf{p}})$ is envy-free and it holds $\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}})>(\lambda-2 k) \bar{q}$.

Now we stress the fact that, in any envy-free outcome $(\mathbf{X}, \mathbf{p})$ for $I^{\prime}$ such that $\operatorname{rev}(\mathbf{X}, \mathbf{p})>0$, it must be $X_{1} \neq \emptyset$. In fact, assume that there exists an envy-free outcome $(\mathbf{X}, \mathbf{p})$ such that $X_{1}=\emptyset$ and $X_{i} \neq \emptyset$ for some $2 \leq i \leq 5$, then, since $d_{1} \leq d_{i}$ and $v_{1}>v_{i}$ for each $2 \leq i \leq 5$, it follows that there exists a subset of $d_{1}$ items $T$ such that $u_{1}>u_{i} \geq 0$, which contradicts the envy-freeness of ( $\mathbf{X}, \mathbf{p}$ ). As a consequence of this fact and of the definition of the demand vector, it follows that each possible envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) for $I^{\prime}$ can only fall into one of the following three cases:

1. $X_{1} \neq \emptyset$ and $X_{i}=\emptyset$ for each $2 \leq i \leq 5$,
2. $X_{1}, X_{3} \neq \emptyset$ and $X_{2}, X_{4}, X_{5}=\emptyset$,
3. $X_{1}, X_{3}, X_{5} \neq \emptyset$ and $X_{2}, X_{4}=\emptyset$.

Note that, for each envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into one of the first two cases, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p}) \leq v_{1} k Q+v_{3} \frac{3}{2} Q \leq Q(2 k+3) \leq(2 k+3) 2(k-1) \frac{3}{2} q_{\text {min }}=$ $6(k+3)(k-1) q_{\text {min }}$. In the remaining of this proof, we will focus only on outcomes falling into case (3).

First of all, we show that, if any such an outcome is envy-free, then the sum of the qualities of the items assigned to buyer 3 cannot exceed $Q$.

Lemma 8. In any envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (3), it holds $\sum_{j \in X_{3}} q_{j} \leq Q$.

Proof. Let ( $\mathbf{X}, \mathbf{p}$ ) be an envy-free outcome falling into case (3) and assume, for the sake of contradiction, that $\sum_{j \in X_{3}}>Q$. Note that, in this case, because of Lemma 1 and the fact that no subset of $k$ items inherited from $I$ can sum a total quality greater than $Q, X_{3}$ must contain the item of quality $Q / 2$ and $X_{1}$ must contain all items of quality $Q$.

By the feasibility of ( $\mathbf{X}, \mathbf{p}$ ), it holds $u_{5} \geq 0$ which implies that there exists one item $j^{\prime} \in X_{5}$ such that $p_{j^{\prime}} \leq q_{j^{\prime}}$. Moreover, by the envy-freeness of $(\mathbf{X}, \mathbf{p})$, for each $j \in X_{3}$, it holds $u_{3 j}=\frac{\overline{\lambda+1}}{\lambda} q_{j}-p_{j} \geq u_{3 j^{\prime}}=\frac{\lambda+1}{\lambda} q_{j^{\prime}}-p_{j^{\prime}} \geq \frac{\lambda+1}{\lambda} q_{j^{\prime}}-q_{j^{\prime}}=\frac{q_{j^{\prime}}}{\lambda}$ which implies $p_{j} \leq \frac{\lambda+1}{\lambda} q_{j}-\frac{q_{j^{\prime}}}{\lambda} \leq \frac{\lambda+1}{\lambda} q_{j}-\frac{\bar{q}}{\lambda}$ for each $j \in X_{3}$. Let $j^{\prime \prime}$ denote the item of quality $Q / 2$. Since $j^{\prime \prime} \in X_{3}$, it follows that $p_{j^{\prime \prime}} \leq \frac{\lambda+1}{\lambda} \frac{Q}{2}-\frac{\bar{q}}{\lambda}$. Again, by the envy-freeness of $(\mathbf{X}, \mathbf{p})$, for each $j \in X_{1}$, it holds $u_{1 j}=2 Q-p_{j} \geq u_{1 j^{\prime \prime}}=$ $Q-p_{j^{\prime \prime}} \geq Q-\frac{\lambda+1}{\lambda} \frac{Q}{2}+\frac{\bar{q}}{\lambda}$ which implies $p_{j} \leq \frac{3 Q \lambda+Q-2 \bar{q}}{2 \lambda}$.

Define $T=X_{1} \cup X_{3} \cup X_{5}$ and let us compute the utility that buyer 2 achieves if she is assigned set $T$ such that $|T|=\lambda=d_{2}$. It holds

$$
\begin{aligned}
u_{2} & =\sum_{j \in T}\left(v_{2} q_{j}-p_{j}\right) \\
& =v_{2} \sum_{j \in X_{5}} q_{j}-\sum_{j \in X_{5}} p_{j}+v_{2} \sum_{j \in X_{3}} q_{j}-\sum_{j \in X_{3}} p_{j}+v_{2} \sum_{j \in X_{1}} q_{j}-\sum_{j \in X_{1}} p_{j} \\
& \geq\left(\frac{1}{\lambda} \frac{Q-2 k \bar{q}+\frac{k Q}{2}(\lambda+1)}{Q+(\lambda-2 k) \bar{q}}\right) \sum_{j \in X_{5}} q_{j}+\left(v_{2}-v_{3}\right) \sum_{j \in X_{3}} q_{j}+\frac{k \bar{q}}{\lambda}+k\left(v_{2} Q-\frac{3 Q \lambda+Q-2 \bar{q}}{2 \lambda}\right) \\
& >\frac{(\lambda-2 k)\left(Q-2 k \bar{q}+\frac{k Q}{2}(\lambda+1)\right)}{\lambda(Q+(\lambda-2 k) \bar{q})}+\frac{(Q k(\lambda-1)-2 \lambda \bar{q}) Q}{2 \lambda(Q(k+1)+(\lambda-2 k) \bar{q})}+\frac{k \bar{q}}{\lambda}+k\left(v_{2} Q-\frac{3 Q \lambda+Q-2 \bar{q}}{2 \lambda}\right) \\
& =0,
\end{aligned}
$$

where the first inequality comes from the fact that, for each $j \in X_{1}$, it holds $q_{j}=Q$ and $p_{j} \leq \frac{3 Q \lambda+Q-2 \bar{q}}{2 \lambda}$, the fact that $u_{5} \geq 0$ implies $\sum_{j \in X_{5}} q_{j} \geq \sum_{j \in X_{5}} p_{j}$ and the fact that $p_{j}<\frac{\lambda+1}{\lambda} q_{j}-\frac{\bar{q}}{\lambda}$ for each $j \in X_{3}$, while the second inequality comes from the fact that $\sum_{j \in X_{5}} q_{j} \geq(\lambda-2 k) \bar{q}$ and $\sum_{j \in X_{3}} q_{j}>Q$.

Hence, since there exists a subset of $d_{2}$ items for which buyer 2 gets a strictly positive utility and buyer 2 is not a winner in $\mathbf{X}$, it follows that the outcome $(\mathbf{X}, \mathbf{p})$ cannot be envy-free, a contradiction.

On the other hand, we also show that, for any envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (3), the sum of the qualities of the items assigned to buyer 3 cannot be smaller than $Q$.

Lemma 9. In any envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (3), it holds $\sum_{j \in X_{3}} q_{j} \geq Q$.

Proof. Let (X, p) be an envy-free outcome falling into case (3) and assume, for the sake of contradiction, that $\sum_{j \in X_{3}}<Q$.

By the feasibility of $(\mathbf{X}, \mathbf{p})$, it holds $u_{5} \geq 0$ which implies that there exists one item $j^{\prime} \in X_{5}$ such that $p_{j^{\prime}} \leq q_{j^{\prime}}$. Moreover, by the envy-freeness of $(\mathbf{X}, \mathbf{p})$, for each $j \in X_{3}$, it holds $u_{3 j}=\frac{\overline{\lambda+1}}{\lambda} q_{j}-p_{j} \geq u_{3 j^{\prime}}=\frac{\lambda+1}{\lambda} q_{j^{\prime}}-p_{j^{\prime}} \geq \frac{\lambda+1}{\lambda} q_{j^{\prime}}-q_{j^{\prime}}=\frac{q_{j^{\prime}}}{\lambda}$ which implies $p_{j} \leq \frac{\lambda+1}{\lambda} q_{j}-\frac{\hat{q}_{j^{\prime}}}{\lambda} \leq \frac{\lambda+1}{\lambda} q_{j}-\frac{\bar{q}}{\lambda}$ for each $j \in X_{3}$.

Define $T=X_{3} \cup X_{5}$ and let us compute the utility that buyer 4 achieves if she is assigned set $T$ such that $|T|=\lambda-k=d_{4}$. It holds

$$
\begin{aligned}
u_{4} & =\sum_{j \in T}\left(v_{4} q_{j}-p_{j}\right) \\
& =v_{4} \sum_{j \in X_{5}} q_{j}-\sum_{j \in X_{5}} p_{j}+v_{4} \sum_{j \in X_{3}} q_{j}-\sum_{j \in X_{3}} p_{j} \\
& \geq\left(\frac{1}{\lambda} \frac{Q-k \bar{q}}{Q+(\lambda-2 k) \bar{q}}\right) \sum_{j \in X_{5}} q_{j}+\left(v_{4}-v_{3}\right) \sum_{j \in X_{3}} q_{j}+\frac{k \bar{q}}{\lambda} \\
& >\frac{(\lambda-2 k)(Q-k \bar{q}) \bar{q}}{\lambda(Q+(\lambda-2 k)) \bar{q}}-\frac{Q(\lambda-k) \bar{q}}{\lambda(Q+(\lambda-2 k) \bar{q})}+\frac{k \bar{q}}{\lambda} \\
& =0
\end{aligned}
$$

where the first inequality comes from the fact that $u_{5} \geq 0$ implies $\sum_{j \in X_{5}} q_{j} \geq$ $\sum_{j \in X_{5}} p_{j}$ and the fact that $p_{j} \leq \frac{\lambda+1}{\lambda} q_{j}-\frac{\bar{q}}{\lambda}$ for each $j \in X_{3}$, while the second inequality comes from the fact that $\sum_{j \in X_{5}} q_{j} \geq(\lambda-2 k) \bar{q}$ and $\sum_{j \in X_{3}} q_{j}<Q$.

Hence, since there exists a subset of $d_{4}$ items for which buyer 4 gets a strictly positive utility and buyer 4 is not a winner in $\mathbf{X}$, it follows that the outcome $(\mathbf{X}, \mathbf{p})$ cannot be envy-free, a contradiction.

As a consequence of Lemmas 8 and 9, it follows that there exists an envyfree outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (3) only if $\sum_{j \in X_{3}} q_{j}=Q$. Since, as we have already observed, in such a case the item of quality $Q / 2$ has to belong to $X_{3}$, it follows that there exists an envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (3) only if there are $k-1$ items inherited from $I$ whose sum is exactly $Q / 2$, that is, only if $I$ admits a positive solution.

Any envy-free outcome not falling into case (3) can raise a revenue of at most $6(k+3)(k-1) q_{\text {min }}$. Hence, if there exists a positive answer to $I$, then, by Lemma 7 there exists a solution to $I^{\prime}$ of revenue greater than $(\lambda-2 k) \bar{q}$, while, if there is no positive answer to $I$, then there exists no solution to $I^{\prime}$ of revenue more than $6(k+3)(k-1) q_{\text {min }}$.

Thus, if there exists an $r$-approximation algorithm for the RMPSD with $r \leq \frac{(\lambda-2 k) q_{\text {min }}}{600(k+3)(k-1) q_{\text {min }}}$, it is then possible to decide in polynomial time the Constrained Partition problem, thus implying $P=N P$. Since, by the definition of $\alpha, \frac{\lambda-2 k}{600(k+3)(k-1)}=O\left(k^{\alpha-2}\right)=O\left(m^{1-2 / \alpha}\right)$ and $m^{1-\epsilon}<m^{1-2 / \alpha}$, the following theorem holds.

Theorem 1. For any $\epsilon>0$, the RMPSD cannot be approximated to a factor $O\left(m^{1-\epsilon}\right)$ unless $\mathrm{P}=\mathrm{NP}$.

We stress that this inapproximability result heavily relies on the presence of two useless buyers, namely buyers 2 and 4 , who cannot be winners in any envyfree solution. This situation suggests that better approximation guarantees may be possible for proper instances, as we will show in the next section.

### 4.2 The Approximation Algorithm

In this subsection, we design a simple $m$-approximation algorithm for the generalization of the RMPSD in which the buyers have unrelated valuations. The inapproximability result given in Theorem 1 shows that, asymptotically speaking, this is the best approximation one can hope for unless $P=N P$.

For each $i \in N$, let $T_{i}=\operatorname{argmax}_{T \subseteq M:|T|=d_{i}}\left\{\sum_{j \in T} v_{i j}\right\}$ be the set of the $d_{i}$ best items for buyer $i$ and define $R_{i}=\left(\sum_{j \in T_{i}} v_{i j}\right) / d_{i}$. Let $i^{*}$ be the index of the buyer with the highest value $R_{i}$. Consider the algorithm best which returns the outcome $(\overline{\mathbf{X}}, \overline{\mathbf{p}})$ such that $\bar{X}_{i^{*}}=T_{i^{*}}, \bar{X}_{i}=\emptyset$ for each $i \neq i^{*}, \bar{p}_{j}=R_{i^{*}}$ for each $j \in T_{i^{*}}$ and $\bar{p}_{j}=\infty$ for each $j \notin T_{i^{*}}$. It is easy to see that the computational complexity of Algorithm best is $O(n m)$.

Theorem 2. Algorithm best returns an m-approximate solution for the RMPSD with unrelated valuations.
Proof. It is easy to see that the outcome ( $\overline{\mathbf{X}}, \overline{\mathbf{p}}$ ) returned by Algorithm best is feasible.In order to prove that it is also envy-free, we just need to show that, for each buyer $i \neq i^{*}$ with $d_{i} \leq d_{i^{*}}$ and each $T_{i} \subseteq T_{i^{*}}$ of cardinality $d_{i}$, it holds $\sum_{j \in T_{i}}\left(v_{i j}-p_{j}\right) \leq 0$. Assume, for the sake of contradiction, that there exists a set $T_{i}$ of cardinality $d_{i}$ such that $\sum_{j \in T_{i}}\left(v_{i j}-\bar{p}_{j}\right)>0$.

We obtain $0<\sum_{j \in T_{i}}\left(v_{i j}-\bar{p}_{j}\right)=\sum_{j \in T_{i}} v_{i j}-d_{i} R_{i^{*}} \leq d_{i} R_{i}-d_{i} R_{i^{*}}=$ $d_{i}\left(R_{i}-R_{i^{*}}\right)$ which implies $R_{i}>R_{i^{*}}$, a contradiction. Hence, $(\overline{\mathbf{X}}, \overline{\mathbf{p}})$ is envy-free.

As to the approximation guarantee, note that $\operatorname{rev}(\overline{\mathbf{X}}, \overline{\mathbf{p}})=d_{i^{*}} R_{i^{*}} \geq R_{i^{*}}$. The maximum possible revenue achievable by any outcome $(\mathbf{X}, \mathbf{p})$, not even an envy-free one, is at most $\sum_{i \in N} \sum_{j \in X_{i}} v_{i j} \leq \sum_{i \in W(\mathbf{X})}\left(d_{i} R_{i}\right) \leq m R_{i^{*}}$, which yields the claim.

## 5 Results for Proper Instances

Given a proper instance $I=(\mathbf{V}, \mathbf{D}, \mathbf{Q})$, denote with $\delta$ the number of different values in $\mathbf{V}$ and, for each $k \in[\delta]$, let $A_{k} \subseteq N$ denote the set of buyers with the $k$ th highest value and $v\left(A_{k}\right)$ denote the value of all buyers in $A_{k}$. For $k \in[\delta]$, define $A_{\leq k}=\bigcup_{h=1}^{k} A_{h}, A_{\geq k}=\bigcup_{h=k}^{\delta} A_{h}, A_{>k}=A_{\geq k} \backslash A_{k}$ and $A_{<k}=A_{\leq k} \backslash A_{k}$, while, for each subset of buyers $A \subseteq N$, define $d(A)=\sum_{i \in A} d_{i}$. Let $\delta^{*} \in[\delta]$ be the minimum index such that $d\left(A_{\leq \delta^{*}}\right)>m$ and let $\widetilde{A} \subset A_{\delta^{*}}$ be a subset of buyers in $A_{\delta^{*}}$ such that

$$
\widetilde{A}=\operatorname{argmax}_{A \subset A_{\delta^{*}}: d(A)+d\left(A_{<\delta^{*}}\right) \leq m}\{d(A)\}
$$

In other words $\widetilde{A}$ is the subset of buyers in $A_{\delta^{*}}$ that feasibly extends $A_{<\delta^{*}}$ (i.e., such that the sum of the requested items of buyers in $A_{<\delta^{*}} \cup \widetilde{A}$ is at most $m$ ) and maximizes the number of allocated items.

Note that any instance $I$ for which $\delta^{*}$ does not exist can be suitably extended with a dummy buyer $n+1$, such that $v_{n+1}<v_{n}$ and $d_{n+1}=m+1$, which is equivalent in the sense that it does not change the set of envy-free outcomes of $I$. Hence, in this section, we will always assume that $\delta^{*}$ is well-defined for each proper instance of the RMPSD.

For our purposes we need to break ties among values of the buyers in $A_{\delta^{*}}$ in such a way that each buyer in $\widetilde{A}$ comes before any buyer in $A_{\delta^{*}} \backslash \widetilde{A}$. In order to achieve this task, we need to explicitly compute the set of buyers $\widetilde{A}$. Such a computation can be done by reducing this problem to the knapsack problem. It is easy to see that, in this case, the well-known pseudo-polynomial time algorithm for knapsack is polynomial in the dimensions of $I$, as $d_{i} \leq m$ for every $i \in N$.

Because of the above discussion, from now on we can assume that ties among values of the buyers in $A_{\delta^{*}}$ are broken in such a way that each buyer in $\widetilde{A}$ comes before any buyer in $A_{\delta^{*}} \backslash \widetilde{A}$. For each $k \in\left[\delta^{*}\right]$, define

$$
\alpha(k)= \begin{cases}\max \left\{i \in A_{k}\right\} & \text { if } k \in\left[\delta^{*}-1\right], \\ \max \{i \in \widetilde{A}\} & \text { if } k=\delta^{*}\end{cases}
$$

By the definition of $\delta^{*}$ and $\widetilde{A}$ and by the tie breaking rule imposed on the buyers in $A_{\delta^{*}}$, it follows that $\sum_{i=1}^{\alpha(k)} d_{i} \leq m$ for each $k \in\left[\delta^{*}\right]$.

We say that an allocation vector $\mathbf{X}$ is an $h$-prefix of $I$, with $h \in\left[\alpha\left(\delta^{*}\right)\right]$, if $\mathbf{X}$ is monotone and $i \in W(\mathbf{X})$ if and only if $i \in[h]$.

### 5.1 Computing an $h$-Prefix of $I$ of Maximum Revenue

Let $\mathbf{X}$ be an $h$-prefix of $I$. We show that $(\mathbf{X}, \widetilde{\mathbf{p}})$ is an envy-free outcome.
Lemma 10. The outcome ( $\mathbf{X}, \widetilde{\mathbf{p}}$ ) is envy-free.
Proof. Since $\mathbf{X}$ is monotone, by exploiting Lemma 5, we only need to prove that for each buyer $i \notin W(\mathbf{X})$ and set $T \subseteq M$ of cardinality $d_{i}$, it holds $\sum_{j \in T} u_{i j} \leq 0$. Note that $i \notin W(\mathbf{X})$ if and only if $i>h$.

For each $i>h$, it holds $v_{i} \leq v_{h}$. Moreover, for each $j$ such that $b(j)=h$, it holds $u_{h j}=0$. Since, because of Lemma 5 $u_{h j} \geq u_{h j^{\prime}}$ for any item $j^{\prime} \in M(\mathbf{X})$, it follows that $u_{h j^{\prime}}=v_{h} q_{j^{\prime}}-\widetilde{p}_{j^{\prime}} \leq 0$ for each $j^{\prime} \in M(\mathbf{X})$. Hence, for each $j^{\prime} \in M(\mathbf{X})$, it holds $u_{i j^{\prime}}=v_{i} q_{j^{\prime}}-\widetilde{p}_{j^{\prime}} \leq v_{h} q_{j^{\prime}}-\widetilde{p}_{j^{\prime}} \leq 0$ and this concludes the proof.

Given an allocation vector $\mathbf{X}$, for each $i \in[\delta]$, denote with $M_{i}(\mathbf{X})=\{j \in$ $\left.M(\mathbf{X}): v_{b(j)}=v\left(A_{i}\right)\right\}$ the set of items allocated to the buyers with the $i$ th highest value in $\mathbf{V}$. Recall that, since $\mathbf{X}$ is an $h$-prefix of $I$, it holds $\beta(\mathbf{X})=h$. The following lemma gives a lower bound on the revenue generated by the outcome (X, $\widetilde{\mathbf{p}}$ ).
$\operatorname{Lemma}$ 11. $\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}}) \geq v_{h} \sum_{j \in M_{h}(\mathbf{X})} q_{j}$.
Proof. By the definition of $\widetilde{\mathbf{p}}$, it follows that $\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}}) \geq \sum_{j \in M_{h}(\mathbf{X})} \widetilde{p}_{j}=$ $v_{h} \sum_{j \in M_{h}(\mathbf{X})} q_{j}$.

We now prove a very important result stating that the price vector $\widetilde{\mathbf{p}}$ is the best one can hope for when overpricing is not allowed. Such a result, of independent interest, plays a crucial role in the proof of the approximation guarantee of the algorithm we define in this section.

Lemma 12. Let $\mathbf{X}$ be an h-prefix of $I$. Then $(\mathbf{X}, \widetilde{\mathbf{p}})$ is an optimal envy-free outcome when overpricing is not allowed.

Proof. It is easy to see that the price vector $\widetilde{\mathbf{p}}$ does not overprice any item in $M(\mathbf{X})$. For any envy-free outcome $(\mathbf{X}, \mathbf{p})$, we show by backward induction that $p_{j} \leq \widetilde{p}_{j}$ for each $j \in M(\mathbf{X})$.

As a base case, for all $j \in M_{h}(\mathbf{X})$, it holds $p_{j} \leq v_{h} q_{j}=\widetilde{p}_{j}$ because $\mathbf{p}$ cannot overprice any item.

For the inductive step, consider an item $j$ such that $b(j)=i<h$ and assume the claim true for each item $j^{\prime}$ such that $b\left(j^{\prime}\right)>i$. By the envy-freeness of $(\mathbf{X}, \mathbf{p})$,
it holds $u_{i j}-u_{i j^{\prime}} \geq 0$ for $j^{\prime}=f(i+1)$. This implies

$$
\begin{aligned}
0 & \leq u_{i j}-u_{i j^{\prime}} \\
& =v_{i} q_{j}-p_{j}-v_{i} q_{j^{\prime}}+p_{j^{\prime}} \\
& \leq v_{i} q_{j}-p_{j}-v_{i} q_{j^{\prime}}+\widetilde{p}_{j^{\prime}},
\end{aligned}
$$

where the last inequality comes from the inductive hypothesis. Hence, we can conclude that

$$
\begin{aligned}
p_{j} & \leq v_{i}\left(q_{j}-q_{j^{\prime}}\right)+\widetilde{p}_{j^{\prime}} \\
& =v_{i}\left(q_{j}-q_{j^{\prime}}\right)+v_{b\left(j^{\prime}\right)} q_{j^{\prime}}-\sum_{k=b\left(j^{\prime}\right)+1}^{h}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& =v_{i}\left(q_{j}-q_{f(i+1)}\right)+v_{i+1} q_{f(i+1)}-\sum_{k=i+2}^{h}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& =v_{i} q_{j}-\sum_{k=i+1}^{h}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right) \\
& =\widetilde{p}_{j}
\end{aligned}
$$

where the second equality comes from $j^{\prime}=f(i+1)$ and $b\left(j^{\prime}\right)=i+1$. This completes the induction and shows the claim.

We design a polynomial time algorithm ComputePrefix which, given a proper instance $I$ and a value $h \in\left[\alpha\left(\delta^{*}\right)\right]$, outputs the $h$-prefix $\mathbf{X}_{h}^{*}$ such that the outcome $\left(\mathbf{X}_{h}^{*}, \widetilde{\mathbf{p}}\right)$ achieves the highest revenue among all possible $h$-prefixes of $I$.

Recall that, by definition of $h$-prefixes of $I$, the set of buyers whose demand is to be satisfied is exactly characterized. Moreover, once fixed a set of items which exactly satisfies the demands of the considered buyers, by the monotonicity of $h$-prefixes of $I$, we know exactly which items must be assigned to each buyer. Hence, in this setting, our task becomes that of determining the set of items maximizing the value $\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}})$.

To this aim, we first show that this problem reduces to that of determining, for each $i \in[h]$, the item $f(i)$. In fact, it holds

$$
\begin{aligned}
\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}}) & =\sum_{j \in M(\mathbf{X})} \widetilde{p}_{j} \\
& =\sum_{i \in[h]} \sum_{j \in X_{i}}\left(v_{i} q_{j}-\sum_{k=i+1}^{h}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right)\right) \\
& =\underbrace{\sum_{i \in[h]}\left(v_{i} \sum_{j \in X_{i}} q_{j}\right)-\sum_{i \in[h]}\left(d_{i} \sum_{k=i+1}^{h}\left(\left(v_{k-1}-v_{k}\right) q_{f(k)}\right)\right)}_{T_{1}} \\
& =\underbrace{\sum_{i \in[h]}\left(v_{i} \sum_{j \in X_{i}} q_{j}\right)}_{T_{2}}-\sum_{i=2}^{\left.\sum_{i=1}^{h}\left(\left(v_{i-1}-v_{i}\right) q_{f(i)}\right) \sum_{k=1}^{i-1} d_{k}\right)}
\end{aligned}
$$

Note that only those items $j$ such that $j=f(i)$ for some $i \in[h]$ contribute to the term $T_{2}$ and that the per quality contribution of each item to the term $T_{1}$ is always strictly positive. This implies that, once fixed all the items $j$ such that $j=f(i)$ for each $i \in[h]$, the remaining $d_{i}-1$ items to be assigned to buyer $i$ in each optimal outcome are exactly the items $j+1, \ldots, j+d_{i}-1$.

Because of the above discussion, we are now allowed to concentrate only on the problem of determining the set of best-quality items assigned to each buyer in [ $h$ ] in an optimal envy-free outcome. Let us denote with $r_{i j}$ the maximum revenue which can be achieved by an envy-free outcome in which the best-quality item of the first $i$ buyers have been chosen among the first $j$ ones. Hence, $r_{i j}$ is defined for $0 \leq i \leq h$ and $\sum_{k=1}^{i-1} d_{k}+1 \leq j \leq m+1-\sum_{k=i}^{h} d_{k}$ and has the following expression:

$$
r_{i j}= \begin{cases}0 & \text { if } i=0 \\ t_{i} q_{j}+\sum_{k=j+1}^{j+d_{i}-1} v_{i} q_{k} & \text { if } i>0 \wedge j=\sum_{k=1}^{i-1} d_{k}+1, \\ \max \left\{r_{i-1, j-1}+t_{i} q_{j} ; r_{i, j-1}\right\}+\sum_{k=j+1}^{j+d_{i}-1} v_{i} q_{k} & \text { if } i>0 \wedge j>\sum_{k=1}^{i-1} d_{k}+1\end{cases}
$$

where $t_{i}=v_{i}-\left(v_{i-1}-v_{i}\right) \sum_{k=1}^{i-1} d_{k}$ is the contribution that item $f(i)$ gives to the revenue per each unit of quality. Clearly, by definition, $r_{h, m+1-d_{h}}$ gives the maximum revenue which can be achieved by an envy-free outcome ( $\mathbf{X}, \widetilde{\mathbf{p}}$ ) such that $W(\mathbf{X})=[h]$. Such a quantity, as well as the allocation vector $\mathbf{X}_{h}^{*}$ realizing it, can be computed by the following dynamic programming algorithm of complexity $O(m h)$.

ComputePrefix(input: instance $I$, integer $h$, output: allocation vector $\mathbf{X}_{h}^{*}$ ): for each $i=0, \ldots, h$ do $r_{i j}:=0$;
for each $i=1, \ldots, h$ do

```
| \(\quad r_{i j}:=t_{i} \cdot q_{j}\) where \(j=\sum_{k=1}^{i-1} d_{k}+1 ;\)
| \(\quad f_{i}:=j\);
for each \(i=1, \ldots, h\) do
| for each \(j=\sum_{k=1}^{i-1} d_{k}+2, \ldots, m+1-\sum_{k=i}^{h} d_{k}\) do
        if \(r_{i, j-1} \geq r_{i-1, j-1}+t_{i} \cdot q_{j}\) then;
        \(r_{i j}:=r_{i, j-1} ;\)
        else
        \(r_{i, j}:=r_{i-1, j-1}+t_{i} \cdot q_{j} ;\)
        \(f_{i}:=j ;\)
for each \(i=1, \ldots, h\) do
\(\mid \quad X_{i}:=\left\{f_{i}, f_{i}+1, \ldots, f_{i}+d_{i}-1\right\} ;\)
return \(\mathbf{X}_{h}^{*}=\left(X_{1}, \ldots, X_{h}\right)\);
```

Let $\mathcal{X}(h)$ be the set of all possible $h$-prefixes of $I$. As a consequence of the analysis carried out in this subsection, we can claim the following result.

Lemma 13. For each $h \in\left[\alpha\left(\delta^{*}\right)\right]$, the h-prefix of $I \mathbf{X}_{h}^{*}$ such that $\operatorname{rev}\left(\mathbf{X}_{h}^{*}, \widetilde{\mathbf{p}}\right)=$ $\max _{\mathbf{X} \in \mathcal{X}(h)}\{\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}})\}$ can be computed in time $O(m h)$.

### 5.2 The Approximation Algorithm

Our approximation algorithm Prefix for proper instances generates a set of prefixes of $I$ for which it computes the allocation of items yielding maximum revenue by exploiting the algorithm ComputePrefix as a subroutine. Then, it returns the solution with the highest revenue among them.

```
Prefix(input: instance \(I\), output: allocation vector \(\mathbf{X}^{*}\) ):
opt \(:=\emptyset ;\) value \(:=-1\);
compute \(\widetilde{A}\);
reorder the buyers in such a way that each \(i \in \widetilde{A}\) comes before any \(i^{\prime} \in A_{\delta^{*}} \backslash \widetilde{A}\);
for each \(h=1, \ldots, \alpha\left(\delta^{*}\right)\) do
\(\mid \quad \mathbf{X}_{h}^{*}:=\) ComputePrefix \((I, h)\);
| if \(\operatorname{rev}\left(\mathbf{X}_{h}^{*}, \widetilde{\mathbf{p}}\right)>\) value then
\(\mid \quad\) opt \(:=\mathbf{X}_{h}^{*} ;\) value \(:=\operatorname{rev}\left(\mathbf{X}_{h}^{*}, \widetilde{\mathbf{p}}\right)\);
for each \(k=0, \ldots, \delta^{*}-1\) do
    for each \(i \in A_{k+1}\) do
        reorder the buyers in \(A_{k+1}\) in such a way that \(i\) is the first buyer in \(A_{k+1}\);
        if \(d\left(A_{\leq k}\right)+d_{i} \leq m\) then \(\mathbf{X}_{k}^{*}:=\) ComputePrefix \(\left(I,\left|A_{\leq k}\right|+1\right)\);
        if \(\operatorname{rev}\left(\mathbf{X}_{k}^{*}, \widetilde{\mathbf{p}}\right)>\) value then
        opt \(:=\mathbf{X}_{k}^{*} ;\) value \(:=\operatorname{rev}\left(\mathbf{X}_{k}^{*}, \widetilde{\mathbf{p}}\right) ;\)
return opt;
```

It is easy to see that the computational complexity of Algorithm Prefix is $O\left(n^{3} m\right)$. As a major positive contribution of this work, we show that it approximates the RMPSD to a factor 2 on proper instance.

Theorem 3. The approximation ratio of Algorithm Prefix is 2 when applied to proper instances.

Proof. Let $I$ be a proper instance and let ( $\mathbf{X}, \mathbf{p}$ ) be its optimal envy-free outcome. We denote with rev(Prefix) the revenue of the outcome returned by Algorithm Prefix. The proof is divided into two cases:

Case (1): $\mathbf{X}$ is an $h$-prefix of $I$ for some $h \in\left[\alpha\left(\delta^{*}\right)\right]$.
Since $\mathbf{X}$ is an $h$-prefix of $I$, the outcome $\left(\mathbf{X}_{h}^{*}, \widetilde{\mathbf{p}}\right)$ has to be considered by algorithm Prefix as a candidate solution. It follows that $\operatorname{rev}(\operatorname{Prefix}) \geq \operatorname{rev}\left(\mathbf{X}_{h}^{*}, \widetilde{\mathbf{p}}\right) \geq$ $v_{h} \sum_{j \in M_{h}\left(\mathbf{X}_{h}^{*}\right)} q_{j}$ by the definition of algorithm Prefix and by Lemma 11 ,

Now, if $\sum_{j \in M_{h}(\mathbf{X})} p_{j} \geq \frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p})$, the claim directly follows since, by the feasibility of $(\mathbf{X}, \mathbf{p})$, it holds $\sum_{j \in M_{h}(\mathbf{X})} p_{j} \leq v_{h} \sum_{j \in M_{h}(\mathbf{X})} q_{j} \leq \operatorname{rev}($ Prefix $)$. Hence, assume that $\sum_{j \in M_{h}(\mathbf{X})} p_{j}<\frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p})$.

Define $i^{\prime}=\max \left\{i \in N: v_{i}>v_{h}\right\}$ (note that $i^{\prime}$ is well-defined because of the assumption) and $\mathbf{X}^{\prime}$ as the $i^{\prime}$-prefix of $I$ such that $X_{i}^{\prime}=X_{i}$ for each $i \in\left[i^{\prime}\right]$. By Lemma 2 it follows that $\left(\mathbf{X}^{\prime}, \mathbf{p}\right)$ is an outcome without overpricing. Because of our assumption, it holds $\operatorname{rev}\left(\mathbf{X}^{\prime}, \mathbf{p}\right)>\frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p})$ and, by Lemma 12 , it also holds $\operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right) \geq \operatorname{rev}\left(\mathbf{X}^{\prime}, \mathbf{p}\right)$. Moreover, since $\mathbf{X}^{\prime}$ is an $i^{\prime}$-prefix of $I$, by the definition of algorithm Prefix and by Lemma 13, it holds rev(Prefix) $\geq$ $\operatorname{rev}\left(\mathbf{X}_{i^{\prime}}^{*}, \widetilde{\mathbf{p}}\right) \geq \operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right)$ which yields the claim.

Case (2): $\mathbf{X}$ is not an $h$-prefix of $I$ for any $h \in\left[\alpha\left(\delta^{*}\right)\right]$.
Let $i^{*}=\min \{i \in N: i \notin W(\mathbf{X})\}$. Since $\mathbf{X}$ is not an $h$-prefix of $I$ for any $h \in\left[\alpha\left(\delta^{*}\right)\right]$, it follows that $\beta(\mathbf{X})>i^{*}$.

Assume that $\sum_{i=1}^{i^{*}-1} \sum_{j \in X_{i}} p_{j} \geq \frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p})$ and define $\mathbf{X}^{\prime}$ as the $\left(i^{*}-1\right)$ prefix of $I$ such that $X_{i}^{\prime}=X_{i}$ for each $i \in\left[i^{*}-1\right]$ (note that our assumption implies that $\left(i^{*}-1\right)$-prefixes of $I$ do exist). By Lemma 2, it follows that $\left(\mathbf{X}^{\prime}, \mathbf{p}\right)$ is an outcome without overpricing. Because of our assumption, it holds $\operatorname{rev}\left(\mathbf{X}^{\prime}, \mathbf{p}\right)=\sum_{i=1}^{i^{*}-1} \sum_{j \in X_{i}} p_{j} \geq \frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p})$ and, by Lemma 12, it also holds $\operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right) \geq \operatorname{rev}\left(\mathbf{X}^{\prime}, \mathbf{p}\right)$. Moreover, since $\mathbf{X}^{\prime}$ is an $\left(i^{*}-1\right)$-prefix of $I$, by the definition of algorithm Prefix and by Lemma 13, it holds $\operatorname{rev}(\operatorname{Prefix}) \geq \operatorname{rev}\left(\mathbf{X}_{i^{*}-1}^{*}, \widetilde{\mathbf{p}}\right) \geq$ $\operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right)$ which yields the claim.

Hence, from now on, we assume that $\sum_{i=1}^{i^{*}-1} \sum_{j \in X_{i}} p_{j}<\frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p})$.
If there does not exist an $i^{*}$-prefix of $I$, then, $\sum_{i>i^{*}: i \in W(\mathbf{X})} d_{i}<d_{i^{*}}$. Assume that there exists a buyer $i^{\prime} \in W(\mathbf{X})$ such that $i^{\prime}<i^{*}$ and $d_{i^{\prime}}>d_{i^{*}}$. Clearly, $i^{\prime}$-prefixes of $I$ do exist. Define $\mathbf{X}^{\prime}$ as the $i^{\prime}$-prefix of $I$ such that $X_{i}^{\prime}=X_{i}$ for each $i \in\left[i^{\prime}\right]$. By the definition of algorithm Prefix and by Lemmas 13 and 11 it holds $\operatorname{rev}($ Prefix $) \geq \operatorname{rev}\left(\mathbf{X}_{i^{\prime}}^{*}, \widetilde{\mathbf{p}}\right) \geq \operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right) \geq v_{i^{\prime}} \sum_{j \in X_{i^{\prime}}^{\prime}} q_{j}$. On the other hand, it holds $\sum_{i>i^{*}} \sum_{j \in X_{i}} q_{j}<d_{i^{*}} q_{\max }$, where $q_{\max }=\max \left\{q_{j}: j \in \bigcup_{i>i^{*}} X_{i}\right\}$. Moreover, $d_{i^{*}} q_{\max }<d_{i^{\prime}} \sum_{j \in X_{i^{\prime}}^{\prime}} q_{j}$ since $d_{i^{\prime}}>d_{i^{*}}$ and $\mathbf{X}$ is monotone. Hence, we
have

$$
\begin{aligned}
\frac{1}{2} r e v(\mathbf{X}, \mathbf{p}) & <\sum_{i>i^{*}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} p_{j} \\
& \leq \sum_{i>i^{*}: i \in W(\mathbf{X})}\left(v_{i} \sum_{j \in X_{i}} q_{j}\right) \\
& \leq \sum_{i>i^{*}: i \in W(\mathbf{X})}\left(v_{i^{*}} \sum_{j \in X_{i}} q_{j}\right) \\
& <v_{i^{*}} d_{i^{*}} q_{\max } \\
& <v_{i^{\prime}} d_{i^{\prime}} q_{\max } \\
& \leq v_{i^{\prime}} \sum_{j \in X_{i^{\prime}}} q_{j}
\end{aligned}
$$

which yields the claim.
Assume that there does not exist any buyer $i^{\prime} \in W(\mathbf{X})$ such that $i^{\prime}<i^{*}$ and $d_{i^{\prime}}>d_{i^{*}}$. Let $k$ be the index such that $i^{*} \in A_{k}$. In this case, by the definition of proper instances, it holds that the allocation vector $\mathbf{X}^{\prime}$ which allocates the bestquality items to the buyers in $A_{<k}$ and to $i^{*}$ is an $h$-prefix of $I$ considered by Algorithm Prefix at line ( $\dagger$ ) for which it holds $\sum_{j \in X_{i *}^{\prime}} q_{j} \geq \sum_{i>i^{*}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} q_{j}$. Hence, we have

$$
\begin{aligned}
\frac{1}{2} r e v(\mathbf{X}, \mathbf{p}) & <\sum_{i>i^{*}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} p_{j} \\
& \leq \sum_{i>i^{*}: i \in W(\mathbf{X})}\left(v_{i} \sum_{j \in X_{i}} q_{j}\right) \\
& \leq \sum_{i>i^{*}: i \in W(\mathbf{X})}\left(v_{i^{*}} \sum_{j \in X_{i}} q_{j}\right) \\
& <v_{i^{*}} d_{i^{*}} q_{\max } \\
& \leq v_{i^{*}} \sum_{j \in X_{i^{*}}^{\prime}} q_{j}
\end{aligned}
$$

which yields the claim.
If $i^{*}$-prefixes of $I$ do exist, define $H=\left\{i \in W(\mathbf{X}): v_{i}=v_{i^{*}}\right\}$ and let $i^{\prime}=\min \{i: i \in H\}$ if $H \neq \emptyset$, otherwise set $i^{\prime}=i^{*}$. Moreover, define $i^{\prime \prime}=$ $\min \left\{i \in W(\mathbf{X}): i>i^{*}\right\}$.

If $v_{i^{*}}>v_{\beta(\mathbf{X})}$, then, by Lemma3, it holds $d_{i^{*}}>\sum_{i>i^{*}: i \in W(\mathbf{X})} d_{i}$. Define $\mathbf{X}^{\prime}$ as the $i^{*}$-prefix of $I$ such that $X_{i}^{\prime}=\left\{1+\sum_{j=1}^{i-1} d_{j}, \ldots, d_{i}+\sum_{j=1}^{i-1} d_{j}\right\}$ for each $i \in$ [ $\left.i^{*}\right]$, i.e., $\mathbf{X}^{\prime}$ assigns the best-quality items to the first $i^{*}$ buyers. Note that the set of buyers $\left[i^{*}-1\right]$ belongs to $W\left(\mathbf{X}^{\prime}\right) \cap W(\mathbf{X})$. Moreover, since $(\mathbf{X}, \mathbf{p})$ is envy-free, by

Lemma 1 and the fact that $\mathbf{X}^{\prime}$ assigns the first $g:=\sum_{i=1}^{i^{*}-1} d_{i}$ best-quality items to the first $i^{*}-1$ buyers, it follows that $\sum_{j=g+1}^{m} q_{j} \geq \sum_{j=f\left(i^{\prime \prime}\right)}^{m} q_{j}$. This inequality, together with $d_{i^{*}}>\sum_{i>i^{*}: i \in W(\mathbf{X})} d_{i}$, implies that $\sum_{i>i^{*}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} q_{j} \leq$ $\sum_{j \in X_{i^{*}}^{\prime}} q_{j}$.

Hence, we have that

$$
\begin{aligned}
\frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p}) & <\sum_{i>i^{*}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} p_{j} \\
& \leq \sum_{i>i^{*}: i \in W(\mathbf{X})}\left(v_{i} \sum_{j \in X_{i}} q_{j}\right) \\
& <\sum_{i>i^{*}: i \in W(\mathbf{X})}\left(v_{i^{*}} \sum_{j \in X_{i}} q_{j}\right) \\
& \leq v_{i^{*}} \sum_{j \in X_{i^{*}}^{\prime}} q_{j} \\
& \leq v_{i^{*}} \sum_{j \in M_{i^{*}}\left(\mathbf{X}^{\prime}\right)} q_{j} \\
& \leq \operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right) \\
& \leq \operatorname{rev}\left(\mathbf{X}_{i^{*}}^{*}, \widetilde{\mathbf{p}}\right) \\
& \leq \operatorname{rev}(\operatorname{Prefix}),
\end{aligned}
$$

which yields the claim.

If $v_{i^{*}}=v_{\beta(\mathbf{X})}$, with $i^{*} \in A_{k}$ for some $k \in\left[\delta^{*}\right]$, define $\mathbf{X}^{\prime}$ as the $\alpha(k)$-prefix of $I$ such that $X_{i}^{\prime}=\left\{1+\sum_{j=1}^{i-1} d_{j}, \ldots, d_{i}+\sum_{j=1}^{i-1} d_{j}\right\}$ for each $i \in[\alpha(k)]$. Note that the set of buyers $\left[i^{\prime}-1\right]$ belongs to $W\left(\mathbf{X}^{\prime}\right) \cap W(\mathbf{X})$. Moreover, since $(\mathbf{X}, \mathbf{p})$ is envyfree, by Lemma 1 and the fact that $\mathbf{X}^{\prime}$ assigns the first $g^{\prime}:=\sum_{i=1}^{i^{\prime}-1} d_{i}$ best-quality items to the first $i^{\prime}-1$ buyers, it follows that $\sum_{j=g^{\prime}+1}^{m} q_{j} \geq \sum_{j=f\left(i^{\prime}\right)}^{m} q_{j}$. This inequality, together with the fact that $\sum_{i=i^{\prime}}^{\alpha(k)} d_{i} \geq \sum_{A \subseteq A_{k}: d\left(A_{<k}\right)+d(A) \leq m} d(A)$ for each $k \in\left[\delta^{*}\right]$ which comes from the definition of $\delta^{*}$ and $\widetilde{A}$, implies that $\sum_{i \geq i^{\prime}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} q_{j} \leq \sum_{i=i^{\prime}}^{\alpha(k)} \sum_{j \in X_{i}^{\prime}} q_{j}$.

Hence, we have that

$$
\begin{aligned}
\frac{1}{2} \operatorname{rev}(\mathbf{X}, \mathbf{p}) & <\sum_{i \geq i^{\prime}: i \in W(\mathbf{X})} \sum_{j \in X_{i}} p_{j} \\
& \leq \sum_{i \geq i^{\prime}: i \in W(\mathbf{X})}\left(v_{k} \sum_{j \in X_{i}} q_{j}\right) \\
& =\sum_{i \geq i^{\prime}: i \in W(\mathbf{X})}\left(v_{i^{*}} \sum_{j \in X_{i}} q_{j}\right) \\
& \leq v_{i^{*}} \sum_{i=i^{\prime}}^{\alpha(k)} \sum_{j \in X_{i}^{\prime}} q_{j} \\
& =v_{i^{*}} \sum_{j \in M_{i^{*}}\left(\mathbf{X}^{\prime}\right)} q_{j} \\
& \leq \operatorname{rev}\left(\mathbf{X}^{\prime}, \widetilde{\mathbf{p}}\right) \\
& \leq \operatorname{rev}\left(\mathbf{X}_{i^{*}}^{*}, \widetilde{\mathbf{p}}\right) \\
& \leq \operatorname{rev}(\operatorname{Prefix})
\end{aligned}
$$

which yields the claim.

We conclude this section by showing that the approximation ratio achieved by Algorithm Prefix is the best possible one for proper instances.

Theorem 4. For any $0<\epsilon \leq 1$, the RMPSD on proper instances cannot be approximated to a factor $2-\epsilon$ unless $\mathrm{P}=\mathrm{NP}$.

Proof. For an integer $k \geq 3$, consider an instance $I$ of the Constrained Partition problem with $2(k-1)$ numbers $q_{1}, \ldots, q_{2(k-1)}$ such that $\sum_{i=1}^{2(k-1)} q_{i}=Q$ and define $q_{\text {min }}=\min _{i \in[2(k-1)]}\left\{q_{i}\right\}$ and $q_{\text {max }}=\max _{i \in[2(k-1)]}\left\{q_{i}\right\}$. Remember that, by definition, $Q$ is even and it holds $\frac{3}{2} q_{\min } \geq q_{\max }$. Also in this case, as observed in the proof of Theorem 1 it holds $q_{j}<Q / 2$ for each $j \in[2(k-1)]$.

For any $0<\epsilon \leq 1$, define

$$
\lambda=\max \left\{600 k^{2} ;\left\lceil\frac{4(k+1)}{\epsilon}+\frac{(5 k+3)(2-\epsilon) Q}{\epsilon \bar{q}}\right\rceil-2\right\} .
$$

We create an instance $I^{\prime}$ of the RMPSD as done in the proof of Theorem 1 with the addition of a buyer 0 , with $v_{0}=\frac{(\lambda-2 k) \bar{q}}{(Q+\bar{q}) k}$ and $d_{0}=k$, and $k+1$ items of quality $Q+\bar{q}$.

We first show that $v_{0}>2=v_{1}$. It holds

$$
\begin{aligned}
v_{0} & =\frac{(\lambda-2 k) \bar{q}}{(Q+\bar{q}) k} \\
& >\frac{(\lambda-2 k) \bar{q}}{\left(3(k-1) q_{\min }+\bar{q}\right) k} \\
& =\frac{(\lambda-2 k) \bar{q}}{(300(k-1) \bar{q}+\bar{q}) k} \\
& \geq \frac{600 k^{2}-2 k}{300 k^{2}-299 k} \\
& >2
\end{aligned}
$$

where the first inequality follows from $Q \leq 2(k-1) q_{\max } \leq 3(k-1) q_{\min }$.
Moreover, note that, in the proof of Theorem 1, we only needed $\lambda>3 k$ in order to show that $v_{i}>v_{i+1}$ for each $i \in[4]$. Hence, we can conclude that $v_{i}>v_{i+1}$ for each $0 \leq i \leq 4$. It follows that, with the addition of buyer 0 and the $k+1$ items of quality $Q+\bar{q}$, the instance $I^{\prime}$ is now proper.

The spirit of the proof is the same of that used in the one of Theorem 1 , i.e., we show that, if $I$ admits a positive answer, then there exists a solution for $I^{\prime}$ with revenue above a certain value, while, if $I$ admits no positive answers, then all the solutions for $I^{\prime}$ must raise a revenue below a certain other value.

First of all, let us determine the set of all possible non-empty allocation vectors able to yield an envy-free outcome. To this aim, we can claim the following set of constraints which come from the fact that $v_{i}>v_{i+1}$ for each $0 \leq i \leq 4$ :
i) Since $d_{0} \leq d_{i}$ for each $i \geq 1$, it must be $X_{0} \neq \emptyset$;
ii) Since $d_{1} \leq d_{i}$ for each $i \geq 2$, it must be $X_{1} \neq \emptyset$ when $\bigcup_{i=2}^{5} X_{i} \neq \emptyset$;
iii) Since $d_{3} \leq d_{i}$ for each $i \geq 4$, it must be $X_{3} \neq \emptyset$ when $X_{4} \cup X_{5} \neq \emptyset$;
iv) Since $d_{2} \leq d_{3}+d_{4}$, it must be $X_{2} \neq \emptyset$ when $X_{3}, X_{4} \neq \emptyset$;

Hence, for each envy-free outcome $(\mathbf{X}, \mathbf{p}), \mathbf{X}$ can only fall into one of the following five cases:

1. $X_{0} \neq \emptyset$ and $X_{i}=\emptyset$ for each $i \geq 1$;
2. $X_{0}, X_{1} \neq \emptyset$ and $X_{i}=\emptyset$ for each $i \geq 2$;
3. $X_{0}, X_{1}, X_{2} \neq \emptyset$ and $X_{i}=\emptyset$ for each $i \geq 3$;
4. $X_{0}, X_{1}, X_{3} \neq \emptyset$ and $X_{2}, X_{4}, X_{5}=\emptyset$;
5. $X_{0}, X_{1}, X_{3}, X_{5} \neq \emptyset$ and $X_{2}, X_{4}=\emptyset$.

When $\mathbf{X}$ falls into case (1), for any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envyfree, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p}) \leq v_{0} k(Q+\bar{q})=(\lambda-2 k) \bar{q}$. When $\mathbf{X}$ falls into case (2), for any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envy-free, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p}) \leq$ $v_{0} k(Q+\bar{q})+2(k Q+\bar{q})=(\lambda-2 k) \bar{q}+2(k Q+\bar{q})$. When $\mathbf{X}$ falls into case (4), for any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envy-free, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p}) \leq$ $v_{0} k(Q+\bar{q})+2(k Q+\bar{q})+\frac{3}{2} v_{3} k Q<(\lambda-2 k) \bar{q}+5 k Q+2 \bar{q}$ since $v_{3}<2$.

When $\mathbf{X}$ falls into case (3), $X_{0}$ can only contain items of quality $Q+\bar{q}$, the remaining item of quality $Q+\bar{q}$, denote it by $j$, must be assigned to $X_{1}$ and $X_{2}$
must contain an item of quality $Q$. For any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envy-free, there must exist an item $j^{\prime} \in X_{2}$ such that $p_{j^{\prime}} \leq v_{2} q_{j^{\prime}}<2 q_{j^{\prime}}$. Moreover, it must be $u_{1 j}=2(Q+\bar{q})-p_{j} \geq u_{1 j^{\prime}}=2 q_{j^{\prime}}-p_{j^{\prime}}>0$ which implies $p_{j} \leq 2(Q+\bar{q})$. Finally, for each item $j^{\prime \prime} \in X_{0}$, it must be $p_{j^{\prime \prime}}=p_{j^{\prime}}$ since $q_{j^{\prime \prime}}=q_{j^{\prime}}$. Hence, it holds

$$
\begin{aligned}
\operatorname{rev}(\mathbf{X}, \mathbf{p}) \leq & 4 k Q+2(k+1) \bar{q}+v_{2}\left(\frac{5}{2} Q+(\lambda-2 k) \bar{q}\right) \\
= & 4 k Q+2(k+1) \bar{q}+\frac{5}{2} Q+(\lambda-2 k) \bar{q} \\
& +\frac{1}{\lambda} \frac{Q-2 k \bar{q}+k Q(\lambda+1) / 2}{Q k+Q-2 k \bar{q}+\lambda \bar{q}}\left(\frac{5}{2} Q+(\lambda-2 k) \bar{q}\right) \\
= & \left(4 k+\frac{5}{2}\right) Q+(\lambda+2) \bar{q}+\frac{(2(\lambda-2 k) \bar{q}+5 Q)(k Q(\lambda+1)+2 Q-4 k \bar{q})}{4 \lambda((\lambda-2 k) \bar{q}+(k+1) Q)} \\
< & (4 k+3) Q+(\lambda+2) \bar{q}+\frac{k Q(\lambda+1)+2 Q}{2 \lambda} \\
< & (4 k+3) Q+(\lambda+2) \bar{q}+k Q \\
= & (5 k+3) Q+(\lambda+2) \bar{q}
\end{aligned}
$$

where the first strict inequality follows from $2(k+1)>5$ and the second one follows from $k+2<k \lambda$.

Hence, we can conclude that, when $\mathbf{X}$ falls into one of the cases from (1) to (4), for any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envy-free, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p})<$ $(5 k+3) Q+(\lambda+2) \bar{q}$.

In the remaining of this proof, we restrict to the case in which $\mathbf{X}$ falls into case (5).

Lemma 14. If there exists a positive answer to $I$, then there exists an envy-free outcome for $I^{\prime}$ of revenue greater than $2(\lambda-2 k) \bar{q}$.

Proof. Consider the allocation vector $\mathbf{X}$ such that $X_{0}$ contains the $k$ items of quality $Q+\bar{q}, X_{1}$ contains $k$ items of quality $Q, X_{3}$ contains the item of quality $Q / 2$ plus the $k-1$ items forming a positive answer to $I, X_{5}$ contains the $\lambda-2 k$ items of quality $\bar{q}$ and $X_{2}=X_{4}=\emptyset$. Note that $\mathbf{X}$ is monotone. We show that the outcome ( $\mathbf{X}, \widetilde{\mathbf{p}}$ ) is envy-free.

According to the price vector $\widetilde{\mathbf{p}}$, it holds $\widetilde{p}_{j}=\frac{(\lambda-2 k) \bar{q}}{k}+\left(3+\frac{1}{\lambda}\right) \frac{Q}{2}-\frac{\bar{q}}{\lambda}$ for each $j \in X_{0}, \widetilde{p}_{j}=\frac{(3 \lambda+1) Q-2 \bar{q}}{2 \lambda}$ for each $j \in X_{1}, \widetilde{p}_{j}=\frac{(\lambda+1) q_{j}-\bar{q}}{\lambda}$ for each $j \in X_{3}$ and $\widetilde{p}_{j}=\bar{q}$ for each $j \in \stackrel{2}{X}_{5}$.

Because of Lemma 5, in order to show that ( $\mathbf{X}, \widetilde{\mathbf{p}}$ ) is envy-free, we only need to prove that, for each buyer $i \notin W(\mathbf{X})$ and $T \subseteq M$ with $|T|=d_{i}$, it holds $\sum_{j \in T} u_{i j} \leq 0$. Note that the buyers not belonging to $W(\mathbf{X})$ are buyers 2 and 4 .

For buyer 2 , for each pair of items $\left(j, j^{\prime}\right)$ with $j \in X_{0}$ and $j^{\prime} \in X_{1}$, it holds $u_{2 j}<u_{2 j^{\prime}}$, for each pair of items $\left(j^{\prime}, j^{\prime \prime}\right)$ with $j^{\prime} \in X_{1}$ and $j^{\prime \prime} \in X_{3}$, it holds $u_{2 j^{\prime}}<u_{2 j^{\prime \prime}}$ and, for each pair of items $\left(j^{\prime}, j^{\prime \prime \prime}\right)$ with $j^{\prime} \in X_{1}$ and $j^{\prime \prime \prime} \in X_{5}$, it
holds $u_{2 j^{\prime}}<u_{2 j^{\prime \prime \prime}}$. In fact, we have

$$
\begin{aligned}
u_{2 j^{\prime}}-u_{2 j} & =\frac{\lambda \bar{q}}{k}-2 \bar{q}-v_{2} \bar{q} \\
& >\left(\frac{\lambda}{k}-4\right) \bar{q} \\
& >0,
\end{aligned}
$$

where the first inequality follows from $v_{2}<2$ and the second one follows from $\lambda>4 k$;

$$
\begin{aligned}
u_{2 j^{\prime \prime}}-u_{2 j^{\prime}} & =v_{2} q_{j}-q_{j}-\frac{q_{j}}{\lambda}-v_{2} Q+\frac{3}{2} Q+\frac{Q}{2 \lambda} \\
& >\frac{1}{\lambda}\left(\frac{Q}{2}-q_{j}\right) \\
& >0
\end{aligned}
$$

where the first inequality follows from $1<v_{2}<3 / 2$ and the second one follows from $q_{j}<Q / 2$ for each $j \in X_{3}$; and

$$
\begin{aligned}
u_{2 j^{\prime \prime \prime}}-u_{2 j^{\prime}} & =v_{2} \bar{q}-\bar{q}-v_{2} Q+\frac{3}{2} Q+\frac{Q}{2 \lambda}-\frac{\bar{q}}{\lambda} \\
& >0,
\end{aligned}
$$

where the inequality follows from $1<v_{2}<3 / 2$ and $\bar{q}<Q / 2$.
Hence, the set of items of cardinality $d_{2}$ which gives the highest utility to buyer 2 is $T=X_{1} \cup X_{3} \cup X_{5}$. It holds

$$
\begin{aligned}
& \sum_{j \in T}\left(v_{2} q_{j}-\widetilde{p}_{j}\right) \\
= & k\left(v_{2} Q-\frac{3}{2} Q-\frac{Q}{2 \lambda}+\frac{\bar{q}}{\lambda}\right)+v_{2} Q-Q-\frac{Q}{\lambda}+\frac{k \bar{q}}{\lambda}+(\lambda-2 k)\left(v_{2} \bar{q}-\bar{q}\right) \\
= & 0 .
\end{aligned}
$$

For buyer 4 , for each pair of items $\left(j, j^{\prime}\right)$ with $j \in X_{0}$ and $j^{\prime} \in X_{1}$, it holds $u_{4 j}<u_{4 j^{\prime}}$, for each pair of items $\left(j^{\prime}, j^{\prime \prime}\right)$ with $j^{\prime} \in X_{1}$ and $j^{\prime \prime} \in X_{3}$, it holds $u_{4 j^{\prime}}<u_{4 j^{\prime \prime}}$ and, for each pair of items $\left(j^{\prime}, j^{\prime \prime \prime}\right)$ with $j^{\prime} \in X_{1}$ and $j^{\prime \prime \prime} \in X_{5}$, it holds $u_{4 j^{\prime}}<u_{4 j^{\prime \prime \prime}}$. In fact, we have

$$
\begin{aligned}
u_{4 j^{\prime}}-u_{4 j} & =\frac{\lambda \bar{q}}{k}-2 \bar{q}-v_{4} \bar{q} \\
& >\left(\frac{\lambda}{k}-4\right) \bar{q} \\
& >0
\end{aligned}
$$

where the first inequality follows from $v_{4}<2$ and the second one follows from $\lambda>4 k$;

$$
\begin{aligned}
u_{4 j^{\prime \prime}}-u_{4 j^{\prime}} & =v_{4} q_{j}-q_{j}-\frac{q_{j}}{\lambda}-v_{4} Q+\frac{3}{2} Q+\frac{Q}{2 \lambda} \\
& >\frac{1}{\lambda}\left(\frac{Q}{2}-q_{j}\right) \\
& >0
\end{aligned}
$$

where the first inequality follows from $1<v_{4}<3 / 2$ and the second one follows from $q_{j}<Q / 2$ for each $j \in X_{3}$; and

$$
\begin{aligned}
u_{4 j^{\prime \prime \prime}}-u_{4 j^{\prime \prime}} & =v_{4} \bar{q}-\bar{q}-v_{4} q_{j}+q_{j}+\frac{q_{j}}{\lambda}-\frac{\bar{q}}{\lambda} \\
& =\left(q_{j}-\bar{q}\right)\left(1+\frac{1}{\lambda}-v_{4}\right) \\
& >0
\end{aligned}
$$

where the inequality follows from $v_{4}<1+1 / \lambda$ and $q_{j}>\bar{q}$ for each $j \in X_{3}$.
Hence, the set of items of cardinality $d_{4}$ which gives the highest utility to buyer 4 is $T=X_{3} \cup X_{5}$. It holds

$$
\begin{aligned}
& \sum_{j \in T}\left(v_{4} q_{j}-\widetilde{p}_{j}\right) \\
= & v_{4} Q-Q-\frac{Q}{\lambda}+\frac{k \bar{q}}{\lambda}+(\lambda-2 k)\left(v_{4} \bar{q}-\bar{q}\right) \\
= & 0
\end{aligned}
$$

Hence, we can conclude that the outcome ( $\mathbf{X}, \widetilde{\mathbf{p}}$ ) is envy-free and it holds $\operatorname{rev}(\mathbf{X}, \widetilde{\mathbf{p}})>2(\lambda-2 k) \bar{q}$.

We continue by showing that, for any envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (5) and such that $X_{1}$ contains an item of quality $Q+\bar{q}$, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p})<$ $(\lambda+2) \bar{q}+(4 k+3) Q$.

Note that, in such a case, by Lemma 1. $X_{0}$ can only contain items of quality $Q+\bar{q}$. For any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envy-free, there must exist an item $j^{\prime} \in X_{5}$ such that $p_{j^{\prime}} \leq q_{j^{\prime}}$. Let $j^{\prime \prime}$ be the index of the item of quality $Q+\bar{q}$ belonging to $X_{1}$. By the envy-freeness of $(\mathbf{X}, \mathbf{p})$, it holds $u_{1 j^{\prime \prime}}=2(Q+\bar{q})-p_{j^{\prime \prime}} \geq$ $2 q_{j^{\prime}}-p_{j^{\prime}}=q_{j^{\prime}}$ which implies $p_{j^{\prime \prime}}<2(Q+\bar{q})$. Clearly, since $(\mathbf{X}, \mathbf{p})$ is envy-free, for each item $j \in X_{0}$, it must be $p_{j}=p_{j^{\prime \prime}}$ since $q_{j}=q_{j^{\prime \prime}}$. Hence, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p})<4 k Q+2(k+1) \bar{q}+\frac{3}{2} v_{3} Q+(\lambda-2 k) \bar{q}<(\lambda+2) \bar{q}+(4 k+3) Q$ because $v_{3}<2$.

Since it holds $(\lambda+2) \bar{q}+(4 k+3) Q<(\lambda+2) \bar{q}+(5 k+3) Q$, it follows that, either when $\mathbf{X}$ falls into case (5) and $X_{1}$ contains an item of quality $Q+\bar{q}$ or $\mathbf{X}$ falls into one of the cases from (1) to (4), for any pricing vector $\mathbf{p}$ such that $(\mathbf{X}, \mathbf{p})$ is envy-free, it holds $\operatorname{rev}(\mathbf{X}, \mathbf{p}) \leq(\lambda+2) \bar{q}+(5 k+3) Q$.

Now we are only left to consider envy-free outcomes ( $\mathbf{X}, \mathbf{p}$ ) such that $\mathbf{X}$ falls into case (5) and $X_{1}$ does not contain any item of quality $Q+\bar{q}$.

Assume that $\sum_{j \in X_{3}}>Q$. This can only happen when buyer 3 is assigned an item of quality at least $Q / 2$. In such a case, since $X_{1}$ does not contain any item of quality $Q+\bar{q}$, it can only be the case that each item in $X_{1}$ is of quality $Q$ and $X_{3}$ gets the item of quality $Q / 2$. This means that the items allocated by $\mathbf{X}$ to buyers 1,3 and 5 are drawn from the same instance $I^{\prime}$ considered in the proof of Theorem 1. Hence, we can replicate the arguments used in the proof of Lemma 8 to show that $\sum_{j \in X_{3}}>Q$ yields a contradiction.

Similarly, assume that $\sum_{j \in X_{3}}<Q$. This can only happen when the items allocated by $\mathbf{X}$ to buyers 3 and 5 are drawn from the same instance $I^{\prime}$ considered in the proof of Theorem [1 Hence, we can replicate the arguments used in the proof of Lemma 9 to show that $\sum_{j \in X_{3}}<Q$ yields a contradiction.

We can conclude that there exists an envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (5) in which no item of quality $Q+\bar{q}$ belongs to $X_{1}$ only if $\sum_{j \in X_{3}} q_{j}=Q$. Since, as we have already observed, in such a case the item of quality $Q / 2$ has to belong to $X_{3}$, it follows that there exists an envy-free outcome ( $\mathbf{X}, \mathbf{p}$ ) falling into case (5) in which no item of quality $Q+\bar{q}$ belongs to $X_{1}$ only if there are $k-1$ items inherited from $I$ whose sum is exactly $Q / 2$, that is, only if $I$ admits a positive solution.

Any other envy-free outcome can raise a revenue of at most $(\lambda+2) \bar{q}+(5 k+3) Q$. Hence, if there exists a positive answer to $I$, then, by Lemma 14 there exists a solution to $I^{\prime}$ of revenue strictly greater than $2(\lambda-2 k) \bar{q}$, while, if there is no positive answer to $I$, then there exists no solution to $I^{\prime}$ of revenue more than $(\lambda+2) \bar{q}+(5 k+3) Q$.

Thus, if there exists an $r$-approximation algorithm for the RMPSD on continuous instances with $r \leq \frac{2(\lambda-2 k) \bar{q}}{(\lambda+2) \bar{q}+(5 k+3) Q}$, it is then possible to decide in polynomial time the Constrained Partition problem, thus implying $\mathrm{P}=\mathrm{NP}$. By $\lambda \geq \frac{4(k+1)}{\epsilon}+\frac{(5 k+3)(2-\epsilon) Q}{\epsilon \bar{q}}-2$, it follows $\frac{2(\lambda-2 k) \bar{q}}{(\lambda+2) \bar{q}+(5 k+3) Q} \geq 2-\epsilon$ which implies the claim.

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