Improving Inconsistency Resolution by Considering Global Conflicts

Cristhian Ariel David Deagustini[1,3], Maria Vanina Martiínez[2], Marcelo A. Falappa[1,3], and Guillermo Ricardo Simari[1]

[1] Artificial Intelligence Research and Development Laboratory
Department of Computer Science and Engineering
Universidad Nacional del Sur - Alem 1253, (8000) Bahía Blanca, Buenos Aires
[2] Department of Computer Science
University of Oxford - Wolfson Building, Parks Road, Oxford OX13QD, UK
[3] Consejo Nacional de Investigaciones Científicas y Técnicas

Abstract. Over the years, inconsistency management has caught the attention of researchers of different areas. Inconsistency is a problem that arises in many different scenarios, for instance, ontology development or knowledge integration. In such settings, it is important to have adequate automatic tools for handling potential conflicts. Here we propose a novel approach to belief base consolidation based on a refinement of kernel contraction that accounts for the relation among kernels using clusters. We define cluster contraction based consolidation operators as the contraction by *falsum* on a belief base using cluster incision functions, a refinement of (smooth) kernel incision functions. A cluster contraction-based approach to belief bases consolidation can successfully obtain a belief base satisfying the expected consistency requirement. Also, we show that the application of cluster contraction-based consolidation operators satisfy minimality regarding loss of information and are equivalent to operators based on maxichoice contraction.

Keywords: Inconsistency Management, Belief Consolidation, Minimal Loss of Information.

1 Introduction

Inconsistency management is admittedly an important problem that has to be faced, *e. g.,* when knowledge provided by different users is expected to be exploited by a reasoning process. Although the integrated knowledge may be inconsistent, it is obvious there is still value in that information even in the presence of (potential) conflicts, and it is highly possible the existence of information that is not related and/or affected by those conflicts. Consider the following simple example that we use in the rest of the paper as the running example.

Suppose that we are gathering information about sports activities of early alumnus of a college and some official records have been lost. We are particularly interested in several remarkable students for which we wish to compile their

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doings and achievements in the college. As the first step for this activity we ask for help from staff and faculty members; for a particular alumni, called Martin, we obtain the following information from three different people that were in the college at the same time as Martin:

- Staff member S_1 tells us Martin used to play soccer, and he thinks he remembers he also coached the school's basketball team; let's denote the first proposition with p and the second with q .
- P.E. professor S_2, who used to be one of Martin's college mates, states that he thinks Martin used to play in the basketball team; we will refer to this proposition as r.
- An old class mate of Martin is not sure but she remembers the soccer team used to be very proud and demanding at that time, so definitely if Martin played soccer he did not play basketball; let this be proposition s.

We have not yet provided a formal definition of consistency, however, it is rather intuitive that it is not possible for all these statements to hold together. Several important approaches used to address the handling of inconsistency had been proposed in Artificial Intelligence (AI), specially in the areas of *belief revision* and *argumentation.* In particular, belief revision deals with the general problem of the dynamics of knowledge, *i.e.,* how belief states change and evolve through time, solving possible inconsistencies in the process. One particular way to deal with the above situation is to try to modify the information contained in the knowledge base as little as possible in order to make it consistent; this is known as knowledge consolidation in the belief revision community. In this work, we define consolidation operators that takes an inconsistent belief base and apply special functions, called incisions functions, so that inconsistencies are resolved. The main contributions of this paper are:

– We first analyze a class of consolidation operators based on kernel contraction [11,12]; these operators make *incisions* on the minimal conflictive subsets of the inconsistent belief base. In Section 3 we demonstrate the operators' behavior and show there are cases in which such operators may not yield minimal loss of information; we also show that this problem arises from treating inconsistency in a localized way, isolating minimal conflicting sets in the consistency restoration process.
– In order to prevent unnecessary loss of information, we develop an alternative and novel class of consolidation operators, called *cluster contraction-based consolidation operators*; these operators aim to address conflicts globally by means of the use of *clusters* [18] instead of minimal conflictive sets.
– In Section 4, we show that cluster incision functions are refinements of smooth kernel incision functions [11], therefore the application of cluster contraction-based consolidation operators produces in general, the deletion of a smaller number of formulæ from the original knowledge base than any smooth kernel contraction-based operators would produce.
– Finally, in Section 5 we show that cluster incision functions satisfy the minimality requirement regarding loss of information and we conclude that a

consolidation operator is a cluster contraction-based consolidation operator if and only if it is a maxichoice contraction-based consolidation operator, completing in this way the spectrum of possibilities arising from the treatment of inconsistency by means of minimal conflicts.

2 Preliminaries

We begin by introducing the notation necessary for our presentation and the required concepts that will be used throughout the paper. Also, we present the research context of belief change theory from which revision operators had arisen.

We assume a propositional language $\mathcal{L}$ built from a set of propositional symbols $\mathcal{P}$. This language is closed under the classical propositional logic symbols $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\leftrightarrow$ (equivalence). We denote propositional letters using lower-case Latin letters, possibly using subscripts (*e. g.,* a, b, c, a_1, a_2) and propositional formulæ using lower-case Greek letters, possibly using subscripts (*e. g.,* $\alpha, \beta, \gamma, \alpha_1, \alpha_2$); but, we reserve ρ and ϱ to represent incision functions.

An interpretation is a total function from $\mathcal{P}$ to $\{0, 1\}$, and the set of all interpretations is denoted with $\mathcal{W}$. An interpretation $\omega \in \mathcal{W}$ is a model of a formula α iff it makes α true in the classical way, denoted with $\omega \models \alpha$. The set of all models of a formula α is denoted with $\text{mods}(\alpha)$, *i.e.,* $\text{mods}(\alpha) = \{\omega \in \mathcal{W} \mid \omega \models \alpha\}$. Finally, $\vdash$ stands for the usual deduction relation on propositional logic, and $\perp$ stands for an arbitrary contradiction.

We assume finite sets of propositional formulæ $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, which are called belief bases and are denoted with upper-case Latin letters, usually K. We extend the notion of models of a formula to sets of formulæ in the natural way, *i.e.,* $\text{mods}(K) = \{\omega \in \mathcal{W} \mid \omega \models \alpha \text{ for all } \alpha \in K\}$. Additionally, $K_{\mathcal{L}}$ denotes the set of every belief base K containing formulæ in $\mathcal{L}$. Finally, a consistent belief base K must have at least one model; formally, we say that K is consistent iff $\text{mods}(K) \neq \emptyset$. Also, K is inconsistent iff K is not consistent.

The work of Alchourrón, Gärdenfors and Makinson where the *AGM* model is presented [1], is currently considered the cornerstone from which belief change theory has evolved (see [19]). In the AGM model, three basic change operators are defined; these can be defined over a knowledge base K as follows: the result of *expanding* K by a sentence α is a possibly larger set that infers α, the result of *contracting* K by α is a possibly smaller set that does not infer α, and finally, the result of *revising* K by α is a set K' that infers α and possibly neither extends nor is part of the set K. In particular, if K infers $\neg\alpha$ then the result of the revision of K by α is a consistent set K' that infers α. AGM provides an axiomatic characterizations of contraction and revision in terms of rationality postulates. AGM contractions can be realized by *partial meet contractions*, which are based on a selection among (maximal) subsets of K that do not imply α (the input sentence). Particular cases of partial meet contractions are *full meet contractions* and *maxichoice contractions*. The former stands for an approach that is as cautious as possible (*i.e.,* only retaining formulæ that belong to every maximal consistent subset), while the latter has the desirable property that it minimizes the loss of information, in the sense that it preserves as most formulæ as possible, since basically it selects one among all maximal consistent subsets. Another possible approach for contraction is based on a selection among the (minimal) subsets of K that contribute to make K imply α; *kernel contraction* [11] is one of such approaches and it is known to be more general than partial meet contraction, and hence to the AGM approach to contraction [11,12]. Finally, Hansson presents a refinement of kernel contraction, known as *smooth kernel contraction*, that aims to solve a problem attached to the generality of the former, as sometimes kernel contraction may produce unnecessary deletions.

In this work we focus on a different belief change operation called *consolidation*; this operation is inherently different from contraction and revision as the ultimate goal of consolidation is to obtain a consistent belief base rather than revising the knowledge base by a specific formula or removing a particular formula from it. A natural way of achieving this is to take an inconsistent belief base and restore its consistency by attending every conflict in it, a process that is known in the belief revision literature as contraction by *falsum* [10].

3 Kernel and Cluster Contraction-based Belief Base Consolidation

The work of Hansson in [11] describes how a contraction operation on belief bases can be modeled by defining *incision functions*. These functions contract a belief base, by a formula α by taking minimal sets that entail α (called α-*kernels*) and producing "incisions" on those sets so they no longer entail α. The resulting belief base is formed by the union of all formulæ that are not removed by the function. This approach is known as *kernel contraction*.

Here, we define the consolidation process as the application of incision functions over the minimal inconsistent subsets of a belief base. Following the terminology proposed by Hansson [11] we will call such sets $\perp$-kernels, or kernels for short; in the following we recall the formal definition from [11].

Definition 1 (Kernels). *Let K be a belief base. The set of kernels of K, denoted $K^{\perp\!\!\perp}\perp$, is the set of all $X \subseteq K$ such that* $\text{mods}(X) = \emptyset$ *and for every $X' \subsetneq X$ it holds that* $\text{mods}(X') \neq \emptyset$.

Example 1. Consider the inconsistent belief base $K = \{a, b \to \neg a, b, c, \neg c, d\}$. For K we have two kernels: $K^{\perp\!\!\perp}\perp = \{\kappa_1, \kappa_2\}$, with $\kappa_1 = \{a, b \to \neg a, b\}$ and $\kappa_2 = \{c, \neg c\}$. As expected by the definition of kernels, if we remove at least one formula from them, the result is consistent.

Once the set of kernels is identified, we need to establish how the inconsistencies are to be resolved. A *kernel incision function* takes a set of kernels and selects formulæ in them to be deleted from K [11].

Definition 2 ((Smooth) Kernel Incision Function). *Let K be a belief base and $K^{\perp\!\!\perp}\bot$ be the set of kernels for K. A* kernel incision function *is a function $\rho : 2^{K_{\mathcal{L}}} \mapsto K_{\mathcal{L}}$ such that the following conditions hold:*

- *$\rho(K^{\perp\!\!\perp}\bot) \subseteq \bigcup(K^{\perp\!\!\perp}\bot)$, and*
- *for all $X \in K^{\perp\!\!\perp}\bot$, if $X \neq \emptyset$ then $(X \cap \rho(K^{\perp\!\!\perp}\bot)) \neq \emptyset$.*

A kernel incision function ρ is said to be smooth if and only if for all $X \subset K$ such that $X \vdash \beta$ and $\beta \in \rho(K^{\perp\!\!\perp}\bot)$, we have then $X \cap \rho(K^{\perp\!\!\perp}\bot) \neq \emptyset$.

The second condition on Definition 2 requires from the incision function to select *at least* one formula to be deleted from every kernel. An incision function may remove several formulæ from a kernel; however, note that given the minimality of kernels, removing only one formula from each kernel suffices to restore its consistency. The last condition ensures that a kernel incision function is smooth [11]. Smoothness is characterized by the relative closure postulate [11] that aims to retain as much from the original knowledge base as possible; it states that the result of contracting a knowledge base K must contain those of its own logical consequences that are also elements of K. Intuitively, smoothness captures the set of incisions that yield contractions that can be obtained by performing the contraction by any incision function and then adding back the elements from K that were unnecessarily dropped by the incision function.

Based on (smooth) kernel incision functions we define kernel contraction-based belief consolidation operators as follows.

Definition 3 (Kernel Contraction-based Consolidation Operator). *Given belief base K, let $K^{\perp\!\!\perp}\bot$ be the set of kernels for K and ρ a kernel incision function. A kernel contraction-based consolidation operator Υ_ρ for K is defined as:*

$$\Upsilon_\rho(K) = K \setminus \rho(K^{\perp\!\!\perp}\bot)$$

Furthermore, if ρ is a smooth kernel incision function then Υ_ρ is a smooth kernel contraction-based consolidation operator.

Note that operator $\Upsilon_\rho(\cdot)$ is parameterized by the incision function ρ; the result of applying such operator will be a consistent belief base since every conflict is attended to by the kernel incision function. However, if we strive for minimal loss of information (as it is usually assumed in the management of inconsistent information), then this operator defined as it is, has the important drawback of solving conflicts locally to every kernel; even if the function only removes one formula from each kernel, the incisions may be too drastic from a global point of view and the operator might end up giving up more formulæ than the ones that are absolutely necessary. To see this problem, consider the following example:

Example 2. Consider $K = \{p, q, r, p \rightarrow \neg r, \neg(q \wedge r)\}$. This KB comes from our running example, regarding Martin's sports activities. The fourth proposition corresponds to proposition s. Furthermore, we have added one more proposition, namely $\neg(q \wedge r)$; it is common sense to assume that it is not possible for the same person to be both a player and the coach of a basketball team. Clearly, K

is inconsistent. As we want to obtain a consistent belief base, we will apply a kernel contraction-based consolidation operator. For belief base K we have that $K^{\perp\!\!\perp}\bot = \{\kappa_1, \kappa_2\}$, where $\kappa_1 = \{p, r, p \to \neg r\}$ and $\kappa_2 = \{q, r, \neg(q \wedge r)\}$.

The following table shows all possible incision functions that delete exactly one formula for each kernel (other incisions are possible deleting more than one formula from each kernel):

Possible Kernel Incision Functions	
$\rho(\kappa_1) = \{p\}$ and $\rho(\kappa_2) = \{q\}$	$\rho(\kappa_1) = \{p\}$ and $\rho(\kappa_2) = \{r\}$
$\rho(\kappa_1) = \{p\}$ and $\rho(\kappa_2) = \{\neg(q \wedge r)\}$	$\rho(\kappa_1) = \{r\}$ and $\rho(\kappa_2) = \{q\}$
$\rho(\kappa_1) = \{r\}$ and $\rho(\kappa_2) = \{r\}$	$\rho(\kappa_1) = \{r\}$ and $\rho(\kappa_2) = \{\neg(q \wedge r)\}$
$\rho(\kappa_1) = \{p \to \neg r\}$ and $\rho(\kappa_2) = \{q\}$	$\rho(\kappa_1) = \{p \to \neg r\}$ and $\rho(\kappa_2) = \{r\}$
$\rho(\kappa_1) = \{p \to \neg r\}$ and $\rho(\kappa_2) = \{\neg(q \wedge r)\}$	

All the above possibilities restore consistency in K, but clearly there are some choices that are better with respect to the amount of information lost in the process. For instance, suppose we choose the functions that perform the following incisions $\rho(\kappa_1) = \{r\}$ and $\rho(\kappa_2) = \{q\}$; we then have that: $\Upsilon_\rho(K) = K \setminus \rho(K^{\perp\!\!\perp}\bot) = K \setminus \{q, r\}$. As we can see, for κ_2 we have deleted q from K in order to solve the conflict. However, this is not actually necessary, as r (*i.e.,* the proposition that says that Martin played at the school's basketball team) will not be in the final belief base anyway, since it is deleted to solve the conflict in κ_1, and thus the conflict in kernel κ_2 is already resolved, that is there is no need to further remove propositions from κ_2. The reason behind this choice is that a kernel contraction-based operator solves conflicts *locally* to the kernels and there is no mechanism in its definition to consider any interaction among them.

Clearly, it is possible to address the problem described above by analyzing all possible incisions and computing the combination that makes the best choice globally. However, this would involve traversing the (possibly) enormous search space of all possible incision functions; in the following we present an approach that avoids this by contemplating only incisions that are globally optimal with respect to the amount of information loss. The proposal is based on the use of *clusters*, first introduced in [18] and further analyzed as a foundation for inconsistency management in [16,17]. This construction will allow us to have a more global vision of conflicts, and, as we shall see latter, will also have a direct impact on the consolidation process. Clusters are obtained by defining an overlapping relation among kernels.

Definition 4 (Overlapping Kernels, Equivalence). *Let K be a belief base, and $K^{\perp\!\!\perp}\bot$ be the set of kernels for K. Given kernels κ_1, $\kappa_2 \in K^{\perp\!\!\perp}\bot$ we say they overlap, denoted $\kappa_1\theta\kappa_2$, iff for some $\alpha \in \kappa_1$ and $\beta \in \kappa_2$ it holds that $\alpha \models \beta$. Furthermore, we denote as θ^* the equivalence relation obtained over $K^{\perp\!\!\perp}\bot$ through the reflexive and transitive closure of θ.*

Example 3. Consider a belief base K such that $K^{\perp\!\!\perp}\bot = \{\kappa_1, \kappa_2\}$ where $\kappa_1 = \{a, \neg a \wedge \neg b\}$ and $\kappa_2 = \{b \vee a, \neg a \wedge \neg b\}$. Clearly: $\kappa_1\theta\kappa_2$, as $\neg a \wedge \neg b \models \neg a \wedge \neg b$.

As another example of overlapping, consider belief base K' such that $K'^{\perp\!\!\perp}\bot = \{\kappa_1, \kappa_2, \kappa_3\}$ where $\kappa_1 = \{a \wedge b, \neg b\}$, $\kappa_2 = \{a, \neg a\}$ and $\kappa_3 = \{a \wedge b, \neg a\}$. Then, it holds that for instance $\kappa_1 \theta \kappa_2$ because $a \wedge b \models a$.

Below, we recall the notion of *clusters*, that formalizes the way in which conflicts will be structured; intuitively, a cluster groups together kernels that stand for related conflicts, in a transitive way.

Definition 5 (Clusters [18]). *Let K be a belief base, $K^{\perp\!\!\perp}\bot$ be the set of kernels for K, and θ the overlapping relation. A cluster of K is a set $\varsigma = \bigcup_{\kappa \in [\kappa]} \kappa$, where $[\kappa] \in K^{\perp\!\!\perp}\bot/\theta^*$. We use $K^{\perp\!\!\perp\!\!\perp}\bot$ to denote the set of all clusters for K.*

Example 4. Consider a belief base K such that $K^{\perp\!\!\perp}\bot = \{\kappa_1, \kappa_2, \kappa_3\}$, with $\kappa_1 = \{\alpha, \beta\}, \kappa_2 = \{\beta, \gamma\}$, and $\kappa_3 = \{\delta, \epsilon\}$. Then, we have the following set of clusters $K^{\perp\!\!\perp\!\!\perp}\bot = \{\varsigma_1, \varsigma_2\}$, where $\varsigma_1 = \{\alpha, \beta, \gamma\}$ and $\varsigma_2 = \{\delta, \epsilon\}$. Note that, κ_3 does not overlap with any other kernel in $K^{\perp\!\!\perp}\bot$, but $[\kappa_3] \in K^{\perp\!\!\perp}\bot/\theta^*$ is such that $[\kappa_3] = \{\kappa_3\}$, then it constitutes a cluster in itself (*i.e.*, $\varsigma_2 = \{\delta, \epsilon\}$).

The use of clusters instead of kernels can help in preventing situations like the one in Example 2 since the cluster structure allows us to identify kernels that overlap; thus, we can contemplate incisions that make global considerations of optimality. Moreover, the proposed notion of overlapping helps to identify only useful clusters. To see this consider belief base K'' such that $K''^{\perp\!\!\perp}\bot = \{\kappa_1, \kappa_2\}$ where $\kappa_1 = \{a \wedge b, \neg b \wedge \neg c\}$ and $\kappa_2 = \{b \wedge c, \neg b \wedge \neg d\}$. Formulæ $a \wedge b$ and $b \wedge c$ share models, however they do not overlap under Definition 4. Considering these two kernels together does not help in improving the consistency restoration process as, for instance, the removal of $a \wedge b$ does not resolve the conflict in κ_2. We have chosen to not consider these cases as overlaps in this work, but clearly this decision depends directly on the way conflicts are allowed to be resolved; for a consistency restoration technique not based on deleting entire formulæ from the clusters a different notion of overlapping could prove more useful.

Remember that by design the simple removal of any single formula within a kernel makes the set no longer inconsistent; however, this is not necessarily the case for clusters [16]. Therefore, in order to define incision functions over clusters, we cannot simply reuse Definition 2, as the following example shows.

Example 5. Continuing with Example 4, consider a kernel incision function ρ and the cluster $\varsigma_1 \in K^{\perp\!\!\perp\!\!\perp}\bot$. We could have, for instance, that $\rho(\varsigma_1) = \{\alpha\}$. Then, the intersection between the cluster and the result of the incision function is non empty and the selected formula belongs to the union of clusters, fulfilling the conditions on the definition of kernel incision functions, but the inconsistency remains as $\kappa_2 = \{\beta, \gamma\}$ still is an inconsistent set.

We now introduce cluster incision functions; these functions are refinements of the ones introduced earlier in the paper.

Definition 6 (Cluster Incision Function). *Let K be a belief base and $K^{\perp\!\!\perp}\bot$ and $K^{\perp\!\!\perp\!\!\perp}\bot$ be the set of kernels and clusters for K, respectively. A* cluster incision function *is a function $\varrho : 2^{K_{\mathcal{L}}} \mapsto K_{\mathcal{L}}$ such that:*

– *$\varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\perp) \subseteq \bigcup(K^{\perp\!\!\!\perp\!\!\!\perp}\perp)$, and*
– *for all $X \in K^{\perp\!\!\!\perp\!\!\!\perp}\perp$ and $Y \in K^{\perp\!\!\!\perp}\perp$ such that $Y \subseteq X$ it holds that for some $\alpha \in Y$, $Y \cap \varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\perp) = \{\alpha\}$.*

From Definition 6, we have that for any kernel Y included in some cluster X in a belief base a cluster incision function selects exactly one formula to remove, (*i.e.,* $\{Y \cap \varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\perp)\}$ is a singleton). Now, based on cluster incision functions we define a new operator, namely cluster contraction-based consolidation operator.

Definition 7 (Cluster Contraction-based Consolidation Operator). *Given a belief base K, let $K^{\perp\!\!\!\perp}\perp$ and $K^{\perp\!\!\!\perp\!\!\!\perp}\perp$ be the set of kernels and clusters for K, respectively, and ϱ be a cluster incision function. A cluster contraction-based operator Ψ_ϱ for K is defined as follows:*

$$\Psi_\varrho(K) = K \setminus \varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\perp)$$

The last condition in Definition 6 ensures that all conflicts are resolved once we delete the selected formulæ. Example 6 shows the behavior of a cluster contraction-based operator Ψ_ϱ over the belief base K from Example 2.

Example 6. Consider once again belief base K from Example 2; we have the following set of clusters $K^{\perp\!\!\!\perp\!\!\!\perp}\perp = \{\varsigma_1\}$, with $\varsigma_1 = \{p, q, r, p \rightarrow \neg r, \neg(q \wedge r)\}$, because r belongs to both kernels in $K^{\perp\!\!\!\perp}\perp$; thus, $r \models r$. For a cluster contraction-based operator Ψ_ϱ based on a cluster incision function ϱ, we have the following possible incisions, narrowing the previous ones shown in Example 2:

Possible Cluster Incision Functions	
$\varrho(\varsigma_1) = \{p, q\}$	$\varrho(\varsigma_1) = \{p, \neg(q \wedge r)\}$
$\varrho(\varsigma_1) = \{r\}$	$\varrho(\varsigma_1) = \{p \rightarrow \neg r, q\}$
$\varrho(\varsigma_1) = \{p \rightarrow \neg r, \neg(q \wedge r)\}$	

Consider option $\varrho(\varsigma_1) = \{p, q\}$. Thus, $\Psi_\varrho(K) = K \setminus \varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\perp) = K \setminus \{p, q\}$. Note that, even if we prefer proposition r over proposition q, the minimal loss of information principle is still fulfilled, as the non-minimal options in Example 2 are not even considered by any cluster incision function. For example, if we were to choose r for deletion then to also choose q (*i.e.,* the option considered in Example 2) is no longer a viable option for a cluster incision function, as if we choose both formulæ then the set $\varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\perp) \cap \kappa_2$ will no longer be a singleton set, violating the second condition from Definition 6.

Proposition 1 shows that the consistency restoration process based on cluster contraction fulfils the consistency requirement.

Proposition 1. *Let K be a belief base, and Ψ_ϱ be a cluster contraction-based consolidation operator. Then,* $\mathrm{mods}(\Psi_\varrho(K)) \neq \emptyset$.

For space reasons we do not include the proof of results. The formal proof for Proposition 1 relies on the fact that, by definition, cluster incision function select one formula for every kernel that composes every clusters, effectively resolving the inconsistency for each one since kernels are minimal inconsistent sets.

4 Relationship with Kernel Contraction-Based Consolidation

In the previous section we introduced an approach for belief base consolidation that works as a refinement of the approach based on kernel contraction. In this section we focus on the relationship between (smooth) kernel contraction-based consolidation and cluster contraction-based consolidation. Specifically, we seek to establish this relationship from the point of view of loss of information.

We first show that cluster incision functions are refinements of kernel incision functions, that is, we have that every cluster contraction-based consolidation operators is also a kernel contraction-based one.

Proposition 2. *Let Ψ_ϱ be a cluster contraction-based consolidation operator. Then, Ψ_ϱ is a kernel contraction-based consolidation operator.*

To prove that Ψ_ϱ is a kernel contraction-based merging operator it is enough to show that cluster incision functions are also kernel incision functions; if we consider the tables in Examples 5 and 6 showing all possible kernel and cluster incisions, respectively, we can see that every possible cluster incision is indeed a kernel incision. The converse from Proposition 2 does not hold, as kernel incision functions are not necessarily cluster incision functions, since the former not always satisfy the last condition from Definition 6 as we illustrate below.

Example 7. Consider belief base K from Example 2 and suppose we have $\Upsilon_\rho(K) = K \setminus \rho(K^{\perp\!\!\perp}\bot) = K \setminus \{q, r\}$. The kernel incision function that gives rise to this operator performs the following incisions: $\rho(\kappa_1) = \{r\}$ and $\rho(\kappa_2) = \{q\}$. Note that ρ is not a valid cluster incision function since $|\kappa_2 \cap \rho(K^{\perp\!\!\perp\!\!\perp}\bot)| = \{q, r\}$, *i.e.*, $|\kappa_2 \cap \rho(K^{\perp\!\!\perp\!\!\perp}\bot)| > 1$. Furthermore, it is not possible for any valid cluster incision function to yield this result.

As hinted by Examples 2 and 6, a benefit of using cluster-based consolidation operators over (smooth) kernel-based ones is that unnecessary deletions can be avoided by clustering conflicts. The characteristics of the choices made by cluster incision functions have an important impact on the number of formulæ deleted: we can show that for any kernel contraction-based consolidation operator there is a cluster contraction-based one that removes at most the same number of formulæ than the former; the following proposition formalizes the result.

Proposition 3. *Let K be a belief base. Then, for any kernel contraction-based consolidation operator Υ_ρ over K there exists a cluster contraction-based consolidation operator Ψ_ϱ over K such that $\Upsilon_\rho(K) \subseteq \Psi_\varrho(K)$.*

Above, we have shown that cluster-based operators are refinements of "pure" kernel contraction-based ones. We can shown that cluster incision functions refines smooth incision functions as well, and hence the operators based on them.

Proposition 4. *If Ψ_ϱ is a cluster contraction-based consolidation operator, then Ψ_ϱ is a smooth kernel contraction-based consolidation operator.*

The converse of Proposition 4 does not hold; consider the following example.

Example 8. Consider a belief base $K = \{p, q, r, p \rightarrow \neg q, p \rightarrow \neg r\}$. We have that $K^{\perp\!\!\!\perp}\bot = \{\kappa_1, \kappa_2\}$ where $\kappa_1 = \{p, q, p \rightarrow \neg q\}$ and $\kappa_2 = \{p, r, p \rightarrow \neg r\}\}$, and thus $\perp\!\!\!\perp\!\!\!\perp(K) = \{\varsigma_1\}$ where $\varsigma_1 = \{p, q, r, p \rightarrow \neg q, p \rightarrow \neg r\}$. Now, consider an incision function $\rho(K^{\perp\!\!\!\perp}\bot) = \{p, r\}$. We can see that ρ satisfies smoothness (cf. Def. 2). However, we have that $\rho(K^{\perp\!\!\!\perp}\bot) \cap \kappa_2$ is not a singleton. Then, ρ is not a cluster incision function. Note that, once we choose to remove p from K, it is unnecessary to remove r, and any valid cluster incision function will avoid that.

From the previous example we can conclude that smooth incision functions, although a proper refinement of kernel incision functions, can still produce unnecessary loss of information. In the next section we characterize a notion of optimality of incision functions and position our proposal with respect to AGM-based approaches to consolidation.

5 Connection with Maxichoice Contraction-Based Consolidation

In this section we further analyze cluster incision functions and the consolidation operators based on them; particularly, we focus in the relationship with maxichoice contraction-based consolidation, as maxichoice contractions [1] are as conservative as possible. Maxichoice contraction is based on the use of selection functions that select one among all possible maximal consistent subsets of the knowledge base. Maxichoice contraction-based consolidation operators are those based on maxichoice contraction in the same manner as the operators defined previously. To formally characterize optimality of incision functions, we recall the notion of *minimality* from [9] and adapt it for cluster incision functions.

Definition 8 (Minimality). *An incision function ϱ for a belief base K is minimal if no proper subset of $\varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\bot)$ defines an incision function.*

Next we show that cluster incision functions are minimal.

Proposition 5. *Let ϱ be an incision function. Then, ϱ is a cluster incision function iff it is a minimal incision function.*

The proof for Proposition 5 is based on the fact that for every cluster $X \in K^{\perp\!\!\!\perp\!\!\!\perp}\bot$ such that $X \cap \varrho(K^{\perp\!\!\!\perp\!\!\!\perp}\bot) = A$ no proper subset of A in itself restores consistency, and hence no subset of it gives raise to a proper incision function.

The relationship between selection and incision functions was previously analyzed in [9]. As noted there, minimality of incision functions corresponds to maximality of contraction; a direct result of this is the following proposition.

Proposition 6 (adapted from [9]). *Let ϱ be an incision function, Ψ_ϱ be its associated consolidation operator and K a belief base. Then, ϱ is a minimal incision function iff there exists $H \subseteq K$ such that $H = K \setminus \Psi_\varrho(K)$ where (1) H is consistent, and (2) there is no $H \subsetneq H' \subsetneq K$ such that H' is consistent.*

Proposition 6 states that there is a one-to-one correlation between maximal consistent subsets of a K and minimal incision functions. More specifically, the result in [9] shows that any kernel contraction-based on minimal incision functions is a maxichoice contraction. The proof in [9] can be slightly modified for our setting; intuitively the validity of this result relies on the fact that if there is a subset H of K that is maximally consistent, then adding any further formulæ will make it inconsistent, thus, any incision ϱ such that $\varrho \cap K = H$ must be minimal; if this were not the case then there would exist a subset $\varrho' \subsetneq \varrho$ such that $K \setminus \varrho'$ is consistent, and therefore H would not be maximally consistent since $K \setminus H \subsetneq K \setminus \varrho'$. Conversely, if an incision ϱ is minimal then it generates a maximally consistent subset because no proper subset of ϱ is an incision function, *i.e.,* for any $A \subsetneq \varrho$ it holds that $K \setminus A$ is inconsistent.

Although Falappa *et al.* elaborate on the relationship between kernel and maxichoice contractions further, no class of incision functions satisfying minimality (thus corresponding to maxichoice contractions) is identified. As a corollary of the previous results we can conclude that our approach is equivalent to consolidation through maxichoice contraction, but arising from minimal incision functions, which means that operators retain as much information as possible.

Corollary 1. *Ψ_ϱ is a cluster contraction-based consolidation operator iff Ψ_ϱ is a maxichoice contraction-based consolidation operator.*

Discussion. Corollary 1 completes the spectrum of possibilities arising from the treatment of minimal conflicts. Nevertheless, it is important to note that, although our approach is equivalent to consolidation through maxichoice contraction in terms of the final belief base obtained, there is still importance in the difference in how this belief base is obtained. While maxichoice operators have to deal with maximal consistent subsets, ours deal with minimal inconsistent ones. There is an interesting ongoing discussion about which approach is better. In [20] examples are shown that indicate that for some instances it is faster to use kernel contraction while for others it is faster to use partial meet (or maxichoice) contraction. As noticed in [20], whether it is possible to detect when it is better to use one or the other method is still an open problem. It can be argued that the final choice will depend on the application environment and the language selected; clearly, different needs from the point of users prompt choosing one approach over the other. As an example, consider the setting from [17], where knowledge bases are relational databases considered together with functional dependencies, there it is possible to efficiently (polynomial time in the number of tuples in the database, assuming a fixed schema) compute and maintain the set of clusters by means of indexes; alternatively, in such setting an inconsistency management approach based on the manipulation of maximal consistent subsets (other than simply computing one of them) would require higher computational effort and possibly the utilization of tools outside the DBMS.

6 Related Work

The problem of inconsistency handling has been addressed differently over the years in diverse environments, *e. g.,* relational databases, propositional knowledge bases or fragments of first order logics such as logic programming.

As stated in the introduction, the area of Belief Revision has undoubtedly produced great advances in the handling of inconsistencies. Particularly, the work by Hansson [11] is the foundation and inspiration for this paper. More recently, the work in [13] presents an approach for merging belief bases, stemming from inconsistency minimization, which removes exactly one formula in each minimal inconsistent subset of formulas. As shown, to remove one formula from minimal inconsistent sets may not be sufficient to ensure that nothing is given up without reason: it is still important *how* such formula is chosen, considering other minimal inconsistent sets as well (cf. Examples 2 and 6). As shown in the paper, the structure of cluster helps in such choice by only considering optimal incisions, minimizing loss of information. In [8] an approach for revising a propositional knowledge base by a set of sentences is presented, where every sentence in the input set can be independently accepted but there may exist inconsistencies when considering the whole set. The main difference between this work and ours is that they first solve inconsistencies in the set of sentences, in this manner they can decide which subset of it will characterize the revision. Furthermore, in our approach no preponderance to particular formulæ is given in the process as our proposal is based on consolidation instead of revision.

Also within Artificial Intelligence, the works by Baral *et al.* in knowledge bases combination [3], Brewka's preferred subtheories [7], and several other works on entailment from inconsistent knowledge bases such as [4,5], are based on the idea of selecting maximal subsets of the knowledge base (or the combination of several ones) that are consistent *w.r.t.* a set the integrity constraints. All of these approaches can be defined in the AGM framework as specific partial meet contraction functions by adequately specifying the selection function. As shown in the previous section, our operators are equivalent to operators based on maxichoice contraction, a specific class of partial meet contraction.

In the area of Databases one of the most influential works is the one by Arenas *et al.* [2] on *Consistent Query Answering.* Their treatment of inconsistencies does not attempt to obtain a consistent database, instead, the consistent answers to a query correspond to the set of (classical) answers to the query in *every repair* of the inconsistent database, which are the consistent subsets (or supersets, depending on the type of integrity constraints) of the original database that differs minimally from it. Similar in spirit are some of the syntactic approaches analyzed in [6]. Unlike our approach, these approaches can be seen as an “on the fly” consistency restoration, guided for particular queries, targeting the subset of the knowledge that matters for that query and not the whole knowledge base.

Finally, regarding the use of clusters the most closely related research to the work presented here is the one by Lukasiewicz *et al.* [16]. There, the authors define a general framework for inconsistency-tolerant query answering in Datalog+/− ontologies based on the notion of incision functions. Besides the obvious difference in the language, the aims of their work and ours are clearly different; their work follows the same idea of Arenas *et al.* [2], focusing on enforcing consistency at query time obtaining (lazy) consistent answers. Clearly, this process must be carried on for every query posed to the system, while our approach allows to obtain a new knowledge base that can be queried without considering inconsistency issues. As usual the choice of one approach over the other heavily depend on the application environment.

7 Conclusions and Future Work

In this paper we focus on an approach to consistency restoration (consolidation) of belief bases defined on terms of belief base contractions [1]. We developed a new class of belief consolidation operators, called cluster contraction-based operators, based on incision functions that aims for a globally efficient conflict resolution. The results show that a cluster contraction-based consolidation operator do not only yields a consistent belief base, as expected, but also does it satisfying minimality requirements regarding loss of information.

This family of operators are defined based on *cluster incision functions*. We have shown that cluster incision functions are refinements of smooth kernel incision functions, which implies that cluster contraction-based operators are at least as efficient as (smooth) kernel contraction-based operators from the point of view of minimal loss of information in the consolidation process. Furthermore, we show that our operators are equivalent to consolidation operators based on maxichoice contraction, completing the spectrum of possibilities for approaches arising from the treatment of minimal conflicts. As recent findings indicates [20], in some cases it is better to use kernel-based approaches, while in others the contrary holds. Clearly, the choice of one approach over the other depend on particular aspects of the application environment.

For future work, we plan to implement the different operators and perform empirical trials over different scenarios. Also, in this first step towards the formalization of this new class of consolidation operators we have not considered any form of ranking in the definition of the incision functions. In the future, we plan to define entrenchment relations in terms of orderings among formulæ based on generic measures, and to study the merits of extensions of cluster incision function to account for such orderings. Furthermore, we intend to analyze the behavior of the operators for particular measures, for instance measures of amount of information in a knowledge base in the presence of inconsistency (*e. g.,* [15]) and measures of the degree of inconsistency (as considered in [14]).

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