An Evolutionary Optimization Approach to Risk Parity Portfolio Selection

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Abstract

In this paper we present an evolutionary optimization approach to solve the risk parity portfolio selection problem. While there exist convex optimization approaches to solve this problem when long-only portfolios are considered, the optimization problem becomes non-trivial in the long-short case. To solve this problem, we propose a genetic algorithm as well as a local search heuristic. This algorithmic framework is able to compute solutions successfully. Numerical results using real-world data substantiate the practicability of the approach presented in this paper.

1 Introduction

The portfolio selection problem is concerned with finding an optimal portfolio x of assets from a given set of n risky assets out of a pre-specified asset universe such that the requirements of the respective investor are met. In general, investors seek to optimize their portfolio in regard of the trade-off between return and risk, such that the meta optimization problem can be formulated as shown in Eq. (1).

$$\begin{array}{ll} \text{minimize} & \text{Risk}(x) \\ \text{maximize} & \text{Return}(x) \end{array} \tag{1}$$

This bi-criteria optimization problem is commonly reduced to a single-criteria problem by just focusing on the risk and constraining the required mean, i.e. the investor sets a lower expected return target μ , which is shown in Eq. (2).

minimize
$$\operatorname{Risk}(x)$$

subject to $\operatorname{Return}(x) \ge \mu$ (2)

Markowitz [12] pioneered the idea of risk-return optimal portfolios using the standard deviation of the portfolios profit and loss function as risk measure. In this case, the optimal portfolio x is computed by solving the quadratic optimization problem shown in Eq. 3. The investor needs to estimate a vector of expected returns r of the assets under consideration as well as the covariance matrix \mathbb{C} . Finally the minimum return target μ has to be defined. Any standard quadratic programming solver can be used to solve this problem numerically.

$$\begin{array}{ll} \text{minimize} & x^T \mathbb{C} x \\ \text{subject to} & r \times x \ge \mu \\ & \sum x = 1 \end{array} \tag{3}$$

While this formulation has been successfully applied for a long time, criticism has sparked recently. This is especially due to the problem of estimating the mean vector. To overcome this problem one seeks optimization model formulations that solely depend on the covariance matrix. Sometimes even simpler approaches are favored, e.g. the 1-over-N portfolio, which equally weights every asset under consideration. It has been shown that there are cases, where this simple strategy outperforms clever optimization strategies, see e.g. DeMiguel et al. [7].

Of course, the Markowitz problem can be simplified to a model without using returns easily by dropping the minimum return constraint. In this case one receives the Minimum Variance Portfolio (MVP), which is overly risk-averse.

One important technique used for practical portfolio purposes are risk-parity portfolios, where the assets are weighted such that they equally contribute risk to the overall risk of the portfolio. The properties of such portfolios are discussed by Maillard et al. [11] and alternative solution approaches are shown by Chaves et al., see [5] and [6], as well as Bai et al. [1].

In this paper, an evolutionary optimization approach to compute optimal risk parity portfolios will be presented. Evolutionary optimization approaches have been shown to be useful for solving a wide range of different portfolio optimization problems, see e.g. [15] or [8] and the references therein. See also the series of books on Natural Computing in Finance for more examples [2], [3], [4].

This paper is organized as follows. Section 2 describes the risk-parity problem in detail, Section 3 presents the evolutionary algorithm developed for solving the problem, and Section 4 presents numerical results. Finally, Section 5 concludes the paper.

2 Risk Parity Portfolio Selection

The type of risk-parity portfolios discussed in this paper are also called Equal Risk Contribution (ERC) portfolios. The idea is to find a portfolio where the assets are weighted such that they equally contribute risk to the overall risk of the portfolio.

We follow Maillard et al. [11] in their definition of risk contribution, i.e. reconsider the above mentioned portfolio $x = (x_1, x_2, ..., x_n)$ of n risky assets. Let \mathbb{C} be the covariance matrix, σ_i^2 the variance of asset i, and σ_{ij} the covariance between asset i and j. Let $\sigma(x)$ be the risk (i.e. standard deviation) of the portfolio as defined in Eq. (4).

$$\sigma(x) = \sqrt{x^T \mathbb{C}x} = \sum_i x_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} x_i x_j \sigma_{ij}. \tag{4}$$

Then the marginal risk contributions $\partial_{x_i}\sigma(x)$ of each asset i are defined as follows

$$\partial_{x_i}\sigma(x) = \frac{\partial\sigma(x)}{\partial x_i} = \frac{x_i\sigma_i^2 + \sum_{j\neq i} x_j\sigma_{ij}}{\sigma(x)}.$$

If we are considering long-only portfolios then the optimal solution can be written as an optimization problem containing a logarithmic barrier term which is shown in Eq. (5) and where c is an arbitrary positive constant. See e.g. also [16] for an alternative formulation. In this long-only case, a singular optimal solution can be computed.

minimize
$$x^T \mathbb{C}x - c \sum_{i=1}^n \ln x_i$$

subject to $x_i > 0$. (5)

However, if we want to include short positions then we need to find solutions in other orthants than in the non-negative orthant. See Bai et al. [1] for a log-barrier approach in this case, which is shown in Eq. (6).

minimize
$$x^T \mathbb{C} x - c \sum_{i=1}^n \ln \beta_i x_i$$

subject to $\beta_i x_i > 0$, (6)

where $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \{-1, 1\}^n$ defines the orthant where the solution should be computed. For each choice of β the above optimization problem is convex and can be solved optimally. However, as shown in [1] there are 2^n different solutions. Investors may add additional constraints to specify their needs, however this cannot be modeled as one convex optimization problem, which is why an evolutionary approach is presented here. The general formulation of the long-short risk parity portfolio problem can be formulated as Eq. (7) as shown in [11].

minimize
$$\sum_{i=1,j=1}^{n} (x_i(\mathbb{C}x)_i - x_j(\mathbb{C}x)_j)^2$$
subject to
$$a_i \leq x_i \leq b_i,$$
$$\sum_{i=1}^{n} x_i = 1.$$
 (7)

3 Implementation

The solution is computed in two steps. First, a genetic algorithm will be employed and afterwards a local search algorithm will be applied.

3.1 Genetic Algorithm

We are using a standard genetic algorithm to compute risk-parity optimal portfolios. The algorithm was implemented using the statistical computing language R [13].

The fitness definition in the risk-parity setting is given by the deviance of each risk contribution from the mean of all risk contributions. Let us use the shorthand notation of $\Delta_i = \partial_{x_i} \sigma(x)$, so we compute the expectation $\Delta = \mathbb{E}(\Delta_i)$ and define the fitness f as the sum of the quadratic distance of each risk contribution from the mean. This non-negative fitness value f has to be minimized, where

$$f = \sum_{i} (\Delta_i - \Delta)^2$$

We use a genotype-phenotype equivalent formulation, i.e. we use chromosomes of length n which contain the specific portfolio weights of the n risky assets. Thus, an important operator is the repair operator, i.e. the sum of the portfolio is normalized to 1 after each operation.

The genetic operators used in the algorithm can be summarized as follows:

Elitist selection: The best n_{ES} chromosomes of each population are kept in the population.

Mutation: A random selection of n_M chromosomes of the parent population will be mutated. Up to a number of 15% of the length of the respective chromosome will be changed to a random value between the portfolio bounds. Let ℓ be the length of the chromosome. First, a random number between 0 and 0.15 is drawn. This number is multiplied by ℓ and rounded up to the next integer value. This value represents the number of genes to be mutated. The mutation positions will be chosen randomly. Afterwards the randomly selected positions will be replaced with a random value between the upper and the lower investment limit of the respective asset.

Random addition: n_R new and completely random chromosomes are added to each new population.

Intermediate crossover: Two chromosomes from the parent population will be randomly selected for an intermediate crossover. The mixing parameter between the two chromosomes will also be chosen randomly. n_{IC} crossover children will be added to the next population. Let the mixing parameter be α and the two randomly chosen parent chromosomes p_1 and p_2 with genes $p_{1,1}, \ldots, p_{1,\ell}$ and $p_{2,1}, \ldots, p_{2,\ell}$ respectively, where ℓ is the length of the chromosome. An intermediate crossover will result in a child chromosome c where the genes are set to

$$c_i = \alpha p_{1,i} + (1 - \alpha) p_{2,i} \quad \forall i = 1, \dots, \ell.$$

3.2 Local Search

In a second step, a local search algorithm is applied to the best solution of the genetic algorithm. Thereby, within each iteration of the algorithm each asset weight of the n assets of the portfolio is increased or decreased by a factor ε . Each of these $(2 \times n)$ new portfolios is normalized and if one exhibits a lower fitness value then this new portfolio will be used subsequently. The algorithm terminates if no local improvement is possible anymore or the maximum number of iterations has been reached.

4 Numerical Results

In this section the above described algorithm will be applied to real-world financial data to obtail numerical results, which can be used for practical portfolio optimization purposes. The first test using stock data from the DJIA index is described in Section 4.1 and both the long-only case (Section 4.2) as well as the long-short case (Section 4.3) is discussed. To check for scalability the algorithm is tested on all stocks of the S&P 100 index in Section 4.4 afterwards.

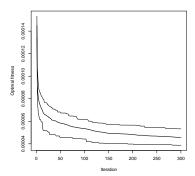
4.1 Financial Data and Setup

We use data from all stocks from the Dow Jones Industrial Average (DJIA) index using the composition of September 20, 2013, i.e. using the stocks with the ticker symbols AXP, BA, CAT, CSCO, CVX, DD, DIS, GE, GS, HD, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, NKE, PFE, PG, T, TRV, UNH, UTX, V, VZ, WMT, XOM.

Using the R package quantmod [14] we obtain daily adjusted closing data from Yahoo! Finance. We use data from the beginning of 2010 until the beginning of November 2014 to compute the Variance-Covariance matrix, i.e. the matrix is entirely based on historical data. The data is solely used for comparison purposes such that a clever approximation algorithm for the Variance-Covariance matrix like those presented e.g. by [9] and [10] is not necessary for the purpose of

Table 1: Parameters for the Genetic Algorithm.

Parameter	Value
Initial population size	200
Maximum iterations	300
Elitist selection	10 top chromosomes from parent population
Random addition	50 new chromosomes
Mutation	100 chromosomes from parent population
Intermediate crossover	100 pairs of chromosomes from parent population



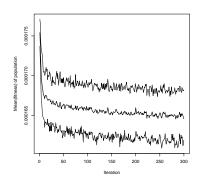


Figure 1: Convergence of the genetic algorithm in the long-only case, i.e. the best (left) and the mean (right) fitness value of each iteration along with the 5% as well as the 95% quantile of 100 instances.

this study. However it should be noted that the matrix is the important input parameter for the calculation.

The parameters used for the genetic algorithm are shown in Table 1. The local search algorithm was started twice, once with $\varepsilon = 0.01$ and subsequently with $\varepsilon = 0.001$. The number of maximum local search steps has been set to 500.

4.2 Computing DJIA Long-Only Portfolios

First, we compute a set if various long-only portfolios without using expected returns, i.e. the Minimum Variance Portfolio (MVP), the 1/N portfolio as well as the risk-parity portfolio using the algorithm developed in this paper and described above. The results is shown in Table 2. Please note that the risk contribution has been normalized to 1. The fitness of the 1/N portfolio is 0.002253031, while the MVP exhibits a fitness of 0.00057129. The algorithm managed to find the Risk Parity portfolio with a fitness of 0.0005019655. A lower fitness is not possible due to the long-only constraint.

Furthermore, the convergence results in the long-only case can be seen in Fig. 1. The left picture shows the best fitness over 300 iterations, while the right picture shows the mean of the population fitness. The middle line depicts the mean of 100 instances while the upper and the lower line depict the 5% as well as the 95% quantile of the instances.

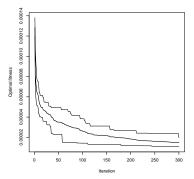
In the long-only case, a simple random multi-start local search algorithm like the one described in Section 3.2 above leads to the same result. We tested this by running it 100 times and figured out that both the GA+Local as well as the Random+Local approach led to the same

Table 2: DJIA - Long Only - MVP, 1/N, and Risk Parity.								
	x(MVP)	RCn(MVP)	x(1/N)	RCn(1/N)	x(RP)	RCn(RP)		
AXP	0.0000	0.0408	0.0300	0.0444	0.0000	0.0404		
BA	0.0000	0.0374	0.0300	0.0411	0.0000	0.0366		
~ A m	0.0000	0.0400	0.0000	0.0404	0.0000	0.0419		

AXP 0.0000 0.0408 0.0300 0.0444 0.0000 0.0404 BA 0.0000 0.0374 0.0300 0.0411 0.0000 0.0366 CAT 0.0000 0.0420 0.0300 0.0484 0.0000 0.0413 CSCO 0.0000 0.0338 0.0300 0.0382 0.0000 0.0341 DD 0.0000 0.0382 0.0300 0.0410 0.0000 0.0371 DIS 0.0000 0.0382 0.0300 0.0410 0.0000 0.0376 DIS 0.0000 0.0383 0.0300 0.0416 0.0000 0.0384 GE 0.0000 0.0370 0.0300 0.0416 0.0000 0.0323 GS 0.0000 0.0323 0.0300 0.0416 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0416 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0283 0.0000 0.0323 JN		x(MVP)	RCn(MVP)	x(1/N)	RCn(1/N)	x(RP)	RCn(RP)
CAT 0.0000 0.0420 0.0300 0.0484 0.0000 0.0413 CSCO 0.0000 0.0338 0.0300 0.0382 0.0000 0.0329 CVX 0.0000 0.0345 0.0300 0.0357 0.0000 0.0341 DD 0.0000 0.0382 0.0300 0.0410 0.0000 0.0376 DIS 0.0000 0.0383 0.0300 0.0416 0.0000 0.0384 GE 0.0000 0.0370 0.0300 0.0451 0.0000 0.0356 HD 0.0000 0.0370 0.0300 0.0451 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0319 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0319 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0353 0.0000 0.0312 JPM 0.0000 0.0424 0.0300 0.0218 0.0376 0.0257 J	AXP	0.0000	0.0408	0.0300	0.0444	0.0000	0.0404
CSCO 0.0000 0.0338 0.0300 0.0382 0.0000 0.0349 CVX 0.0000 0.0345 0.0300 0.0357 0.0000 0.0341 DD 0.0000 0.0382 0.0300 0.0410 0.0000 0.0376 DIS 0.0000 0.0383 0.0300 0.0394 0.0000 0.0384 GE 0.0000 0.0395 0.0300 0.0416 0.0000 0.0395 GS 0.0000 0.0370 0.0300 0.0451 0.0000 0.0356 HD 0.0000 0.0323 0.0300 0.0319 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0283 0.0000 0.0272 INTC 0.0000 0.0312 0.0300 0.0353 0.0000 0.0305 JPM 0.0000 0.04244 0.0300 0.0218 0.0376 0.0257 JPM 0.0000 0.04244 0.0300 0.0255 0.0275 0.0341 <td< td=""><td>BA</td><td>0.0000</td><td>0.0374</td><td>0.0300</td><td>0.0411</td><td>0.0000</td><td>0.0366</td></td<>	BA	0.0000	0.0374	0.0300	0.0411	0.0000	0.0366
CVX 0.0000 0.0345 0.0300 0.0357 0.0000 0.0341 DD 0.0000 0.0382 0.0300 0.0410 0.0000 0.0376 DIS 0.0000 0.0383 0.0300 0.0394 0.0000 0.0384 GE 0.0000 0.0395 0.0300 0.0416 0.0000 0.0356 HD 0.0000 0.0323 0.0300 0.0451 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0283 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0283 0.0000 0.0328 IBM 0.0207 0.0285 0.0300 0.0283 0.0000 0.0272 INTC 0.0000 0.0312 0.0300 0.0283 0.0000 0.0305 JNJ 0.2015 0.0285 0.0300 0.0218 0.0376 0.0257 JPM 0.0000 0.0424 0.0300 0.0502 0.0000 0.0417 K	CAT	0.0000		0.0300		0.0000	0.0413
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JPM 0.0000 0.0424 0.0300 0.0502 0.0000 0.0417 KO 0.0038 0.0285 0.0300 0.0255 0.0275 0.0334 MCD 0.2421 0.0285 0.0300 0.0195 0.2333 0.0288 MMM 0.0000 0.0345 0.0300 0.0359 0.0000 0.0340 MRK 0.0000 0.0301 0.0300 0.0274 0.0000 0.0299 MSFT 0.0000 0.0308 0.0300 0.0327 0.0000 0.0307 NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 TRV 0.0000 0.0317 0.0300 0.0328 0.0330 0.0228 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361	INTC	0.0000	0.0312	0.0300	0.0353	0.0000	0.0305
KO 0.0038 0.0285 0.0300 0.0255 0.0275 0.0334 MCD 0.2421 0.0285 0.0300 0.0195 0.2333 0.0288 MMM 0.0000 0.0345 0.0300 0.0359 0.0000 0.0340 MRK 0.0000 0.0301 0.0300 0.0274 0.0000 0.0299 MSFT 0.0000 0.0308 0.0300 0.0327 0.0000 0.0307 NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0324 0.0000 0.0322 UNH 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V<	JNJ	0.2015	0.0285	0.0300	0.0218	0.0376	0.0257
MCD 0.2421 0.0285 0.0300 0.0195 0.2333 0.0288 MMM 0.0000 0.0345 0.0300 0.0359 0.0000 0.0340 MRK 0.0000 0.0301 0.0300 0.0274 0.0000 0.0299 MSFT 0.0000 0.0308 0.0300 0.0327 0.0000 0.0307 NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ<	$_{ m JPM}$	0.0000	0.0424	0.0300	0.0502	0.0000	0.0417
MMM 0.0000 0.0345 0.0300 0.0359 0.0000 0.0340 MRK 0.0000 0.0301 0.0300 0.0274 0.0000 0.0299 MSFT 0.0000 0.0308 0.0300 0.0327 0.0000 0.0307 NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT<	KO	0.0038	0.0285	0.0300	0.0255	0.0275	0.0334
MRK 0.0000 0.0301 0.0300 0.0274 0.0000 0.0299 MSFT 0.0000 0.0308 0.0300 0.0327 0.0000 0.0307 NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0364 0.0300 0.0324 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	MCD	0.2421	0.0285	0.0300	0.0195	0.2333	0.0288
MSFT 0.0000 0.0308 0.0300 0.0327 0.0000 0.0307 NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	MMM	0.0000	0.0345	0.0300	0.0359	0.0000	0.0340
NKE 0.0000 0.0343 0.0300 0.0365 0.0000 0.0347 PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	MRK	0.0000	0.0301	0.0300	0.0274	0.0000	0.0299
PFE 0.0000 0.0306 0.0300 0.0289 0.0000 0.0300 PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	MSFT	0.0000	0.0308	0.0300	0.0327	0.0000	0.0307
PG 0.1890 0.0285 0.0300 0.0187 0.3050 0.0322 T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	NKE	0.0000	0.0343	0.0300	0.0365	0.0000	0.0347
T 0.0745 0.0285 0.0300 0.0228 0.0330 0.0288 TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	PFE	0.0000	0.0306	0.0300	0.0289	0.0000	0.0300
TRV 0.0000 0.0317 0.0300 0.0308 0.0000 0.0322 UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0176 0.2565 0.0312 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312		0.1890	0.0285	0.0300	0.0187	0.3050	0.0322
UNH 0.0000 0.0305 0.0300 0.0324 0.0000 0.0293 UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0222 0.1072 0.0304 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	${ m T}$	0.0745	0.0285	0.0300	0.0228	0.0330	
UTX 0.0000 0.0364 0.0300 0.0382 0.0000 0.0361 V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0222 0.1072 0.0304 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	TRV	0.0000	0.0317	0.0300	0.0308	0.0000	0.0322
V 0.0000 0.0330 0.0300 0.0360 0.0000 0.0320 VZ 0.0554 0.0285 0.0300 0.0222 0.1072 0.0304 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312	UNH	0.0000	0.0305	0.0300	0.0324	0.0000	0.0293
VZ 0.0554 0.0285 0.0300 0.0222 0.1072 0.0304 WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312		0.0000	0.0364	0.0300	0.0382	0.0000	0.0361
WMT 0.2130 0.0285 0.0300 0.0176 0.2565 0.0312		0.0000	0.0330	0.0300	0.0360	0.0000	0.0320
			0.0285	0.0300	0.0222	0.1072	
XOM 0.0000 0.0325 0.0300 0.0326 0.0000 0.0323	WMT	0.2130	0.0285	0.0300	0.0176	0.2565	0.0312
	XOM	0.0000	0.0325	0.0300	0.0326	0.0000	0.0323

Table 3: DJIA - Long-Short - Risk Parity.

	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD
X	-0.065	-0.010	-0.039	0.000	-0.015	-0.042	-0.060	-0.071	0.034	0.019
RCn	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033
	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE
X	0.073	0.024	0.247	-0.050	0.010	0.257	0.015	0.012	0.021	-0.004
RCn	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033
	PFE	PG	Τ	TRV	UNH	UTX	V	VZ	WMT	XOM
X	0.016	0.185	0.102	0.027	0.016	-0.020	0.013	0.076	0.211	0.019
RCn	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033



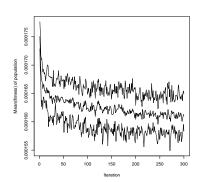


Figure 2: Convergence of the genetic algorithm in the long-short case, i.e. the best (left) and the mean (right) fitness value of each iteration along with the 5% as well as the 95% quantile of 100 instances.

optimal portfolio in all cases. However, the optimal solution of the genetic algorithm needed significantly less iterations compared to starting from random solutions. A statistical t-test returned t=-60.5674 (df=183.198) and a p-value of 0 with respect to the number of local search iterations. However, this is different in the long-short case, which is described in the next section.

4.3 Computing DJIA Long-Short Portfolios

In the long-short case, a random multi-start local search heuristic does not return any useful result. However, the evolutionary approach works well. The long-short result with a lower bound of -0.2 is shown in Table 3. The convergence results in the long-short case can be seen in Fig. 2.

4.4 Scalability

To test for scalability of the algorithm, we used stocks from the S&P 100 index as of March 21, 2014. Again, we use historical data from the beginning of 2010 until the beginning of November 2014 to compute our Variance-Covariance matrix. Four stocks have been excluded due to data issues, i.e. ABBV, FB, GM, and GOOG, such that the stocks with the following ticker symbols have been considered: AAPL, ABT, ACN, AIG, ALL, AMGN, AMZN, APA, APC, AXP, BA,

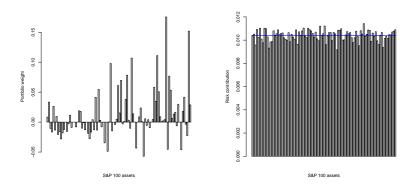


Figure 3: S&P 100 - portfolio (left) and risk contribution (right).

BAC, BAX, BIIB, BK, BMY, BRK.B, C, CAT, CL, CMCSA, COF, COP, COST, CSCO, CVS, CVX, DD, DIS, DOW, DVN, EBAY, EMC, EMR, EXC, F, FCX, FDX, FOXA, GD, GE, GILD, GS, HAL, HD, HON, HPQ, IBM, INTC, JNJ, JPM, KO, LLY, LMT, LOW, MA, MCD, MDLZ, MDT, MET, MMM, MO, MON, MRK, MS, MSFT, NKE, NOV, NSC, ORCL, OXY, PEP, PFE, PG, PM, QCOM, RTN, SBUX, SLB, SO, SPG, T, TGT, TWX, TXN, UNH, UNP, UPS, USB, UTX, V, VZ, WAG, WFC, WMT, XOM.

The lower bound was set to -0.2. Fig. 3 shows the resulting portfolio as well as the risk contribution of the assets. It can be seen that the algorithm arrives at a solution, which exhibits a rather exact risk parity solution with only slight differences from a perfect solution, which can be observed in the right plot of Fig. 3. To get a more detailed picture on the scalability, a clearer analysis of the proportion between the contribution of the evolutionary solution as well as the local search to the final solution would have to be accomplished, but this will be left out for future research. From an investor's perspective the optimal portfolio solution exhibits quite a few number of assets, which would have to be shorted. To make the solution more realistic at least a net exposure constraint would have to be added. A cardinality constraint on the number of shorted assets would also be an option. Both constraints can be integrated rather easily in the evolutionary context, see e.g. [19], [17], and [18]. However, such constraints would disable the possibility to obtain a perfect risk parity solution, which was the aim of the algorithm presented in this paper.

5 Conclusion

In this paper, we presented an evolutionary approach to compute optimal risk parity portfolios. This algorithm was designed to overcome the problem that only the long-only case can be solved conveniently using convex optimization models. A two-step approach using a genetic algorithm as well as a local search technique proved to be successful, especially in the long-short case. Another advantage is that further constraints can be integrated directly into the algorithm and this approach can be extended to other risk measures as well.

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