Two-Point Boundary Problems with One Mild Singularity and an Application to Graded Kirchhoff Plates

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Abstract. We develop a new theory for treating boundary problems for linear ordinary differential equations whose fundamental system may have a singularity at one of the two endpoints of the given interval. Our treatment follows an algebraic approach, with (partial) implementation in the Theorema software system (which is based on Mathematica). We study an application to graded Kirchhoff plates for illustrating a typical case of such boundary problems.

Keywords: Linear boundary problems, singular boundary problems, generalized Green's operator, Green's functions, integro-differential operators, ordinary differential equations, Kirchhoff plates, functionally graded materials

1 Introduction

The treatment of boundary problems in symbolic computation was initiated in the PhD thesis [8] under the guidance of Bruno Buchberger in cooperation with Heinz Engl; see also [10,9] and for the further development [11,7,12]. Its implementation was originally carried out within the TH∃OREM∀ project [1].

Up to now, we have always assumed differential equations without singularity or, equivalently, monic differential operators (leading coefficient function being unity). In this paper, we develop for the first time an *algebraic theory* for treating boundary problems with a (mild) *singularity at one endpoint*. For details, we refer to Section 3. Our approach is very different from the traditional analysis setting in terms of the Weyl-Titchmarsh theory (limit points and limit circles). It would be very interesting to explore the connections between our approach and the classical treatment; however, this must be left for future work.

Regarding the general setup of the *algebraic language for boundary problems*, we refer to the references mentioned above, in particular [12]. At this point, let us just recall some notation. We start from a fixed integro-differential algebra $(\mathcal{F}, \partial, {\textstyle\int})$. The formulation of (local) boundary conditions is in terms of *evaluations*, which are by definition multiplicative linear functionals in $\mathcal{F}^*$. We write $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1$ is a *subspace* of $\mathcal{F}_2$; the same notation is used for subspaces of the dual space $\mathcal{F}^*$. The *orthogonal* of a subspace $\mathcal{B} \leq \mathcal{F}^*$ is defined as

$$\mathcal{B}^\perp = \{u \in \mathcal{F} \mid \beta(u) = 0 \text{ for all } \beta \in \mathcal{B}\}$$

and similarly for the orthogonal of a subspace $\mathcal{A} \leq \mathcal{F}$. We write $[f_1, f_2, \dots]$ for the *linear span* of (possibly infinitely many) elements $f_1, f_2, \dots \in \mathcal{F}$; the same notation is used for linear spans within $\mathcal{F}^*$. The *zero vector space* of any dimension is denoted by O.

We write

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1)$$

for the *falling factorial*, where $n \in \mathbb{C}$ could be arbitrary (but will be an integer for our purposes) while k is taken to be a natural number. Note that $n^{\underline{k}} = 0$ for $k > n$. Our main example of an integro-differential algebra will be the space $C^\omega(I)$ of analytic functions on a closed interval $I = [a, b]$. Recall that by definition this is the space of functions that are analytic in some open set containing $[a, b]$.

The development of the new algebraic theory of two-point boundary problems with a mild singularity (whose treatment is really just broached in this paper) was triggered by a collaboration between a *symbolic computation* team (consisting of the first, second and fourth author) and a researcher in *engineering mechancs* (the third author). This underlines the importance and fruitfulness of collaborations between theoretical developments and practical applications. We present the example that had originally led to this research in Section 4.

From a methodological point of view, this research on the symbolic algorithm for linear boundary value problems has particular relevance and is drawing from the Theorema Project, see [1]. This project aims at supporting a new paradigm for doing (algorithmic) mathematical research: In the phase of doing research on new theorems and algorithms, TH∃OREM∀ provides a formal language (a version of predicate logic) and an automated reasoning system by which the exploration of the theory is supported. In the phase in which algorithms based on the new theory should be implemented and used in computing examples, TH∃OREM∀ allows to program and execute the algorithms *in the same language*. In the particular case of our approach to solving linear boundary value problems, the fundamental theorem on which the approach is based was proved automatically by checking that the rewrite rules for integro-differential operators forms a Gröbner basis. In a second step, the algorithm for solving linear boundary value problems is expressed again in the TH∃OREM∀ language and can then be called by the users by inputting the linear boundary value problems in a user-friendly notation.

In its current version, the engine for solving boundary problems is bundled in the `GreenGroebner` package of TH∃OREM∀. As an example, consider the boundary problem

$$\boxed{\begin{array}{l} u'' + \frac{1}{x}u' - \frac{1}{x^2}u = f, \\ u(0) = u(1) = 0, \end{array}}$$

which we shall consider in greater detail in Section 2. This can be given to TH∃OREM∀ in the form

$$\mathtt{BPSolve}\left[\boxed{\begin{array}{l}\mathtt{u}'' + \frac{1}{\mathtt{x}}\,\mathtt{u}' - \frac{1}{\mathtt{x}^2}\,\mathtt{u} = \mathtt{f}\\ \mathtt{u}[0] = \mathtt{u}[1] = 0\end{array}}, \mathtt{u}, \mathtt{x}, 0, 1\right]$$

leading either to the solution for $u(x)$ as

$$-\frac{1}{2}\frac{1}{\mathtt{x}}\left(\mathtt{x}^2\int_0^1 \mathtt{f}[\xi]\,\mathbf{d}\xi - \mathtt{x}^2\int_0^{\mathtt{x}} \mathtt{f}[\xi]\,\mathbf{d}\xi - \mathtt{x}^2\int_0^1 \xi^2\mathtt{f}[\xi]\,\mathbf{d}\xi + \int_0^{\mathtt{x}} \xi^2\mathtt{f}[\xi]\,\mathbf{d}\xi\right)$$

or to the Green's function $g(x,\xi)$ as

$$\begin{cases}\frac{1}{2}\frac{1}{\mathtt{x}}\,(-1+\mathtt{x}^2)\,\xi^2 & \Leftarrow \quad \xi \le \mathtt{x}\\ \frac{1}{2}\,\mathtt{x}\,(-1+\xi^2) & \Leftarrow \quad \mathtt{x} < \xi\end{cases}$$

at the user's request.

In addition to the `GreenGroebner` package in TH∃OREM∀, a Maple package named `IntDiffOp` is also available [5]. This package was developed in the frame of Anja Korporal's PhD thesis, supervised by Georg Regensburger and the first author. The Maple package supports also generalized boundary problems (see Section 2 for their relevance to this paper). One advantage of the TH∃OREM∀ system is that both the research phase and the application phase of our method can be formulated and supported within the same logic and software system—which we consider to be quite a novel and promising paradigm for the future.

2 A Simple Example

For illustrating the new ideas, it is illuminating to look at a simple example that exhibits the kind of phenomena that we have to cope with in the Kirchhoff plate boundary problem.

Example 1. Let us start with the *intuitive but mathematically unprecise* statement of the following example: Given a "suitable" forcing function f on the unit interval $I = [0, 1]$, we want to find a "reasonable" solution function u such that

$$\boxed{\begin{array}{l}u'' + \frac{1}{x}u' - \frac{1}{x^2}u = f,\\ u(0) = u(1) = 0.\end{array}} \tag{1}$$

But note that the differential operator $T = D^2 + \frac{1}{x}D - \frac{1}{x^2}$ is singular at the left boundary point $x = 0$ of the interval I under consideration. Hence the first boundary condition $u(0) = 0$ should be looked at with some suspicion. And what function space are we supposed to consider in the first place? If the $\frac{1}{x}$ and $\frac{1}{x^2}$ are to be taken literally, the space $C^\infty[0, 1]$ will clearly not do. On the other hand, we need functions that are smooth (or at least continuous) at $x = 0$ to make sense of $u(0)$.

In the rest of this section, we shall give one possible solution to the dilemma outlined above. Of course we could resort to using different function space for u and f, and this is in fact the approach one usually takes in Analysis. For our present purposes, however, we prefer to keep the simple paradigm of integro-differential algebras as outlined in Section 1, but we shall modify it to accommodate singularities such as in $\frac{1}{x}$ and $\frac{1}{x^2}$. Note that these are just poles, so we can take $\mathcal{F}$ to be the subring of the field $\mathcal{M}(I)$ consisting of complex-valued *meromorphic functions* that are regular at all $x \in I$ except possibly $x = 0$. In other words, these are functions that have a Laurent expansion at $x = 0$ with finite principal part, converging in a complex annulus $0 < |z| < \rho$ with $\rho > 1$. In fact, we will only use the real part $[-1, 1] \setminus \{0\}$ of this annulus. Note that we have of course $\frac{1}{x}, \frac{1}{x^2} \in \mathcal{F}$.

The ring $\mathcal{F}$ is an integro-differential algebra over $\mathbb{C}$ if we use the standard *derivation* $\partial = \frac{d}{dx}$ and the *Rota-Baxter operator*

$$\textstyle\int f := \int_1^x f(\xi)\, d\xi$$

initialized at the regular point $x = 1$.

We have now ensured that the differential operator of (1) has a clear algebraic interpretation $T \in \mathcal{F}[\partial]$. However, the boundary condition $u(0) = 0$ is still dubious. For making it precise, note that the integro-differential algebra $(\mathcal{F}, \partial, \textstyle\int)$ has only the second of the two boundary evaluations $L, R\colon \mathcal{F} \to \mathbb{C}$ with $L(f) = f(0)$ and $R(f) = f(1)$ in the usual sense of a total function. So while we can interpret the second boundary condition algebraically by $Ru = 0$, the same does not work on the left endpoint. Instead of an evaluation at $x = 0$ we shall introduce the map

$$\mathrm{pp}\colon \sum_{n=N}^{\infty} a_n x^n \mapsto \sum_{n=N}^{-1} a_n x^n$$

that extracts the *principal part* of a function written in terms of its Laurent expansion at $x = 0$. Here and henceforth we assume such expansions of nonzero functions are written with $a_N \neq 0$. If $N \geq 0$ the function is regular at $x = 0$, and the above sum is to be understood as zero. Clearly, $\mathrm{pp}\colon \mathcal{F} \to \mathcal{F}$ is a linear projector, with the complementary projector $\mathrm{reg} := 1_{\mathcal{F}} - \mathrm{pp}$ extracting the *regular part* at $x = 0$. Incidentially, pp and reg are also Rota-Baxter operators of weight -1, which play a crucial role in the renormalization theory of perturbative quantum field theory [3, Ex. 1.1.10].

Finally, we define the functional $C\colon \mathcal{F} \to \mathbb{C}$ that extracts the *constant term* a_0 of a meromorphic function expanded at $x = 0$. Combining C with the monomial multiplication operators, we obtain the coefficient functionals $\langle x^n \rangle := Cx^{-n}$ $(n \in \mathbb{Z})$ that map $\sum_n a_n x^n$ to a_n. In particular, the residue functional is given by $\langle x^{-1} \rangle = Cx$.

Note that for functions regular at $x = 0$, the functional C coincides with the evaluation at the left endpoint, $L\colon \mathcal{F} \to \mathbb{C}, f \mapsto f(0)$. However, for general meromorphic functions, L is undefined and C is *not multiplicative* since for example $C(x \cdot \frac{1}{x}) = 1 \neq 0 = C(x)\, C(\frac{1}{x})$. Hence we refer to C only as a functional but not as an evaluation.

We can now make the boundary condition $u(0) = 0$ precise for our setting over $\mathcal{F}$. What we really mean is that $\lambda(u) = 0$, where $\lambda := \mathrm{pp}{+}C$ is the projector that extracts the principal part together with the constant term. Extending the algebraic notion of boundary problem to allow for boundary conditions that are not functionals, we may thus view (1) as $(D^2 + \frac{1}{x}D - \frac{1}{x^2}, [\lambda, R]^\perp)$. In this way we have given a precise meaning to the *formulation of the boundary problem.*

But how are we to go about its *solution* in a systematic manner? Let us first look at the adhoc *standard method* way of doing this—determining the general homogeneous solution, then add the inhomogeneous solution via variation of constants, and finally adapt the integration constants to accommodate the boundary conditions. In our case, one sees immediately that $\mathrm{Ker}(T) = [x, \frac{1}{x}]$ so that the general solution of the homogeneous differential equation is $u(x) = c_1 x + \frac{c_2}{x}$, where $c_1, c_2 \in \mathbb{C}$ are integration constants. Variation of constants [2, p. 74] then yields

$$u(x) = c_1 x + \frac{c_2}{x} + \int_1^x \big(\frac{x}{2} - \frac{\xi^2}{2x}\big)\, f(\xi)\, d\xi \tag{2}$$

for the inhomogeneous solution. Note that $f \in \mathcal{F}$ may also have singularities as $x = 0$ or any other point $x \in I$ apart from $x = 1$.

Now we need to impose the *boundary conditions.* From $u(1) = 0$ we obtain immediately $c_1{+}c_2 = 0$. For the boundary condition at $x = 0$ we have to proceed a bit more cautiously, obtaining

$$\begin{aligned} u(0) &= \lim_{x\to 0} \Big(c_1 x + \frac{c_2}{x} + \frac{x}{2}\int_1^x f(\xi)\, d\xi - \frac{1}{2x}\int_1^x \xi^2 f(\xi)\, d\xi\Big) \\ &= \lim_{x\to 0} \Big(\frac{c_2}{x} - \frac{1}{2x}\int_1^x \xi^2 f(\xi)\, d\xi\Big) \end{aligned}$$

where we assume that f is regular at 0. It is clear that the remaining limit can only exist if the integral tends to a finite limit as $x \to 0$, and this is the case by our assumption on f. But then we may apply the boundary condition to obtain $c_2 = \frac{1}{2}\int_1^0 \xi^2 f(\xi)\, d\xi = -\frac{1}{2}\int_0^1 \xi^2 f(\xi)\, d\xi$. This gives the overall solution

$$u(x) = \Big(\frac{x}{2}\int_0^1 \xi^2 - \frac{1}{2x}\int_0^1 \xi^2 - \frac{x}{2}\int_x^1 + \frac{1}{2x}\int_x^1 \xi^2\Big) f(\xi)\, d\xi, \tag{3}$$

which one may write in the standard form $u(x) = \int_0^1 g(x,\xi)\, f(\xi)\, d\xi$ where the *Green's function* is defined as

$$g(x,\xi) = \begin{cases} \frac{x\xi^2}{2} - \frac{\xi^2}{2x} & \text{if } \xi \le x \\ \frac{x\xi^2}{2} - \frac{x}{2} & \text{if } \xi \ge x \end{cases} \tag{4}$$

in the usual manner.

How are we to make sense of this in an algebraic way, i.e. without (explicit) use of limits and hence topology? The key to this lies in the projector pp and the functional L, which serve to distill into our algebraic setting what we need from the topology (namely $f(x) = (\mathrm{pp}\, f)(x) + O(1)$ as $x \to 0$). However, there is another complication when compared to boundary problems without singularities as presented in Section 1: We cannot expect a solution $u \in \mathcal{F}$ to (1) for every given forcing function $f \in \mathcal{F}$. In other words, this boundary problem is *not regular* in the sense of [11].

In the past, we have also used the term *singular boundary problem* for such situations (here this seems to be suitable in a double sense but we shall be careful to separate the second sense by sticking to the designation "boundary problems with singularities"). The theory of singular boundary problems was developed in an abstract setting in [4]; applications to boundary problems (without singularities) have been presented in [5]. At this point we shall only recall a few basic facts.

A boundary problem $(T, \mathcal{B})$ is called *semi-regular* if $\mathrm{Ker}(T) \cap \mathcal{B}^\perp = O$. It is easy to see that the boundary problem $(D^2 + \frac{1}{x}D - \frac{1}{x^2}, [\lambda, R]^\perp)$ is in fact semi-regular. Since any $u \in \mathrm{Ker}(T)$ can be written as $u(x) = c_1 x + \frac{c_2}{x}$, the condition $Ru = 0$ implies $c_2 = -c_1$ and hence $u(x) = c_1(x - \frac{1}{x})$. But then $(\lambda u)(x) = -\frac{c_1}{x} = 0$ forces $c_1 = 0$ and hence $u = 0$.

If $(D^2 + \frac{1}{x}D - \frac{1}{x^2}, [\lambda, R]^\perp)$ were a regular boundary problem, we would have $\mathrm{Ker}(T) \dotplus [\lambda, R]^\perp = \mathcal{F}$. However, it is easy to see that there are elements $u \in \mathcal{F}$ that do not belong to $\mathrm{Ker}(T) + [\lambda, R]^\perp$, for example $u(x) = \frac{1}{x^2}$. Hence we conclude that the boundary problem (1) is in fact overdetermined. For such boundary problems $(T, \mathcal{B})$ one can always select a *regular subproblem* $(T, \tilde{\mathcal{B}})$, in the sense that $\tilde{\mathcal{B}} < \mathcal{B}$. In our case, a natural choice is $\tilde{\mathcal{B}} = [\langle x^{-1} \rangle, R]$. This is regular since the evaluation matrix

$$(\langle x^{-1} \rangle, R)(\tfrac{1}{x}, x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is regular, and the associated kernel projector is $P = \frac{1}{x}\,\langle x^{-1} \rangle + x\,(R - \langle x^{-1} \rangle)$ by [7, Lem. A.1].

For making the boundary problem (1) well-defined on $\mathcal{F}$ we need one more ingredient: We have to fix a complement $\mathcal{E}$ of $T(\mathcal{B}^\perp)$, the so-called *exceptional space*. Intuitively speaking, this comprises the "exceptional functions" of $\mathcal{F}$ that we decide to discard in order to render (1) solvable. Let us first work out what $T(\mathcal{B}^\perp) \leq \mathcal{F}$ looks like. Since $u(x) = \sum_{n \geq N}^{\infty} a_n x^n \in \mathcal{B}^\perp$ forces the principal part and constant term to vanish, we may start from $u(x) = \sum_{n>0} a_n x^n$, with the additional proviso that $\sum_{n>0} a_n = 1$. Applying $T = D^2 + \frac{1}{x}D - \frac{1}{x^2}$ to this $u(x)$ yields $\sum_{n>1} a_n\,(n^2 - 1)\,x^{n-2}$, which represents an arbitrary element of $C^\omega(I) = [\mathrm{pp}]^\perp$ for a suitable choice of coefficients $(a_n)_{n>1}$ since the additional condition $\sum_{n>0} a_n = 1$ can always be met by choosing $a_1 = 1 - \sum_{n>1} a_n$. But then it is very natural to choose $\mathcal{E} = [\mathrm{reg}]^\perp$ as the required complement. Clearly, the elements of this space $\mathcal{E}$ are the Laurent polynomials of $\mathbb{C}[\frac{1}{x}]$ without constant term.

By Prop. 2 of [5], the *Green's operator* of a generalized boundary problem $(T, \mathcal{B}, \mathcal{E})$ is given by $G = \tilde{G}Q$, where $\tilde{G}$ is the Green's operator of some regular subproblem $(T, \tilde{\mathcal{B}})$ and Q is the projector onto $T(\mathcal{B}^\perp)$ along $\mathcal{E}$. In our case, the latter projector is clearly $Q = \mathrm{reg}$, while the Green's operator $\tilde{G} = (1 - P)T^\Diamond$

by [11, Thm. 26], with the kernel projector P as above and the fundamental right inverse $T^\Diamond$ given according to [11, Prop. 23] by

$$T^\Diamond = \tfrac{1}{2}\, A_1 x - \tfrac{1}{2x}\, A_1 x^2,$$

which is essentially just a reformulation of the inhomogeneous part in (2). Following the style of [9] we write here $\int_1^x$ as $A_1 \in \mathcal{F}[\partial, \smallint]$ to emphasize its role in the integro-differential operator ring. Similarly, we write $F := -LA_1 = \int_0^1$ for the definite integral over the full interval I, which we may regard as a linear functional $C^\omega(I) \to \mathbb{C}$.

Putting things together, it remains to compute

$$\begin{aligned} G = (1-P)\,T^\Diamond Q &= \big(1 - \tfrac{1}{x}\,\langle x^{-1}\rangle + x\,\langle x^{-1}\rangle - xR\big)\big(\tfrac{1}{2}\,A_1x - \tfrac{1}{2x}\,A_1x^2\big)\,\mathrm{reg} \\ &= \big(-\tfrac{x}{2}A_1 + \tfrac{1}{2x}\,A_1x^2 - \tfrac{1}{2x}\,Fx^2 + \tfrac{x}{2}\,Fx^2\big)\mathrm{reg}, \end{aligned}$$

which may be done by the usual *rewrite rules* [11, Tbl. 1] for the operator ring $\mathcal{F}[\partial, \smallint]$, together with the obvious extra rules that on $\mathrm{Im}(\mathrm{reg}) = C^\omega(I)$ the residual $\langle x^{-1}\rangle$ vanishes and $C = \langle x^0\rangle$ coincides with the evaluation L at zero. Using the standard procedure for extracting the *Green's function*, one obtains exactly (4) when restricting the forcing functions to $f \in C^\omega(I)$. We have thus succeeded in applying the algebraic machinery to regain the solution previoulsy determined by analysis techniques. More than that: The precise form of accessible forcing functions is now fully settled, whereas the regularity assumption in (3) was left somewhat vague (a sufficient condition whose necessity was left open).

3 Two-Point Boundary Problems with One Singularity

Let us now address the general question of specifying and solving boundary problems (as usual: relative to a given fundamental system) that have only *one singularity*. The case of multipliple singularities is left for future investigations. Using a scaling transformation (and possibly a reflection), we may thus assume the same setting as in Section 2, with the *singularity at the origin* and the other boundary point at 1.

For the scope of this paper, we shall also restrict ourselves to a certain subclass of Stieltjes boundary problems $(T, \mathcal{B})$: First of all, we shall allow *only local boundary conditions* in $\mathcal{B}$. This means multi-point conditions and higher-order derivatives (leading to ill-posed boundary problems with distributional Green's functions) are still allowed, but no global parts (integrals); for details we refer to [13, Def. 1]. The second restriction concerns the differential operator $T \in \mathbb{C}(x)[\partial]$, which we require to be *Fuchsian without resonances*. The latter means the differential equation $Tu = 0$ is of Fuchsian type (the singularity is regular), and has fundamental solutions $x^{\lambda_1}\varphi_1(x), \dots, x^{\lambda_n}\varphi_n(x)$ with each $\varphi_i \in C^\omega(I)$ having order 0, where $\lambda_1, \dots, \lambda_n$ are the roots of the indicial equation [2, p. 127]. In other words, we do not require logarithms for the solutions. (A sufficient—but not necessary—condition for this is that the roots λ_i are all distinct and do not differ by integers.)

Definition 2. We call $(T, \mathcal{B})$ a *boundary problem with mild singularity* if $T \in \mathbb{C}(x)[\partial]$ is a nonresonant Fuchsian operator and $\mathcal{B}$ a local boundary space.

For a fixed $(T, \mathcal{B})$, we shall then enlarge the *function space* $\mathcal{F}$ of Section 2 by adding $x^{\mu_1}, \dots, x^{\mu_n}$ as algebra generators, where each μ_i is the fractional part of the corresponding indicial root λ_i. Every element of $\mathcal{F}$ is then a sum of series $x^\mu \sum_{n \geq N} a_n x^n$, with $\mu \in \{\mu_1, \dots, \mu_n\}$ and $N \in \mathbb{Z}$. The integro-differential structure on $(\mathcal{F}, \partial, \textstyle\int)$ is determined by setting $\partial x^\mu = \lambda_i \, x^{\mu-1}$, as usual, and by using for $\textstyle\int$ the integral $\int_1^x$ as we did also in Section 2.

As *boundary functionals* in $\mathcal{B}$, we admit derivatives $\mathtt{E}_\xi D^l$ $(0 < \xi \leq 1, l \geq 0)$ and coefficient functionals $\langle x^{k+\mu} \rangle$ $(k \in \mathbb{Z})$ whose action is $x^\mu \sum_{n \geq N} a_n x^n \mapsto a_k$. For functions $f \in C^\omega(I)$ we have of course $\langle x^{k+\mu} \rangle \, x^\mu f = f^{(k)}(0)/k!$. Since the projectors reg and pp can be expressed in terms of the coefficient functionals, we shall henceforth regard the latter just as convenient abbrevations; boundary spaces are always written in terms of $\mathtt{E}_\xi$ and $\langle x^k \rangle$ only but of course they can be infinite-dimensional. For example, in Section 2 we had the "regularized boundary condition" $\lambda(u) = 0$, which is equivalent to $Lu = 0$ and $\mathrm{pp}(u) = 0$ and hence to $\langle x^k \rangle \, u = 0$ $(k \leq 0)$. Its full boundary space is therefore $\mathcal{B} = [R, \langle x^k \rangle \mid k \leq 0]$.

The first issue that we must now address is the choice of *suitable boundary conditions*: Unlike in the "smooth case" (without singularities), we may not be able to impose n boundary conditions for an n-th order differential equation. The motivating example of Section 2 was chosen to be reasonably similar to the smooth case, so the presence of a singularity was only seen in replacing the boundary evaluation L by its regularized version λ. As explained above, we were effectively adding the extra condition $\mathrm{pp}(u) = 0$ to the standard boundary conditions $u(0) = u(1) = 0$. In other cases, this will not do as the following simple example shows.

Example 3. Consider the nonresonant Fuchsian differential equation $Tu(x) := u'' + \frac{4}{x} \, u' + \frac{2}{x^2} \, u = 0$. Note that here the indicial equation has the roots $\lambda_1 = -2$ and $\lambda_2 = -1$, which differ by the integer 1. Nevertheless, we may take $\{\frac{1}{x}, \frac{1}{x^2}\}$ as a fundamental system, so T is indeed nonresonant.

Trying to impose the same (regularized) boundary space $\tilde{\mathcal{B}} = [\mathrm{pp}, L, R]$ as in Example 1, one obtains

$$T(\tilde{\mathcal{B}}^\perp) = \Big\{ \sum_{n=-1}^{\infty} b_n x^n \mid \sum_{n=-1}^{\infty} \tfrac{b_n}{(n+3)(n+4)} = 0 \Big\}$$

after a short calculation. But this means that the forcing functions f in the boundary problem

$$\boxed{\begin{array}{l} u'' + \frac{4}{x} \, u' + \frac{2}{x^2} \, u = f \\ \mathrm{pp}(u) = u(0) = u(1) = 0 \end{array}}$$

must satisfy an awkward extra condition (viz. the one on the right-hand side of $T(\tilde{\mathcal{B}}^\perp)$ above). This is not compensated by the slightly enlarged generality of allowing f to have a simple pole at $x = 0$.

In the present case, we could instead impose *initial conditions* at 0 so that $\tilde{\mathcal{B}} = [\mathrm{pp}, L, LD]$. In this case one gets $T(\tilde{\mathcal{B}}^{\perp}) = C^{\omega}(I)$, so there is a unique solution for every analytic forcing function.

For a given nonresonant Fuchsian operator $T \in \mathbb{C}(x)[\partial]$, a better approach appears to be the following (we will make this more precise below):

1. We compute first some boundary functionals $\beta_1, \dots, \beta_n$ that ensure a *regular subproblem* $(T, \mathcal{B})$ with $\mathcal{B}_n := [\beta_1, \dots, \beta_n]$. Adding extra conditions (vanishing of all $\langle x^{k+\mu_i} \rangle$ for sufficiently small k) we obtain a boundary space $\mathcal{B} = \mathcal{B}_n + \cdots$ such that $(T, \mathcal{B})$ is semi-regular.
2. If a *particular boundary condition* is desired, it may be "traded" against one of the β_i; if this is not possible, it can be "annexed" to the extra conditions. After these amendments, the subproblem $(T, \mathcal{B}_n)$ is still regular, and $(T, \mathcal{B})$ still semi-regular.
3. Next we compute the corresponding *accessible space* $T(\mathcal{B}^{\perp})$. This space might not contain $C^{\omega}(I)$, as we saw in Example 3 above when we insisted on the conditions $u(0) = u(1)$.
4. We determine a complement $\mathcal{E}$ of $T(\mathcal{B}^{\perp})$ as *exceptional space* in $(T, \mathcal{B}, \mathcal{E})$.

Once these steps are completed, we have a regular generalized boundary problem $(T, \mathcal{B}, \mathcal{E})$ whose *Green's operator* G can be computed much in the same way as in Section 2. In detail, we get $G = \tilde{G}Q$, where $\tilde{G}$ is the Green's operator of the regular subproblem $(T, \mathcal{B}_n)$ and Q the projector onto $T(\mathcal{B}^{\perp})$ along $\mathcal{E}$. As we shall see, the operators G and Q can be computed as in the usual setting [11]. Let us first address Step 1 of the above program.

Lemma 4. *Let $T \in \mathbb{C}(x)[\partial]$ be a nonresonant Fuchsian differential operator of order n. Then there exists a fundamental system $u_1, \dots, u_n \in \mathcal{F}$ of T and n coefficient functionals $\beta_1 := \langle x^{\mu_1+k_1} \rangle, \dots, \beta_n := \langle x^{\mu_n+k_n} \rangle$ ordered as $k_1 + \mu_1 < \cdots < k_n + \mu_n$ so that $\beta(u) \in \mathbb{C}^{n \times n}$ is a lower unitriangular matrix.*

Proof. We start from an arbitrary fundamental system

$$u_1 = x^{\mu_1} \sum_{k \geq k_1} a_{1,k} x^k, \dots, u_n = x^{\mu_n} \sum_{k \geq k_n} a_{k \geq k_n} x^k$$

of the Fuchsian operator T, where we take $\mu_1, \dots, \mu_n$ fracational as before and we may assume that $a_{1,k_1}, \dots, a_{n,k_n} = 1$ so that each fundamental solution u_i has order k_i. (The order of a series $u = x^{\mu} \sum_{k \geq N} a_k \, x^k$ is defined as the smallest integer k such that $\langle x^k \rangle \, u \neq 0$.) We order the fundamental solutions such that $k_1 + \mu_1 \leq \cdots \leq k_n + \mu_n$. We can always achieve strict inequalities as follows. If $i < n$ is the first place where $k_i + \mu_i = k_{i+1} + \mu_{i+1}$ we must also have $\mu_i = \mu_{i+1}$ since $0 \leq \mu_i, \mu_{i+1} < 1$. Therefore we have $k_i = k_{i+1}$, and we can replace u_{i+1} by $u_{i+1} - u_i$ and make it monic so as to ensure $k_i + \mu_i < k_{i+1} + \mu_{i+1}$. Repeating this process at most $n-1$ times we obtain $k_1 + \mu_1 < \cdots < k_n + \mu_n$. Choosing now the boundary functionals $\beta_1 := \langle x^{\mu_1+k_1} \rangle, \dots, \beta_n := \langle x^{\mu_n+k_n} \rangle$ as in the statement of the lemma, we have clearly $\beta_i(u_i) = 1$ and $\beta_i(u_j) = 0$ for $j > i$ as claimed.

In particular we see that $E_n := (\beta_1, \dots, \beta_n)(u_1, \dots, u_n)$ has unit determinant, so it is regular. Setting $\mathcal{B}_n := [\beta_1, \dots, \beta_n]$, we obtain a regular boundary problem $(T, \mathcal{B}_n)$. Note that some of the μ_i may coincide. For each $\mu \in M := \{\mu_1, \dots, \mu_n\}$ let k_μ be the smallest of the k_i with $\mu = \mu_i$. We expand the n boundary functionals by suitable curbing constraints to the full boundary space

$$\mathcal{B} := \mathcal{B}_n + [\langle x^{k+\mu} \rangle \mid \mu \in M,\ k < k_\mu] \tag{5}$$

since the inhomogeneous solutions should be at least as smooth (in the sense of pole order) as the homogeneous ones. Note that $(T, \mathcal{B})$ is clearly a semi-regular boundary problem. This achieves Step 1 in our program.

Now for Step 2. Suppose we want to *impose a boundary condition* β, assuming it is of the type discussed above (composed of derivatives $\mathrm{E}_\xi D^l$ and coefficient functionals $\langle x^{k+\mu} \rangle$ for fractional parts μ of indicial roots). We must distinguish two cases:

Trading. If the row vector $r := \beta(u_i)_{i=1,\dots,n} \in \mathbb{C}^n$ is nonzero, we can express it as a $\mathbb{C}$-linear combination $c_1 r_1 + \cdots + c_n r_n$ of the rows $r_1, \dots, r_n$ of E_n. Let k be the largest index such that $c_k \neq 0$. Then we may express r_k as a $\mathbb{C}$-linear combination of r and the remaining rows r_i $(i \neq k)$, hence we may exchange β_k with β without destroying the regularity of $E_n = (\beta_1, \dots, \beta_n)(u_1, \dots, u_n)$.

Annexation. Otherwise, we have $\operatorname{Ker}(T) \leq \beta^\perp$. Together with $\operatorname{Ker}(T) \dotplus \mathcal{B}_n^\perp = \mathcal{F}$ this implies $\big([\beta] \cap \mathcal{B}_n\big)^\perp = \beta^\perp + \mathcal{B}_n^\perp = \mathcal{F}$ and hence $\beta \notin \mathcal{B}_n$ by the identities of [7, App. A]. Furthermore, Lemma 4.14 of [4] yields

$$T(\mathcal{B}_n + \beta^\perp) = T(\mathcal{B}_n^\perp \cap \beta^\perp) = T(\mathcal{B}_n^\perp) \cap T(\beta^\perp) = T(\beta^\perp)$$

since we have $T(\mathcal{B}_n^\perp) = \mathcal{F}$ from the regularity of $(T, \mathcal{B})$. But this means that adding β to $\mathcal{B}$ as a new boundary condition necessarily cuts down the space of accessible functions unless β happens to be in the span of the curbing constraints $\langle x^{k+\mu} \rangle \in \mathcal{B}$ added to $\mathcal{B}_n$ in (5).

In the sense of the above discussion (see Example 3), the first case signifies a "natural" choice of boundary condition while the second case means we insist on imposing an extra condition (unless it is a redundant curbing constraint). Repeating these steps as the cases may be, we can successively impose any (finite) number of given boundary conditions. This completes Step 2.

For Step 3 we require the computation of the accessible space $T(\mathcal{B}^\perp)$. We shall now sketch how this can be done algorithmically, starting with a finitary description of the *admissible space* $\mathcal{B}^\perp$ as given in the next proposition. The proof is unfortunately somewhat tedious and long-wided but the basic idea is simple enough: We substitute a series ansatz into the boundary conditions specified in $\mathcal{B}$ to determine a number of lowest-order coefficients. The rest is just some bureaucracy for making sure that everything works out (most likely this could also be done in a more effective way).

Proposition 5. *Let $(T, \mathcal{B})$ be a semi-regular boundary problem of order n with mild singularity such that $(T, \mathcal{B}_n)$ is a regular subproblem. Let $M = \{\mu_1, \dots, \mu_n\}$ be the fractional parts of the indicial roots for T. Then we have a direct decomposition of the admissible space $\mathcal{B}^\perp = \bigoplus_{\mu \in M} x^\mu \mathcal{A}_\mu$ with components*

$$\mathcal{A}_\mu = [p_{\mu 1}(x), \dots, p_{\mu\, l_\mu}(x)] + P_\mu\big(C^\omega(I)^M\big) \qquad (\mu \in M). \tag{6}$$

Here $p_{\mu 1}(x), \dots, p_{\mu\, l_\mu}(x) \in \mathbb{C}[x, \frac{1}{x}]$ are linearly independent Laurent polynomials and the linear operators $P_\mu \colon C^\omega(I)^M \to C^\omega(I)$ are defined by

$$P_\mu\big(b(x)\big) = x^{j_\mu} b_\mu(x) + \sum_{\nu \in M} \sum_{\xi, j} q_{\mu\nu\xi j}(x)\, \mathtt{E}_\xi D^j(b_\nu),$$

with $j_\mu \in \mathbb{Z}$ and Laurent polynomials $q_{\mu\nu\xi j}(x) \in \mathbb{C}[x, \frac{1}{x}]$, almost all of which vanish over the summation range $0 < \xi \leq 1$ and $j \geq 0$.

Proof. Any element $u \in \mathcal{F}$ may be written in the form

$$u = \sum_{\mu \in M} \sum_{k=k(\mu)}^{\infty} a_{\mu,k}\, x^{k+\mu},$$

where the $k(\mu) \in \mathbb{Z}$ are *a priori* arbitrary. However, because of the curbing constraints of $\mathcal{B}$ in (5) we may assume $k(\mu) = k_\mu$ for $u \in \mathcal{B}^\perp$. Let us write $\mathcal{B}_+$ for the extra conditions that were "annexed" at Step 2 (if none were added then clearly $\mathcal{B}_+ = O$); thus $\mathcal{B}$ is a direct sum of $\mathcal{B}_n$ and $\mathcal{B}_+$ and the curbing constraints. If s_μ denotes the largest integer such that $\langle x^{s+\mu} \rangle$ occurs in any condition of $\mathcal{B}_n + \mathcal{B}_+$ and if we set $r_\mu := s_\mu + 1 + \mu$, we can also write $u \in \mathcal{B}^\perp$ as

$$u = \sum_{\mu \in M} \sum_{k=k_\mu}^{s_\mu} a_{\mu k} x^{k+\mu} + \sum_{\mu \in M} x^{r_\mu}\, b_\mu(x), \tag{7}$$

where the $b_\mu(x) \in C^\omega(I)$ are convergent power series. Next we impose the conditions $\beta \in \mathcal{B}_n + \mathcal{B}_+$ on u. But each β is a $\mathbb{C}$-linear combintions of coefficient functionals $\langle x^{k+\mu} \rangle$ with $u \mapsto a_{\mu k}$ and derivatives $\mathtt{E}_\xi D^l$ $(0 < \xi \leq 1, l \geq 0)$ whose action on u yields

$$\sum_\mu \sum_k a_{\mu k}\, (k+\mu)^{\underline{l}}\, \xi^{k+\mu-l} + \sum_\mu \sum_{j=0}^{l} \binom{l}{j}\, r_\mu^{\underline{j}}\, \xi^{r_\mu - j}\, b_\mu^{(l-j)}(\xi),$$

Hence $\beta(u)$ yields a $\mathbb{C}$-linear combination of the $a_{\mu k}$ and the $b_\mu^{(j)}(\xi)$. We compile the coefficients $a_{\mu k} \in \mathbb{C}$ into a column vector $\hat{a} \in \mathbb{C}^S$ of size $S := \sum_\mu (s_\mu - k_\mu + 1)$, consisting of $|M|$ contiguous blocks of varying size $s_\mu - k_\mu + 1$. Putting the (finite) set $\varXi$ of evaluation points occurring in $\mathcal{B}_n + \mathcal{B}_+$ into ascending order, we compile also the derivatives $b_\mu(\xi), b'_\mu(\xi), \dots$ into a column vector $\hat{b} \in \mathbb{C}^T$ of size $T := \sum_\mu t_\mu$, consisting of $|M|$ contiguous blocks, each holding t_μ derivatives $b_\mu^{(j)}(\xi)$ for a fixed $\mu \in M$ and certain $\xi \in \Xi$ and $j \geq 0$. Assuming $\mathcal{B}_n + \mathcal{B}_+$ has dimension R, the boundary conditions $\beta(u) = 0$ for $\beta \in \mathcal{B}_n + \mathcal{B}_+$ can be written as

$$\hat{A}\hat{a} = \hat{B}\hat{b} \tag{8}$$

for suitable matrices $\hat{A} \in \mathbb{C}^{R \times S}$ and $\hat{B} \in \mathbb{C}^{R \times T}$ that can be computed from the boundary functionals $\beta \in \mathcal{B}_n + \mathcal{B}_+$.

Regarding the right-hand side as given, let us now put (8) into row echelon form (retaining the same letters for simplicity). If the resulting system contains any rows that are zero on the left but nonzero on the right, this signals constraints amongst the $b_\mu(x)$ whose treatment is postponed until later. For the moment we discard such rows as well rows that are zero on both sides. Let $U \leq R$ be the number of the remaining rows and $p_1, \dots, p_U$ the pivot positions and $V := S - U$ the number of free parameters. Then we can solve (8) in the form $\hat{a} = \tilde{a} + \tilde{A} \cdot \mathbb{C}^V$. Here $\tilde{a} \in \mathbb{C}^S$ is the vector with entry $(\hat{B}\hat{b})_j$ in row p_j and zero otherwise, and $\tilde{A} \in \mathbb{C}^{S \times V}$ consists of the non-pivot columns of the corresponding padded matrix (its j-th row is the i-th row of $\hat{A}$ for pivot indices $j = p_i$, and $-e_j$ for non-pivot indices j). Writing the free parameters as $\hat{v} = (v_1, \dots, v_V) \in \mathbb{C}^V$, we may substitute this solution $\hat{a}$ into the ansatz (7) to obtain

$$u = \sum_{\mu \in M} \sum_{k=k_\mu}^{s_\mu} a_{\mu k}(\hat{v}, \hat{b})\, x^{k+\mu} + \sum_{\mu \in M} x^{r_\mu}\, b_\mu(x), \tag{9}$$

where the $a_{\mu k}(\hat{v}, \hat{b})$ are $\mathbb{C}$-linear combinations of the free parameters $\hat{v}$ and the derivatives $b_\mu^{(j)}(\xi)$ comprising $\hat{b}$. Note that we may regard the $a_{\mu k}(\dots)$ as row vectors in $\mathbb{C}^{1 \times (V+T)}$ whose entries are computed from $\hat{A}$ and $\hat{B}$.

We turn now to the issue of constraints amongst the $b_\mu(x)$, embodied in those rows of the reduced row echelon form of (8) that are zero on the left-hand side but nonzero on the right-hand side. We put the corresponding block of $\hat{B}$ into reduced row echelon form (again retaining the same letters for simplicity). Let μ' the smallest μ occurring in any such row. Then each of the rows containing a derivative $b_{\mu'}^{(j)}(x)$ provides a constraint of the form

$$\sum_{\xi \in \Xi} \sum_j c_{\xi j}\, b_{\mu'}^{(j)}(\xi) + \sum_{\xi \in \Xi} \sum_{\mu > \mu'} \sum_j c_{\xi \mu j}\, b_\mu^{(j)}(\xi) = 0, \tag{10}$$

where the $c_{\xi j}, c_{\xi \mu j} \in \mathbb{C}$ are determined by $\hat{B}$, and with finite sums over j. Let the number of such constraints be X. Collecting the $\big(b_{\mu'}^{(j)}(\xi)\big)_{\xi,l}$ into a vector $\hat{b}_{\mu'} \in \mathbb{C}^Y$ of size $Y := t_{\mu'}$, we can write the constraints (10) as matrix equation $\hat{C}\hat{b}_{\mu'} = \hat{d}$ where the coefficient matrix $\hat{C} \in \mathbb{C}^{X \times Y}$ is determined by the $c_{\xi j}$ of each constraint (10), and the right-hand side $\hat{d} \in \mathbb{C}^X$ by the corresponding $c_{\xi \mu j}$ and the $b_\mu^{(j)}(\xi)$ for $\mu > \mu'$, which for the moment we regard as known. Note that $X < Y$ since $\hat{C}$ is in row echelon form. Let its pivots be in the positions $q_1, \dots, q_X$ and set $Z := Y - X$. Then we can write the solution as $\hat{b}_{\mu'} = \tilde{b}_{\mu'} + \tilde{C} \cdot \mathbb{C}^Z$. Here $\tilde{b}_{\mu'} \in \mathbb{C}^Y$ is the vector with entry $\hat{d}_j$ in row q_j and zero otherwise, while $\tilde{C} \in$

$\mathbb{C}^{Y \times Z}$ consists of the non-pivot columns of the padded matrix (its j-th row is the i-th row of $\hat{C}$ for pivot indices $j = q_i$, and $-e_j$ for non-pivot indices j). Writing $\hat{w} = (w_1, \dots, w_Z) \in \mathbb{C}^Z$ for the corresponding free parameters, we may thus view (10) as providing Y constraints

$$b_{\mu'}^{(j)}(\xi) = b_{\mu'\xi j}(\hat{w}, \hat{b}_+), \tag{11}$$

where the $b_{\mu'\xi j}(\hat{w}, \hat{b}_+)$ are $\mathbb{C}$-linear combinations of the free parameters $\hat{w}$ and certain derivatives $b_\mu^{(j)}(\xi)$ that we have collected into a vector $\hat{b}_+ \in \mathbb{C}^{T-Y}$. Again we may view $b_{\mu'\xi j}(\dots)$ as a row vector in $\mathbb{C}^{1 \times (Z+T-Y)}$ whose entries can ultimately be computed from $\hat{A}$ and $\hat{B}$.

We regard now (11) as determining equations for fixing Y of the coefficients of $b_{\mu'}(x) = \sum b_{\mu' k} x^k$. Indeed, if j is the highest derivative order occurring in (11) we may split according to $b_{\mu'}(x) = \sum_{k \le j} b_{\mu' k} x^k + x^{j+1} \bar{b}_{\mu'}(x)$, with an arbitrary power series $\bar{b}_{\mu'}(x)$, and then substitute this into (11) to obtain

$$b_{\mu' j} = b_{\mu'\xi j}(\hat{w}, \hat{b}_+) - \sum_{i=0}^{j} \binom{j}{i} \frac{j+1}{i+1} \, \xi^{i+1} \, \bar{b}_{\mu'}^{(i)}(\xi) \tag{12}$$

for fixing one coefficient of $b_{\mu'}(x)$. Now we substitute this b_j back into the above ansatz $b_{\mu'}(x) = \sum_{k \le j} b_{\mu' k} x^k + x^{j+1} \bar{b}_{\mu'}(x)$, and we repeat the whole process for all other constraints (11), each time determining the lowest unknown coefficient b_k of $b_{\mu'}(x)$. Of course, it may be necessary to expand the splitting to extract a larger polynomial part (if all the coefficients in the current polynomial part are determined). Eventually, we end up with

$$b_{\mu'}(x) = \sum_{k \le m_{\mu'}} b_{\mu' k}(\hat{w}, \hat{b}_+) \, x^k + x^{m_{\mu'}+1} \, \bar{b}_{\mu'}(x), \tag{13}$$

with some break-off index $m_{\mu'}$ that is specific to $b_{\mu'}(x)$, and with $\hat{b}_+$ enlarged to comprise also the derivatives $\bar{b}_{\mu'}^{(j)}(\xi)$ that were needed in determining the coefficients (12). Substituting (13) into (9), we see that the $b_{\mu' k}(\hat{w}, \hat{b}_+)$ may be combined with the $a_{\mu' k}(\hat{v}, \hat{b})$ if we adjoin the parameters $\hat{w}$ to the parameters $\hat{v}$; then we can also rename the series $\bar{b}_{\mu'}(x)$ back to $b_{\mu'}(x)$. Of course new terms $a_{\mu' k}(\hat{v}, \hat{b}) \, x^{k+\mu'}$ may be created in the ansatz (9), and its polynomial part may be expanded—but its overall form is not altered.

We have now eliminated those constraints amongst the $b_\mu(x)$ occurring in (8) that involve $b_{\mu'}$, where μ' was chosen minimal. Hence we are only left with constraints amongst the $b_\mu(x)$ with $\mu > \mu'$. Repeating the elimination process a finite number of times (at most $|M|$ eliminations are necessary), we obtain the generic form of $u \in \mathcal{B}^\perp$ as given in (9), where the $b_\mu(x)$ are now arbitrary (convergent) power series. Since the terms with distinct factors x^μ are clearly in distinct direct sum components $x^\mu \mathcal{A}_\mu$, the latter may now be described by

$$\mathcal{A}_\mu = \left\{ \sum_{k=k_\mu}^{s_\mu} a_{\mu k}(\hat{v}, \hat{b}) \, x^k + x^{j_\mu} \, b_\mu(x) \;\middle|\; \begin{array}{l} \hat{v} \in \mathbb{C}^V, \\ b(x) \in C^\omega(I)^M \end{array} \right\},$$

where we have set $j_\mu := s_\mu + 1$. Splitting the coefficients $a_{\mu k}(\hat{v}, \hat{b})$ into a $\mathbb{C}$-linear combination of the free parameters $(v_1, \dots, v_V)$ and a $\mathbb{C}$-linear combination of the derivatives $b_\nu^{(j)}(\xi)$ comprising $\hat{b}$, we collect terms whose coefficient is a specific parameter v_i $(i = 1, \dots, V)$ or a specific derivative $b_\nu^{(j)}(\xi)$ $(\nu \in M, \xi \in \Xi, j \geq 0)$. This leads to

$$\mathcal{A}_\mu = \left\{ \sum_{i=1}^{V} v_i \, p_{\mu i}(x) + x^{j_\mu} \, b_\mu(x) + \sum_{\nu \in M} \sum_{\xi, j} b_\nu^{(j)}(\xi) \, q_{\mu\nu\xi j}(x) \;\middle|\; \begin{array}{l} \hat{v} \in \mathbb{C}^V, \\ b(x) \in C^\omega(I)^M \end{array} \right\}$$

and hence to (6) by extracting a $\mathbb{C}$-basis for each $[p_{\mu 1}(x), \dots, p_{\mu V}(x)]$.

We have now established Step 3 of our program since the accessible space $T(B^\perp)$ can be specified by applying $T \in \mathbb{C}[x, \frac{1}{x}$ to the generic functions (6) of the admissible space $B^\perp$. Our next goal is to find a *projector* Q onto $T(B^\perp)$, which then gives the exceptional space $\mathcal{E} := \operatorname{Ker}(Q) = \operatorname{Im}(1 - Q)$ required for Step 4. This will be easy once we have a corresponding projector onto $\mathcal{B}^\perp$. In fact, the operator P_μ in Proposition 5 is not quite a projector (for one thing, it is not even an endomorphism), but in a sense it is not far away from being one. For seeing this, note that we have

$$\mathcal{F} = \bigoplus_{\mu \in M} x^\mu \, \mathbb{C}((x)), \tag{14}$$

where $\mathbb{C}((x))$ denotes the field of Laurent series (converging in the punctured unit disk). The direct decomposition (14) just reflects the fact that the x^μ for distinct $\mu \in M$ are linearly independent. Let us write $\langle \mu \rangle \colon \mathcal{F} \to \mathcal{F}$ for the *indicial projector* onto the component $x^\mu \, \mathbb{C}((x))$ of (14). In other words, $\langle \mu \rangle$ extracts all terms of the form $x^{k+\mu}$ $(k \in \mathbb{Z})$ from a series in $\mathcal{F}$. Note that combinations like $\beta = \mathtt{E}_\xi D^k \langle \mu \rangle$ provide linear functionals $\beta \in \mathcal{F}^*$ for extracting derivatives of the μ-component of a given series in $\mathcal{F}$.

For writing the projector corresponding to Proposition 5, let us also introduce the *auxiliary operator* $x^\mu \colon \mathbb{C}((x)) \to x^\mu \mathbb{C}((x)) \leq \mathcal{F}$ and its inverse $x^{-\mu}$. Then we shall see that the required projector is essentially a "twisted" version of two kinds of projector: one for splitting off the polynomial part of the occurring series, and one for imposing the derivative terms. For convenience, we shall use *orthogonal projectors* in $\mathbb{C}((x))$, where the underlying inner product is defined by $\langle x^k | x^l \rangle = \delta_{kl}$ for all $k, l \in \mathbb{Z}$. Such projectors are always straightforward to compute (using linear algebra on complex matrices).

Corollary 1. *Using the same notation as in Proposition 5, let $R_\mu, S_\mu \colon \mathbb{C}(x)) \to \mathbb{C}(x)$ be the orthogonal projectors onto $[p_{\mu 1}(x), \dots, p_{\mu \, l_\mu}(x)]$ and onto $x^{j_\mu} \, \mathbb{C}((x))$, respectively. Writing $R'_\mu := x^\mu R_\mu x^{-\mu} \langle \mu \rangle$ and $S'_\mu := x^\mu S_\mu x^{-\mu} \langle \mu \rangle$ for their twisted analogs, we define the linear operator $P \colon \mathcal{F} \to \mathcal{F}$ by*

$$P = \sum_{\mu \in M} \Big(R'_\mu + S'_\mu + \sum_{\nu \in M} \sum_{\xi, j} x^\mu \, q_{\mu\nu\xi j}(x) \, \mathtt{E}_\xi D^j x^{-j_\mu - \mu} S'_\nu \Big) \tag{15}$$

is a projector onto $\mathcal{B}^\perp$.

Proof. The action of P on a general series (which we may split at $s_\mu = j_\mu - 1$ in its μ-components)

$$u(x) = \sum_{\mu \in M} x^\mu \, f_\mu(x) = \sum_{\mu \in M} \sum_{k=k_\mu}^{s_\mu} f_{\mu k} x^{k+\mu} + \sum_{\mu \in M} x^{j_\mu + \mu} \, b_\mu(x) \in \mathcal{F}$$

is

$$Pu(x) = \sum_{\mu \in M} x^\mu \Big(R_\mu \Big(\sum_{k=k_\mu}^{s_\mu} f_{\mu k} x^k \Big) + x^{j_\mu} \, b_\mu(x) + \sum_{\nu \in M} \sum_{\xi, j} q_{\mu\nu\xi j}(x) \, b_\nu^{(j)}(\xi) \Big),$$

where we have set $b_\mu(x) := x^{-j_\mu} f_\mu(x) \in C^\omega(I)$. Extracting the x^μ component of Pu yields

$$R_\mu \Big(\sum_{k=k_\mu}^{s_\mu} f_{\mu k} x^k \Big) + x^{j_\mu} \, b_\mu(x) + \sum_{\nu \in M} \sum_{\xi, j} q_{\mu\nu\xi j}(x) \, b_\nu^{(j)}(\xi) \in \mathcal{A}_\mu \tag{16}$$

as one sees by comparing with the last displayed equation in the proof of Proposition 5. Hence we may conclude that $\mathrm{Im}(P) \le \mathcal{B}^\perp$.

It remains to prove that $Pu = u$ for $u \in \mathcal{B}^\perp$ since this implies $\mathrm{Im}(P) \ge \mathcal{B}^\perp$ and $P^2 = P$ so that P is indeed a projector onto $\mathcal{B}^\perp$ as claimed in the corollary. So assume $u(x) \in \mathcal{B}^\perp$ is arbitrary, and split it as above. The orders k_μ of the series f_μ are given by the curbing constraints. Since $u(x)$ already satisfies all boundary conditions in $\mathcal{B}_n + \mathcal{B}_+ \le \mathcal{B}$, the reduced row echelon form of (8) will contain only zero rows. The original ansatz (7) is then left intact; no expansion of the polynomial part is necessary and no $\hat{b}$ are involved in its coefficients. Therefore the original coefficients $a_{\mu k}$ in (7) coincide with the coeffients $a_{\mu k}(\hat{v}, \hat{b}) = a_{\mu k}(\hat{v})$ in (8), which constitute the polynomials of $\mathcal{P}_\mu := [p_{\mu 1}(x), \dots, p_{\mu \, l_\mu}(x)]$. Now let us consider (16). Since the series $\sum_k f_{\mu k} x^k$ is thus already in $\mathcal{P}_\mu = \mathrm{Im}(R_\mu)$, we may omit the action of R_μ and since $a_{\mu k}(\hat{v}, \hat{b}) = a_{\mu k}(\hat{v})$ the triple sum in (16) is zero. But then $Pu(x)$ becomes identical with $u(x)$ as was claimed.

For accomplishing Step 4 of our program, it only remains to determine a projector Q onto the accessible space $T(\mathcal{B}^\perp)$ from the projector P onto the admissible space $\mathcal{B}^\perp$ provided in Corollory 1. This can be done easily since Q is *essentially a conjugate* of P except that we use a fundamental right inverse $T^\Diamond$, for want of a proper inverse. (The formula for $T^\Diamond$ in [11, Prop. 23] and [12, Thm. 20] may be used but recall that in our case $\int = \int_1^x$ so that $\mathtt{E} = \mathtt{E}_1$.)

Proposition 1. *Using the same notation as in Proposition 1, the operator*

$$Q := TPT^\Diamond \colon \mathcal{F} \to \mathcal{F}$$

is a projector onto the accessible space $T(\mathcal{B}^\perp)$.

Proof. Let us first check that Q is a projector. Writing $U := 1 - T^\Diamond T$ for the projector onto $\operatorname{Ker} T$ along $[\mathtt{E}, \mathtt{E}D, \dots, \mathtt{E}D^{n-1}]$ and using $P^2 = P$, we will indeed get

$$Q^2 = TP^2T^\Diamond - TPUPT^\Diamond = TPT^\Diamond = Q$$

provided we can ascertain that $\operatorname{Ker} T \leq \operatorname{Ker} P =: \mathcal{C}$ since then $PU = 0$. We know that $B^\perp \dotplus \mathcal{C} = \mathcal{F}$ since P is a projector onto $\mathcal{B}^\perp$ along $\mathcal{C}$. On the other hand, we have also $\operatorname{Ker}(T) \dotplus \mathcal{B}_n^\perp = \mathcal{F}$ since $(T, \mathcal{B}_n)$ is regular. Intersecting $\mathcal{C}$ in the former decomposition with the latter yields

$$\mathcal{B}^\perp \dotplus (\mathcal{C} \cap \mathcal{B}_n^\perp) \dotplus (\mathcal{C} \cap \operatorname{Ker} T) = \mathcal{F}. \tag{17}$$

But $\mathcal{B}_n \leq \mathcal{B}$ implies $\mathcal{B}^\perp \leq \mathcal{B}_n^\perp$, so intersecting the decomposition $\mathcal{B}^\perp \dotplus \mathcal{C} = \mathcal{F}$ with $\mathcal{B}_n^\perp$ leads to $\mathcal{B}^\perp \dotplus (\mathcal{C} \cap \mathcal{B}_n^\perp) = \mathcal{B}_n^\perp$. Using the other decompostion $\operatorname{Ker}(T) \dotplus \mathcal{B}_n^\perp = \mathcal{F}$ one more time we obtain

$$\mathcal{B}^\perp \dotplus (\mathcal{C} \cap \mathcal{B}_n^\perp) \dotplus \operatorname{Ker} T = \mathcal{F}. \tag{18}$$

Comparing (17) and (18), we can apply the well-known rule [4, (2.6)] to obtain the identity $\mathcal{C} \cap \operatorname{Ker} T = \operatorname{Ker} T$ and hence the required inclusion $\operatorname{Ker} T \leq \mathcal{C}$.

It remains to prove that $\operatorname{Im}(Q) = T(\mathcal{B}^\perp)$. The inclusion from left to right is obivous, so assume $f = Tu$ with $u \in \mathcal{B}^\perp$. Then $T^\Diamond f = u - Uu$ and hence $PT^\Diamond f = Pu - PUu = u$ because P projects onto $\mathcal{B}^\perp$ and $PU = 0$ from the above. But then we have also $Qf = TPT^\Diamond f = Tu = f$ and in particular $f \in \operatorname{Im}(Q)$.

We have now sketched how to carry out the four main steps in our program aimed at the algorithmic treatment of finding/imposing "good" boundary conditions on a Fuchsian differential equation with one (mild) singularity. At the moment we do not have a full implementation of the underlying algorithms in TH∃OREM∀ (or any other system). However, we have implemented a prototype version of some portion of this theory. We shall demonstrate some of its features by with example from engineering mechanics.

4 Application to Functionally Graded Kirchhoff Plates

Circular plates play an important role for many application areas in engineering mechanics and mathematical physics. If the plates are thin (the ratio of thickness to diameter is small enough), one may employ the well-known Kirchhoff-Love plate theory [6], whose mathematical description is essentially two-dimensional (via a linear second-order partial differential equation in two independent variables). We will furthermore restrict ourselves to *circular Kirchhoff plates* so as to have a one-dimensional mathematical model, via a linear ordinary differential equation of second order.

However, we shall not assume homogeneous plates. Indeed, the precise manufacture of *functionally graded* materials is an important branch in engineering mechanics. In the case of Kirchhoff plates, the functional grading is essentially the variable thickness $t = t(r)$ or variable bending rigidity $D = D(r)$ of the plate along its radial profile. (We write r for the radius variable ranging between zero at the center and the outer radius $r = b$.)

Let $w = w(r)$ be the displacement of the plate as a function of its radius. This is the quantity that we try to determine. It induces the *radial and tangential moments* given, respectively by

$$M_r(r) = -D(r)\left(w''(r) + \tfrac{\nu}{r}\, w'(r)\right),$$
$$M_\theta(r) = -D(r)\left(\tfrac{1}{r}\, w'(r) + \nu\, w''(r)\right),$$

where ν is the Poisson's ratio of the plate (which we assume constant). For typical materials, ν may be taken as $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ or even 0. A reasonable constitutive law for the bending rigidity is

$$D(r) = \tfrac{E(r)\, t(r)^3}{12(1-\nu^2)}, \tag{19}$$

where $E = E(r)$ is the variable Young's modulus of the plate.

The *equilibrium equation* can then be written as

$$\frac{dM_r}{dr} + \frac{M_r - M_\theta}{r} = Q_r, \tag{20}$$

where $Q_r = Q_r(r)$ is the cumulative load

$$Q_r(r) = -\frac{1}{2\pi r}\int_0^r q(r)\, 2\pi r\, dr = -\frac{1}{r}\int_0^r q(r)\, r\, dr$$

induced by a certain *loading* $q = q(r)$ that may be thought to describe the weight (or other forces) acting in each ring $[r, r + dr]$.

For the calculational treatment of (20) it it useful to introduce the function $\varphi := -w'(r)$, which represents the (negative) slope of the plate profile. In terms of φ, the equilibrium equation is given by

$$\varphi''(r) + \Big(\frac{1}{r} + \frac{D'(r)}{D(r)}\Big)\,\varphi'(r) + \Big(\nu\,\frac{D'(r)}{D(r)} - \frac{1}{r}\Big)\frac{\varphi(r)}{r} = \frac{Q_r(r)}{D(r)}.$$

A typical example of a useful thickness grading is the *linear ansatz* $t = t_0(1 - \frac{r}{b})$, cut off beyond some $a < b$ close to b; this describes a radially symmetric pointed plate with straight edges (more or less a very flat cone). Suppressing the cut-off for the moment and changing the independent variable r to $\rho = r/b$, we have thus thickness $t(\rho) = t_0 \cdot (1 - \rho)$ and from (19) bending rigidity $D(\rho) = D_0 \cdot (1 - \rho)^3$ with $D_0 := Et_0^3/12(1 - \nu^2)$. The equilibrium equation becomes now

$$\varphi''(\rho) + \Big(\frac{1}{\rho} - \frac{3}{1-\rho}\Big)\varphi'(\rho) - \Big(\frac{1}{\rho} + \frac{1}{1-\rho}\Big)\frac{\varphi(\rho)}{\rho} = \frac{Q_r(\rho)\, b^2}{D_0(1-\rho)^3},$$

where we have set $\nu = \frac{1}{3}$ for simplicity. Note that the right-hand side of this equation, which we shall designate by $f(\rho)$, is not a fixed function of ρ but depends on our choice of the loading. Hence we consider $f(x)$ as a *forcing function.*

For the *boundary conditions* (for once we use this word in its original literal sense!) we shall use $w'(0) = w'(a) = 0$, which translates to $\varphi'(0) = \varphi'(\beta) = 0$ in the $\varphi = \varphi(\rho)$ formulation, with the abbreviation $\beta := a/b < 1$. Physically speaking, this corresponds to a plate that is clamped in the center and left free at its periphery (this comes from translating suitable boundary conditions for the displacement $w = w(r)$ and taking the appropriate limits). In summary, we have the boundary problem

$$\boxed{\begin{array}{l} \varphi''(\rho) + \left(\frac{1}{\rho} - \frac{3}{1-\rho}\right)\varphi'(\rho) - \left(\frac{1}{\rho} + \frac{1}{1-\rho}\right)\frac{\varphi(\rho)}{\rho} = f(\rho), \\ \varphi(0) = \varphi(\beta) = 0, \end{array}} \tag{21}$$

which is indeed of the type discussed in Section 3, by a simple scaling from $I = [0, 1]$ to $[0, \beta]$. Its treatment in the `GreenGroebner` package proceeds as follows [we apologize for the lousy graphics rendering—we plan to fix this as soon as possible]:

$$\mathtt{sol1 = BPSolve}\left[\boxed{\begin{array}{c} \phi''[\rho] + \left(\frac{1}{\rho} - \frac{3}{1-\rho}\right)\phi'[\rho] - \left(\frac{1}{\rho^2} + \frac{1}{(1-\rho)\,\rho}\right)\phi[\rho] == \mathrm{f}[\rho] \\ \phi[\mathrm{L}] == \phi[\mathrm{R}] == 0 \end{array}}\,, \phi, \rho, 0, \beta, \mathtt{Params} \to \{\beta\}, \mathtt{PostProc} \to \mathtt{FullSimplify}\right]$$

$$\begin{aligned}
&\frac{1}{6}\,\frac{1}{(-1+\rho)^2}\,\frac{1}{\rho} \\
&\quad \left(\rho^2(-3+2\rho)\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\mathrm{f}[\xi]\,d\xi\right) + (3-2\rho)\,\rho^2\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\mathrm{f}[\xi]\,d\xi\right) + 3\,(3-2\rho)\,\rho^2\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\xi\,\mathrm{f}[\xi]\,d\xi\right) + 3\,\rho^2(-3+2\rho)\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi\,\mathrm{f}[\xi]\,d\xi\right) - \right. \\
&\quad\ 3\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^2\,\mathrm{f}[\xi]\,d\xi\right) + 3\,(3-2\rho)\,\rho^2\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^2\,\mathrm{f}[\xi]\,d\xi\right) + 11\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^3\,\mathrm{f}[\xi]\,d\xi\right) + \rho^2(-3+2\rho)\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^3\,\mathrm{f}[\xi]\,d\xi\right) - \\
&\quad\ 15\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^4\,\mathrm{f}[\xi]\,d\xi\right) + 9\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^5\,\mathrm{f}[\xi]\,d\xi\right) - 2\left(\int_0^\rho \frac{1}{(-1+\xi)^2}\,\xi^6\,\mathrm{f}[\xi]\,d\xi\right) + (3-2\rho)\,\rho^2\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\xi^2\,\mathrm{f}[\xi]\,d\xi\right)\left(-3-3\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right) + \\
&\quad\ (3-2\rho)\,\rho^2\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\xi^3\,\mathrm{f}[\xi]\,d\xi\right)\left(1+11\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right) + (3-2\rho)\,\rho^2\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\xi^4\,\mathrm{f}[\xi]\,d\xi\right)\left(-15\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right) + \\
&\quad\ \left.(3-2\rho)\,\rho^2\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\xi^6\,\mathrm{f}[\xi]\,d\xi\right)\left(-2\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right) + (3-2\rho)\,\rho^2\left(\int_0^\beta \frac{1}{(-1+\xi)^2}\,\xi^5\,\mathrm{f}[\xi]\,d\xi\right)\left(9\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)\right)
\end{aligned}$$

The output is the Green's operator G of (21) applied to a generic forcing function $f(\rho)$, giving the solution $\varphi(\rho) = Gf\,(\rho)$ as an integral

$$\int_0^1 g(\rho, \xi)\, f(\xi)\, d\xi$$

in terms of the Green's function $g(\rho, \xi)$; the latter can also be retrieved explicitly if this is desired (note that the expression is clipped on the right-hand side so the two case conditions $\xi \leq \rho$ and $\rho < \xi$ labelling the two lines are not visible):

$$\begin{aligned}
&\mathtt{gf = BPSolve}\left[\boxed{\begin{array}{c} \phi''[\rho] + \left(\frac{1}{\rho} - \frac{3}{1-\rho}\right)\phi'[\rho] - \left(\frac{1}{\rho^2} + \frac{1}{(1-\rho)\,\rho}\right)\phi[\rho] == \mathrm{f}[\rho] \\ \phi[\mathrm{L}] == \phi[\mathrm{R}] == 0 \end{array}}\,, \phi, \rho, 0, \beta, \mathtt{Params} \to \{\beta\}, \mathtt{PostProc} \to \mathtt{FullSimplify}, \right. \\
&\quad \left. \mathtt{Output} \to \mathtt{GreensFunction}\right]
\end{aligned}$$

$$\begin{cases}
\frac{1}{6}\,\frac{1}{(-1+\xi)^2}\,\xi^2\,\frac{1}{(-1+\rho)^2}\,\frac{1}{\rho}\left(-3+11\xi-15\xi^2+9\xi^3-2\xi^4+9\rho^2-3\xi\rho^2-6\rho^3+2\xi\rho^3+(3-2\rho)\,\rho^2\left(-3-3\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)+\xi\,(3-2\rho)\,\rho^2\left(1+11\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)\right) \\
-\frac{1}{6}\,\frac{1}{(-1+\xi)^2}\,\frac{1}{(-1+\rho)^2}\,\rho\,(-3+2\rho)\left(-1+3\xi+\xi^2\left(-3-3\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)+\xi^3\left(1+11\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)+\xi^4\left(-15\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)+\xi^6\left(-2\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)+\xi^5\left(9\,\frac{1}{\beta^2}\,\frac{1}{-3+2\beta}\right)\right)
\end{cases}$$

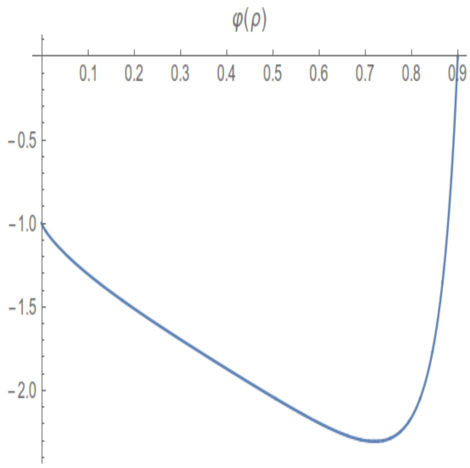
Let us now choose a constant loading $q(\rho) = q_0$. Taking the cut-off into account, this leads to the forcing function $f(\rho) = \frac{\rho+1}{\rho(\rho-1)^2}$. In this case, we can compute the solution $\varphi(\rho)$ of (21) explicitly. For definiteness, let us choose a cut-off at $\beta = 0.9$. In that case, we obtain

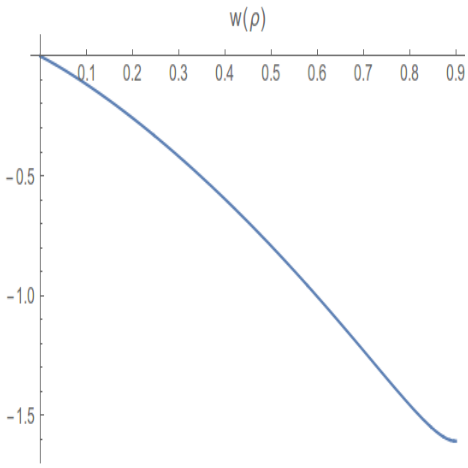
$$\begin{aligned}\varphi(\rho) = \Big(& -2790\rho^3 + 1944\,\rho^3\log(9/\rho-9) + 2000\,\rho^3\log(1-\rho)\\ &-2000\rho^3\log(10-10\rho) + 4671\rho^2 - 2916\rho^2\log(9/\rho-9) - 3000\rho^2\log(1-\rho)\\ &+3000\rho^2\log(10-10\rho) - 1944\rho + 972\log(1-\rho)\Big)\Big/\Big(2916(\rho-1)^2\rho\Big),\end{aligned}$$

and the corresponding displacement $w(\rho) = -\int_0^\beta \varphi(\rho)\,d\rho$ is given by

$$\begin{aligned}w(\rho) = \Big(& -972(\rho-1)\,\mathrm{Li}_2\left((1-\rho)^{-1}\right) + 972(\rho-1)\,\mathrm{Li}_2(1-\rho) - 2790\rho^2\\ &+1944\rho^2\log(9) + 3944\rho^2\log(1-\rho) - 2000\rho^2\log(-10(\rho-1))\\ &-1944\rho^2\log(\rho) + 2853\rho + 500\rho\log^2(10) + 14\rho\log^2(1-\rho)\\ &-500\rho\log^2(-10(\rho-1)) - 14\log^2(1-\rho) + 500\log^2(-10(\rho-1))\\ &-972\rho\log(9) - 1500\rho\log(100) + 486\rho\log(81)\log(1-\rho) - 7825\rho\log(1-\rho)\\ &+4000\rho\log(-10(\rho-1)) + 972\rho\log(1-\rho)\log(\rho) + 972\rho\log(\rho)\\ &-1944\log(9/\rho-9) - 486\log(81)\log(1-\rho) + 6825\log(1-\rho)\\ &-3000\log(-10(\rho-1)) - 972\log(1-\rho)\log(\rho) - 1944\log(\rho) - 500\log^2(10)\\ &+1944\log(9) + 1500\log(100)\Big)\Big/2916(1-\rho)\end{aligned}$$

We have displayed the graphs of these solutions $\varphi(\rho)$ and $w(\rho)$ below.





Of course one could use different functional gradings and/or loading functions, and the integrals would not always come out in closed form. In this case one can resort to numerical integration (which is also supported by *Mathematica*).

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