Permutations of the integers induce only the trivial automorphism of the Turing degrees

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Abstract

Let π be an automorphism of the Turing degrees induces by a homeomorphism φ of the Cantor space 2^ω such that φ preserves all Bernoulli measures. It is proved that π must be trivial. In particular, a permutation of ω can only induce the trivial automorphism of the Turing degrees.

Contents

1 Introduction **2**

2 Universal algebra setup **2**

3 Permutations preserve randomness **3**

4 Cones have small measure **4**

5 Main result **6**

6 Computing the permutation **6**

7 Measure-preserving homeomorphisms of the Cantor set **8**

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1 Introduction

Let $\mathscr{D}_\mathrm{T}$ denote the set of Turing degrees and let $\leq$ denote its ordering. This article gives a partial answer to the following famous question.

Question 1. Does there exist a nontrivial automorphism of $\mathscr{D}_\mathrm{T}$?

Definition 1. A bijection $\pi : \mathscr{D}_\mathrm{T} \to \mathscr{D}_\mathrm{T}$ is an *automorphism* of $\mathscr{D}_\mathrm{T}$ if for all $\mathbf{x}, \mathbf{y} \in \mathscr{D}_\mathrm{T}$, $\mathbf{x} \leq \mathbf{y}$ iff $\pi(\mathbf{x}) \leq \pi(\mathbf{y})$. If moreover there exists an $\mathbf{x}$ with $\pi(\mathbf{x}) \neq \mathbf{x}$ then π is *nontrivial*.

Question 1 has a long history. Already in 1977, Jockusch and Solovay [3] showed that each jump-preserving automorphism of the Turing degrees is the identity above $\mathbf{0}^{(4)}$. Nerode and Shore 1980 [8] showed that each automorphism (not necessarily jump-preserving) is equal to the identity on some cone. Slaman and Woodin [11] showed that each automorphism is equal to the identity on the cone above $\mathbf{0}''$.

Haught and Slaman [2] used permutations of the integers to obtain automorphisms of the polynomial-time Turing degrees in an ideal (below a fixed set).

Theorem 2 (Haught and Slaman [2])**.** *There is a permutation of $2^{<\omega}$, or equivalently of ω, that induces a nontrivial automorphism of*

$$(\mathsf{PTIME}^A, \leq_{\mathrm{pT}}).$$

for some A.

Our result can be seen as a contrast to the following work of Kent.

Definition 3. $A \subset \omega$ is *cohesive* if for each recursively enumerable set W_e, either $A \cap W_e$ is finite or $A \cap (\omega \setminus W_e)$ is finite.

Theorem 4 (Kent [9, Theorem 12.3.IX], [4, 5])**.** *There exists a permutation f such that*

(i) for all recursively enumerable B, $f(B)$ and $f^{-1}(B)$ are recursively enumerable (and hence for all recursive A, $f(A)$ and $f^{-1}(A)$ are recursive);

(ii) f is not recursive.

Proof. Kent's permutation is just any permutation of a cohesive set (and the identity off the cohesive set). $\square$

2 Universal algebra setup

Definition 5. The *pullback* of $f : \omega \to \omega$ is $f^* : \omega^\omega \to \omega^\omega$ given by

$$f^*(A)(n) = A(f(n)).$$

We often write $F = f^*$. Given a set $S \subseteq \omega$ let $\mathscr{D}_S = S^\omega / \equiv_\mathrm{T}$. Thus the elements of $\mathscr{D}_S$ are of the form

$$[g]_S = \{\, h \in S^\omega \mid h \equiv_\mathrm{T} g \,\}, \qquad g \in S^\omega.$$

Given $F : S^\omega \to S^\omega$, let $F_S : \mathscr{D}_S \to \mathscr{D}_S$ be defined by

$$F_S([A]_S) = [F(A)]_S.$$

If $F = f^*_S$ then we say that F_S and F are both *induced* by f.

Lemma 6. *For each $f : \omega \to \omega$ and each $S \subseteq \omega$, the pullback f^* maps S^ω into S^ω.*

Proof.

$$A \in S^\omega,\ n \in \omega \quad \Longrightarrow \quad f^*(A)(n) = A(f(n)) \in S. \qquad \square$$

In light of Lemma 6, we can define:

Definition 7. $f^*_S : \mathscr{D}_S \to \mathscr{D}_S$ is the map given by

$$f^*_S([g]_S) = [f^*(g)]_S.$$

For $S \subseteq \omega$ (with particular attention to $S \in \{2, \omega\}$), let

$$\mathscr{D}_S = S^\omega / \equiv_\mathrm{T}.$$

Our main result concerns $\mathscr{D}_2$; the corresponding result for $\mathscr{D}_\omega$ is much easier:

Theorem 8. *Let $f : \omega \to \omega$ be a bijection and let f^* be its pullback. If f^*_S is an automorphism of $\mathscr{D}_S$ for some infinite computable set S, then f is computable.*

Proof. Let $\eta : \omega \to S$ be a computable bijection between ω and S. Then for all $x \in \omega$,

$$f^*(\eta \circ f^{-1})(x) = (\eta \circ f^{-1})(f(x)) = \eta(f^{-1}(f(x))) = \eta(x).$$

Since $\eta \in S^\omega$ is computable and f^*_S is an automorphism, $\eta \circ f^{-1} \in S^\omega$ must be computable. Hence f is computable. $\square$

3 Permutations preserve randomness

Theorem 9. *If B is f-μ_p-random, $F = f^*$ and $A = F(B)$ or $A = F^{-1}(B)$, then A is f-μ_p-random.*

Proof. First note that f^{-1}-μ_p-randomness is the same as f-μ_p-randomness since $f \equiv_\mathrm{T} f^{-1}$. Thus the result for $A = F^{-1}(B)$ follows from the result for $A = F(B)$. So suppose $A = F(B)$ and A is not f-μ_p-random. So $A \in \cap_n U_n$ where $\{U_n\}_n$ is an f-μ_p-ML test. Then

$$B \in \{X \mid F(X) \in \cap_n U_n\} = \cap_n V_n$$

where

$$V_n = \{X \mid F(X) \in U_n\} = F^{-1}(U_n)$$

We claim that V_n is $\Sigma_1^0(f)$ (uniformly in n) and $\mu_p(V_n) = \mu_p(U_n)$. Write $U_n = \cup_k[\sigma_k]$ where the strings σ_k are all incomparable. Then

$$V_n = \cup_k F^{-1}([\sigma_k])$$

and

$$\mu_p[\sigma_k] = \mu_p F^{-1}([\sigma_k])$$

and the $F^{-1}([\sigma_k])$, $k \in \omega$ are still disjoint and clopen. (If we think of $\sigma \in 2^{<\omega}$ as a partial function from ω to 2 then

$$F^{-1}([\sigma]) = \{X \mid F(X) \in [\sigma]\}$$

$$= \{X \mid X(f(n)) = \sigma(n), n < |\sigma|\} = [\{\langle f(n), \sigma(n)\rangle \mid n < |\sigma|\}].)$$

Thus $\{V_n\}_n$ is another f-μ_p-ML test, and so B is not f-μ_p-random, which completes the proof. □

Theorem 10. $\mu_p(\{A : A \geq_{\mathrm{T}} p\}) = 1$, *in fact if A is μ_p-ML-random then A computes p.*

Proof. Kjos-Hanssen [6] showed that each Hippocratic μ_p-random set computes p. In particular, each μ_p-random set computes p. □

4 Cones have small measure

Definition 11 (Bernoulli measures)**.** For each $n \in \omega$,

$$\mu_p(\{X \in 2^\omega : X(n) = 1\}) = p$$

and $X(0), X(1), X(2), \ldots$ are mutually independent random variables.

Definition 12. An *ultrametric* space is a metric space with metric d satisfying the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Definition 13. A *Polish space* is a separable completely metrizable topological space.

Definition 14. In a metric space, $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$.

Theorem 15 ([7, Proposition 2.10])**.** *Suppose that X is a Polish ultrametric space, μ is a probability measure on X, and $\mathcal{A} \subseteq X$ is Borel. Then*

$$\lim_{\varepsilon \to 0} \frac{\mu(\mathcal{A} \cap B(x, \varepsilon))}{\mu(B(x, \varepsilon))} = 1$$

for μ-almost every $x \in \mathcal{A}$.

Definition 16. For any measure μ define the conditional measure by

$$\mu(\mathcal{A} \mid \mathcal{B}) = \frac{\mu(\mathcal{A} \cap \mathcal{B})}{\mu(\mathcal{B})}.$$

A measurable set $\mathcal{A}$ has density d at X if

$$\lim_n \mu_p(\mathcal{A} \mid [X \upharpoonright n]) = d.$$

Let $\Xi(\mathcal{A}) = \{X : \mathcal{A} \text{ has density 1 at } X\}$.

Theorem 17 (Lebesgue Density Theorem for μ_p)**.** *For Cantor space with Bernoulli(p) product measure μ_p, the Lebesgue Density Theorem holds:*

$$\lim_{n\to\infty} \frac{\mu_p(\mathcal{A} \cap [x \upharpoonright n])}{\mu_p([x \upharpoonright n])} = 1$$

for μ-almost every $x \in \mathcal{A}$.

If $\mathcal{A}$ is measurable then so is $\Xi(\mathcal{A})$. Furthermore, the measure of the symmetric difference of $\mathcal{A}$ and $\Xi(\mathcal{A})$ is zero, so $\mu(\Xi(\mathcal{A})) = \mu(\mathcal{A})$.

Proof. Consider the ultrametric $d(x, y) = 2^{-\min\{n:x(n)\neq y(n)\}}$. It induces the standard topology on 2^ω. Apply Theorem 15. □

Sacks [10] and de Leeuw, Moore, Shannon, and Shapiro [1] showed that each cone in the Turing degrees has measure zero. Here we use Theorem 17 to extend this to μ_p.

Theorem 18. *If $\mu_p(\{X : W_e^X = A\}) > 0$ then A is c.e. in p.*

Proof. Suppose $\mu_p(\{X : W_e^X = A\}) > 0$. Then $S := \{X \mid W_e^X = A\}$ has positive measure, so $\Xi(S)$ has positive measure, and hence by Theorem 15 there is an X such that S has density 1 at X. Thus, there is an n such that $\mu_p(S \mid [X \upharpoonright n]) > \frac{1}{2}$. Let $\sigma = X \upharpoonright n$. We can now enumerate A using p by taking a "vote" among the sets extending σ. More precisely, $n \in A$ iff

$$\mu_p(\{Y : \sigma \prec Y \wedge n \in W_e^Y\}) > \frac{1}{2},$$

and the set of n for which this holds is clearly c.e. in p. □

Theorem 19. *Each cone strictly above p has μ_p-measure zero:*

$$\mu_p(\{A : A \geq_{\mathrm{T}} q\}) = 1 \qquad \Longrightarrow \qquad q \leq_{\mathrm{T}} p.$$

Proof. If A can compute q then A can enumerate both q and the complement of q. Hence by Theorem 18, q is both c.e. in p and co-c.e. in p; hence $q \leq_{\mathrm{T}} p$. □

5 Main result

We are now ready to prove our main result Theorem 20 that no nontrivial automorphism of the Turing degrees is induced by a permutation of ω.

Theorem 20. *If π is an automorphism of $\mathscr{D}_2$ which is induced by a permutation of ω then $\pi(\mathbf{p}) = \mathbf{p}$ for each $\mathbf{p} \in \mathscr{D}_{\mathrm{T}}$.*

Proof. Fix a permutation $f : \omega \to \omega$ and let $F = f^* \upharpoonright 2^\omega$. Let B be f-μ_p-random. We claim that B computes $F(p)$.

By Theorem 10, for any f-μ_p random A, we have $p \leq_{\mathrm{T}} A$, hence $F(p) \leq_{\mathrm{T}} F(A)$. So it suffices to represent B as $F(A)$.

Now $B = F(F^{-1}(B))$. Let $A = F^{-1}(B)$. By Theorem 9, A is f-μ_p-random. Thus every f-μ_p-random computes $F(p)$.

Thus we have completed the proof of our claim that μ_p-almost every real computes $F(p)$.

By Theorem 19 it follows that $F(p) \leq_{\mathrm{T}} p$.

By considering the inverse f^{-1} we also obtain $F^{-1}(p) \leq_{\mathrm{T}} p$ and hence $p \leq_{\mathrm{T}} F(p)$. So $F(p) \equiv_{\mathrm{T}} p$ and F induces the identity automorphism. $\square$

6 Computing the permutation

Theorem 21. *Let $f : \omega \to \omega$ be a permutation. Let $F = f^*$ be its pullback (Definition 5) to 2^ω. If for positive Lebesgue measure many G, $F(G) \leq_T G$, then f is recursive.*

Proof. By the Lebesgue Density Theorem we can get a Φ and a σ such that, if μ_σ denotes conditional probability on σ and $E = \{A : F(A) = \Phi^A\}$, then

$$\mu_\sigma(E) \geq 95\%.$$

For simplicity let us write $p_n(A) = A+n = A \cup \{n\}$ and $m_n(A) = A-n = A \setminus \{n\}$. Then $p_n^{-1}E = \{A : p_n(A) \in E\}$. Note that

$$E \subseteq p_n^{-1}(E) \cup m_n^{-1}(E)$$

and

$$E^c \subseteq p_n^{-1}(E^c) \cup m_n^{-1}(E^c)$$

Then

$$\mu_\sigma(E) \leq \mu_\sigma(p_n^{-1}(E) \cup m_n^{-1}(E)) \leq \mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E))$$

We now have

$$\mu_\sigma\{A : F(A+n) = \Phi^{A+n}\} \geq 90\%$$

and

$$\mu_\sigma\{A : F(A-n) = \Phi^{A-n}\} \geq 90\%;$$

Indeed, the events $m_n^{-1}(A)$, $p_n^{-1}(A)$ are each independent of the event $n \in A$, so for $n > |\sigma|$,

$$\begin{aligned}
95\% \le \mu_\sigma(E) &= \mu_\sigma(p_n^{-1}(E) \mid n \in A)\mu_\sigma(n \in A) + \mu_\sigma(p_n^{-1}(E) \mid n \notin A)\mu_\sigma(n \notin A) \\
&= \frac{1}{2}\left(\mu_\sigma(p_n^{-1}(E) \mid n \in A) + \mu_\sigma(m_n^{-1}(E) \mid n \notin A)\right) \\
&= \frac{1}{2}\left(\mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E))\right)
\end{aligned}$$

which gives

$$1.9 \le \mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E)) \le 1 + \min\{\mu_\sigma(p_n^{-1}(E)), \mu_\sigma(m_n^{-1}(E))\}.$$

Also $F(A-n)$ and $F(A+n)$ differ in exactly one bit, namely $f^{-1}(n)$, for all A:

$$\begin{aligned}
F(A-n)(b) \neq F(A+n)(b) &\iff (A-n)(f(b)) \neq (A+n)(f(b)) \\
&\iff n = f(b) \iff b = f^{-1}(n),
\end{aligned}$$

that is

$$\{A : (\forall b)(F(A+n)(b) \neq F(A-n)(b) \leftrightarrow b = f^{-1}(n))\} = 2^\omega.$$

Let $D_{n,b} = \{A : \Phi^{A+n}(b)\downarrow \neq \Phi^{A-n}(b)\downarrow\}$. For $n > |\sigma|$,

$$\mu_\sigma\left(D_{n,f^{-1}(n)} \setminus \bigcup_{b \neq f^{-1}(n)} D_{n,b}\right) = \mu_\sigma\{A : (\forall b)(A \in D_{n,b} \leftrightarrow b = f^{-1}(n))\} \ge 80\%$$

since

$$\begin{aligned}
&\mu_\sigma\{A : \neg(\forall b)(A \in D_{n,b} \quad \leftrightarrow \quad b = f^{-1}(n))\} \\
&\le \mu_\sigma(\neg p_n^{-1}(E)) + \mu_\sigma(\neg m_n^{-1}(E)) \le 10\% + 10\% = 20\%.
\end{aligned}$$

Therefore, given any n, we can compute $f^{-1}(n)$: enumerate computations until we have found some bit b such that

$$\mu_\sigma D_{n,b} \ge 80\%.$$

Then $b = f^{-1}(n)$.

Thus f^{-1} is computable and hence so is f. □

Theorem 22. *If π is an automorphism of $\mathscr{D}_\mathrm{T}$ which is induced by a permutation f of ω then f is recursive.*

Proof. By Theorem 20, $f^*(G) \equiv_\mathrm{T} G$ for each $G \in 2^\omega$. By Theorem 21, f is recursive. □

7 Measure-preserving homeomorphisms of the Cantor set

Proposition 23. *A permutation of ω induces a homeomorphism of 2^ω that is μ_p-preserving for each p.*

Proposition 24. *There exist homeomorphisms of 2^ω that are μ_p-preserving for each p, but are not induced by a permutation.*

Proof. Map
$$[1] \mapsto [111] \cup [001] \cup [101] \cup [110]$$
(more generally, any collection of cylinders of strings of length 3 including 2 strings of Hamming weight 2 and 1 of Hamming weight 1).

Another way to express this is that the homeomorphism preserves the fraction of 1s in a certain sense.

More precisely,

$$\begin{aligned} 100 &\mapsto 001, \\ 101 &\mapsto 101, \\ 110 &\mapsto 110, \\ 111 &\mapsto 111. \end{aligned}$$

$\square$

Theorem 25. *Suppose φ is a homeomorphism of 2^ω which is μ_p-preserving for all p (it suffices to require this for infinitely many p, or for a single transcendental p). Suppose φ induces an automorphism π of the Turing degrees. Then $\pi = \mathrm{id}$.*

We omit the proof which follows along the same lines as before.

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