Affine-Invariant Orders on the Set of Positive-Definite Matrices

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Abstract. We introduce a family of orders on the set S_n^+ of positive-definite matrices of dimension n derived from the homogeneous geometry of S_n^+ induced by the natural transitive action of the general linear group $GL(n)$. The orders are induced by affine-invariant cone fields, which arise naturally from a local analysis of the orders that are compatible with the homogeneous structure of S_n^+. We then revisit the well-known Löwner-Heinz theorem and provide an extension of this classical result derived using differential positivity with respect to affine-invariant cone fields.

1 Introduction

The question of how one can order the elements of a space in a consistent and well-defined manner is of fundamental importance to many areas of applied mathematics, including the theory of monotone functions and matrix means in which the notion of order plays a defining role [9, 6, 1, 8] . These concepts play an important role in a wide variety of applications across information geometry where one is interested in performing statistical analysis on sets of matrices. In such applications, the choice of order relation is often taken for granted. This choice, however, is of crucial significance since a function that is not monotone with respect to one order, may be monotone with respect to another, in which case powerful results from monotonicity theory would become relevant.

In this paper, we outline an approach to systematically generate orders on homogeneous spaces, which form a class of nonlinear spaces that are ubiquitous in many applications in information engineering and control theory. A homogeneous space is a manifold on which a Lie group acts transitively, in the sense that any point on the manifold can be mapped onto any other point by an element of a group of transformations that act on the space. The geometry of homogeneous spaces, coupled with the observation that cone fields induce conal orders on continuous spaces [7], forms the basis for the approach taken in this paper. The aim is to systematically generate cone fields that are invariant with respect to the homogeneous geometry, thereby defining families of conal orders built upon the underlying symmetries of the space.

The focus of this paper is on ordering the elements of the set of symmetric positive-definite matrices S_n^+ of dimension n. Positive definite matrices arise in

numerous applications, including as covariance matrices in statistics and computer vision, as variables in convex and semidefinite programming, as unknowns in fundamental problems in systems and control theory, as kernels in machine learning, and as diffusion tensors in medical imaging. The space S_n^+ forms a smooth manifold that can be viewed as a homogeneous space admitting a transitive action by the general linear group $GL(n)$, which endows the space with an affine-invariant geometry as reviewed in Section 2. In Section 3, this geometry is used to construct affine-invariant cone fields and new partial orders on S_n^+. In Section 4, we discuss how differential positivity [5] can be used to study and characterize monotonicity on S_n^+ with respect to the invariant orders introduced in this paper. We also state a generalized version of the celebrated Löwner-Heinz theorem [9, 6] of operator monotonicity theory derived using this approach.

2 Homogeneous geometry of S_n^+

The set S_n^+ of symmetric positive definite matrices of dimension n has the structure of a homogeneous space with a transitive $GL(n)$-action. This follows by noting that any $\Sigma \in S_n^+$ admits a Cholesky decomposition $\Sigma = AA^T$ for some $A \in GL(n)$. The Cauchy polar decomposition of the invertible matrix A yields a unique decomposition $A = PQ$ of A into an orthogonal matrix $Q \in O(n)$ and a symmetric positive-definite matrix $P \in S_+^n$. Now note that if Σ has Cholesky decomposition $\Sigma = AA^T$ and A has a Cauchy polar decomposition $A = PQ$, then $\Sigma = PQQ^TP = P^2$. That is, Σ is invariant with respect to the orthogonal part Q of the polar decomposition. Therefore, we can identify any $\Sigma \in S_n^+$ with the equivalence class $[\Sigma^{1/2}] = \Sigma^{1/2} \cdot O(n)$ in the quotient space $GL(n)/O(n)$, where $\Sigma^{1/2}$ denotes the unique positive definite square root of Σ. That is,

$$S_n^+ \cong GL(n)/O(n). \tag{1}$$

The identification in (1) can also be made by noting the transitive action of $GL(n)$ on S_n^+ defined by

$$\tau_A : \Sigma \mapsto A\Sigma A^T \quad \forall A \in GL(n),\ \forall \Sigma \in S_n^+. \tag{2}$$

This action is said to be *almost effective* in the sense that $\pm I$ are the only elements of $GL(n)$ that fix every $\Sigma \in S_n^+$. The isotropy group of this action at $\Sigma = I$ is precisely $O(n)$, since $\tau_Q : I \mapsto QIQ^T = I$ if and only if $Q \in O(n)$. Once again, if $\Sigma \in S_n^+$ has Cholesky decomposition $\Sigma = AA^T$ and A has polar decomposition $A = PQ$, then $\tau_A(I) = AIA^T = P^2 = \Sigma$.

A homogeneous space G/H is said to be reductive if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. Recall that the Lie algebra $\mathfrak{gl}(n)$ of $GL(n)$ consists of the set $\mathbb{R}^{n\times n}$ of all real $n \times n$ matrices equipped with the Lie bracket $[X, Y] = XY - YX$, while the Lie algebra of $O(n)$ is $\mathfrak{o}(n) = \{X \in \mathbb{R}^{n\times n} : X^T = -X\}$. Since any matrix $X \in \mathbb{R}^{n\times n}$ has a unique decomposition $X = \frac{1}{2}(X - X^T) + \frac{1}{2}(X + X^T)$, as a sum of an antisymmetric part and a symmetric part, we have $\mathfrak{gl}(n) = \mathfrak{o} \oplus \mathfrak{m}$, where $\mathfrak{m} = \{X \in \mathbb{R}^{n\times n} : X^T =$

$X\}$. Furthermore, since $\mathrm{Ad}_Q(S) = QSQ^{-1} = QSQ^T$ is a symmetric matrix for each $S \in \mathfrak{m}$, we have $\mathrm{Ad}_{O(n)} = \{QSQ^{-1} : Q \in O(n),\ S \in \mathfrak{m}\} \subseteq \mathfrak{m}$. Hence, $S_n^+ = GL(n)/O(n)$ is indeed a reductive homogeneous space with reductive decomposition $\mathfrak{gl}(n) = \mathfrak{o}(n) \oplus \mathfrak{m}$.

The tangent space $T_oS_n^+$ of S_n^+ at the base-point $o = [I] = I \cdot O(n)$ is identified with $\mathfrak{m}$. For each $\Sigma \in S_n^+$, the action $\tau_{\Sigma^{1/2}} : S_n^+ \to S_n^+$ induces the vector space isomorphism $d\tau_{\Sigma^{1/2}}|_I : T_IS_n^+ \to T_\Sigma S_n^+$ given by

$$d\tau_{\Sigma^{1/2}}\big|_I X = \Sigma^{1/2} X \Sigma^{1/2}, \quad \forall X \in \mathfrak{m}. \tag{3}$$

The map (3) can be used to extend structures defined in $T_oS_n^+$ to structures defined on the tangent bundle TS_n^+ through affine-invariance, provided that the structures in $T_oS_n^+$ are $\mathrm{Ad}_{O(n)}$-invariant. The $\mathrm{Ad}_{O(n)}$-invariance is required to ensure that the extension to TS_n^+ is unique and thus well-defined. For instance, any homogeneous Riemannian metric on $S_n^+ \cong GL(n)/O(n)$ is determined by an $\mathrm{Ad}_{O(n)}$-invariant inner product on $\mathfrak{m}$. Any such inner product induces a norm that is rotationally invariant and so can only depend on the scalar invariants $\mathrm{tr}(X^k)$ where $k \geq 1$ and $X \in \mathfrak{m}$. Moreover, as the inner product is a quadratic function, $\|X\|^2$ must be a linear combination of $(\mathrm{tr}(X))^2$ and $\mathrm{tr}(X^2)$. Thus, any $\mathrm{Ad}_{O(n)}$-invariant inner product on $\mathfrak{m}$ must be a scalar multiple of

$$\langle X, Y \rangle_{\mathfrak{m}} = \mathrm{tr}(XY) + \mu\, \mathrm{tr}(X)\, \mathrm{tr}(Y), \tag{4}$$

where μ is a scalar parameter with $\mu > -1/n$ to ensure positive-definiteness [12]. Therefore, the corresponding affine-invariant Riemannian metrics are generated by (3) and given by

$$\begin{aligned} \langle X, Y \rangle_\Sigma &= \langle \Sigma^{-1/2} X \Sigma^{-1/2}, \Sigma^{-1/2} Y \Sigma^{-1/2} \rangle_{\mathfrak{m}} \\ &= \mathrm{tr}(\Sigma^{-1} X \Sigma^{-1} Y) + \mu\, \mathrm{tr}(\Sigma^{-1} X)\, \mathrm{tr}(\Sigma^{-1} Y), \end{aligned} \tag{5}$$

for $\Sigma \in S_n^+$ and $X, Y \in T_\Sigma S_n^+$. In the case $\mu = 0$, (5) yields the most commonly used 'natural' Riemannian metric on S_n^+, which corresponds to the Fisher information metric for the multivariate normal distribution [4, 13], and has been widely used in applications such as tensor computing in medical imaging.

3 Affine-invariant orders

3.1 Affine-invariant cone fields

A cone field $\mathcal{K}$ on S_n^+ smoothly assigns a cone $\mathcal{K}(\Sigma) \subset T_\Sigma S_n^+$ to each point $\Sigma \in S_n^+$. We say that $\mathcal{K}$ is affine-invariant or homogeneous with respect to the quotient geometry $S_n^+ \cong GL(n)/O(n)$ if

$$\left(d\tau_{\Sigma_2^{1/2}\Sigma_1^{-1/2}}\big|_{\Sigma_1} \right) \mathcal{K}(\Sigma_1) = \mathcal{K}(\Sigma_2), \tag{6}$$

for all $\Sigma_1, \Sigma_2 \in S_n^+$. The procedure we will use for constructing affine-invariant cone fields on S_n^+ is similar to the approach taken for generating the affine-invariant Riemannian metrics in Section 2. We begin by defining a cone $\mathcal{K}(I)$ at

I that is $\mathrm{Ad}_{O(n)}$-invariant:

$$X \in \mathcal{K}(I) \Leftrightarrow \mathrm{Ad}_Q X = d\tau_Q X = QXQ^T \in \mathcal{K}(I), \quad \forall Q \in O(n). \tag{7}$$

Using such a cone, we generate a cone field via

$$\mathcal{K}(\Sigma) = d\tau_{\Sigma^{1/2}}\big|_I \mathcal{K}(I) = \{X \in T_\Sigma S_n^+ : \Sigma^{-1/2} X \Sigma^{-1/2} \in \mathcal{K}(I)\}. \tag{8}$$

The $\mathrm{Ad}_{O(n)}$-invariance condition (7) is satisfied if $\mathcal{K}(I)$ has a spectral characterization; that is, we can check to see if any given $X \in T_I S_n^+ \cong \mathfrak{m}$ lies in $\mathcal{K}(I)$ using only properties of X that are characterized by its spectrum. For instance, $\mathrm{tr}(X)$ and $\mathrm{tr}(X^2)$ are both properties of X that are spectrally characterized and indeed $\mathrm{Ad}_{O(n)}$-invariant. Furthermore, quadratic $\mathrm{Ad}_{O(n)}$-invariant cones are defined by inequalities on suitable linear combinations of $(\mathrm{tr}(X))^2$ and $\mathrm{tr}(X^2)$.

Proposition 1 *For any choice of parameter $\mu \in (0, n)$, the set*

$$\mathcal{K}(I) = \{X \in T_I S_n^+ : (\mathrm{tr}(X))^2 - \mu\,\mathrm{tr}(X^2) \geq 0,\ \mathrm{tr}(X) \geq 0\}, \tag{9}$$

defines an $\mathrm{Ad}_{O(n)}$-invariant cone in $T_I S_n^+ = \{X \in \mathbb{R}^{n\times n} : X^T = X\}$.

Proof. $\mathrm{Ad}_{O(n)}$-invariance is clear since $\mathrm{tr}(X^2) = \mathrm{tr}(QXQ^TQXQ^T)$ and $\mathrm{tr}(X) = \mathrm{tr}(QXQ^T)$ for all $Q \in O(n)$. To prove that (9) is a cone, first note that $0 \in \mathcal{K}(I)$ and for $\lambda > 0$, $X \in \mathcal{K}(I)$, we have $\lambda X \in \mathcal{K}(I)$ since $\mathrm{tr}(\lambda X) = \lambda\,\mathrm{tr}(X) \geq 0$ and

$$(\mathrm{tr}(\lambda X))^2 - \mu\,\mathrm{tr}((\lambda X)^2) = \lambda^2[(\mathrm{tr}(X))^2 - \mu\,\mathrm{tr}(X^2)] \geq 0. \tag{10}$$

To show convexity, let $X_1, X_2 \in \mathcal{K}(I)$. Now $\mathrm{tr}(X_1 + X_2) = \mathrm{tr}(X_1) + \mathrm{tr}(X_2) \geq 0$, and

$$\begin{aligned}(\mathrm{tr}(X_1 + X_2))^2 - \mu\,\mathrm{tr}((X_1 + X_2)^2) &= [(\mathrm{tr}(X_1))^2 - \mu\,\mathrm{tr}(X_1^2)] \\ &+ [(\mathrm{tr}(X_2))^2 - \mu\,\mathrm{tr}(X_2^2)] + 2[\mathrm{tr}(X_1)\,\mathrm{tr}(X_2) - \mu\,\mathrm{tr}(X_1X_2)] \geq 0,\end{aligned} \tag{11}$$

since $\mathrm{tr}(X_1X_2) \leq (\mathrm{tr}(X_1^2))^{\frac{1}{2}}(\mathrm{tr}(X_2^2))^{\frac{1}{2}} \leq \frac{1}{\sqrt{\mu}}\,\mathrm{tr}(X_1)\frac{1}{\sqrt{\mu}}\,\mathrm{tr}(X_2)$, where the first inequality follows by Cauchy-Schwarz. Finally, we need to show that $\mathcal{K}(I)$ is pointed. If $X \in \mathcal{K}(I)$ and $-X \in \mathcal{K}(I)$, then $\mathrm{tr}(-X) = -\mathrm{tr}(X) = 0$. Thus, $(\mathrm{tr}(X))^2 - \mu\,\mathrm{tr}(X^2) = -\mu\,\mathrm{tr}(X^2) \geq 0$, which is possible if and only if all of the eigenvalues of X are zero; i.e., if and only if X = 0. □

The parameter μ controls the opening angle of the cone. If $\mu = 0$, then (9) defines the half-space $\mathrm{tr}(X) \geq 0$. As μ increases, the opening angle of the cone becomes smaller and for $\mu = n$ (9) collapses to a ray. For any fixed $\mu \in (0, n)$, we obtain a unique well-defined affine-invariant cone field given by

$$\mathcal{K}(\Sigma) = \{X \in T_\Sigma S_n^+ : (\mathrm{tr}(\Sigma^{-1}X))^2 - \mu\,\mathrm{tr}(\Sigma^{-1}X\Sigma^{-1}X) \geq 0,\ \mathrm{tr}(\Sigma^{-1}X) \geq 0\}. \tag{12}$$

It should be noted that of course not all $\mathrm{Ad}_{O(n)}$-invariant cones at I are quadratic. Indeed, it is possible to construct polyhedral $\mathrm{Ad}_{O(n)}$-invariant cones that arise as the intersections of a collection of spectrally defined half-spaces in $T_I S_n^+$. The clearest example of such a construction is the cone of positive-semidefinite matrices in $T_I S_n^+$, which of course itself has a spectral characterization $\mathcal{K}(I) = \{X \in T_I S_n^+ : \lambda_i(X) \geq 0,\ i = 1, \ldots, n\}$, where $(\lambda_i(X))$ denote the n real eigenvalues of the symmetric matrix X.

3.2 Visualization of affine-invariant cone fields on S_2^+

It is well-known that the set of positive semi-definite matrices of dimension n forms a cone in the space of symmetric $n \times n$ matrices. Moreover, S_n^+ forms the interior of this cone. A concrete visualization of this identification can be made in the $n = 2$ case, as shown in Figure 1. The set S_2^+ can be identified with the interior of $K = \{(x, y, z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 \geq 0,\ z \geq 0\}$, through the map $\phi : S_2^+ \to K$ given by

$$\phi : \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto (x, y, z) = \left(\sqrt{2}b, \frac{1}{\sqrt{2}}(a - c), \frac{1}{\sqrt{2}}(a + c)\right). \tag{13}$$

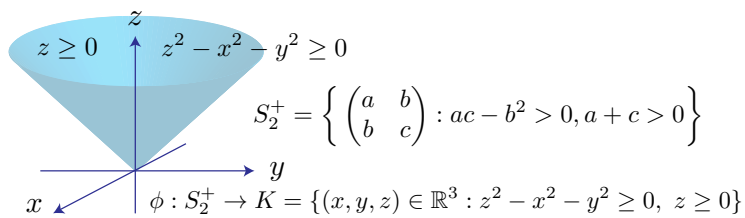

Fig. 1. Identification of S_2^+ with the interior of the closed, convex, pointed cone $K = \{(x, y, z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 \geq 0,\ z \geq 0\}$ in $\mathbb{R}^3$.

Inverting ϕ, we find that $a = \frac{1}{\sqrt{2}}(z+y)$, $b = \frac{1}{\sqrt{2}}x$, $c = \frac{1}{\sqrt{2}}(z-y)$. Note that the point $(x, y, z) = (0, 0, \sqrt{2})$ corresponds to the identity matrix $I \in S_2^+$. We seek to arrive at a visual representation of the affine-invariant cone fields generated from the $\mathrm{Ad}_{O(n)}$-invariant cones (9) for different choices of the parameter μ. The defining inequalities $\mathrm{tr}(X) \geq 0$ and $(\mathrm{tr}(X))^2 - \mu\,\mathrm{tr}(X^2) \geq 0$ in $T_I S_2^+$ take the forms

$$\delta z \geq 0, \quad \text{and} \quad \left(\frac{2}{\mu} - 1\right) \delta z^2 - \delta x^2 - \delta y^2 \geq 0, \tag{14}$$

respectively, where $(\delta x, \delta y, \delta z) \in T_{(0,0,\sqrt{2})} K \cong T_I S_2^+$. Clearly the translation-invariant cone fields generated from this cone are given by the same equations as in (14) for $(\delta x, \delta y, \delta z) \in T_{(x,y,z)} K \cong T_{\phi^{-1}(x,y,z)} S_2^+$.

To obtain the affine-invariant cone fields, note that at $\Sigma = \phi^{-1}(x, y, z) \in S_2^+$, the inequality $\mathrm{tr}(\Sigma^{-1} X) \geq 0$ takes the form

$$\mathrm{tr}\left[\begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} \delta a & \delta b \\ \delta b & \delta c \end{pmatrix}\right] = c\ \delta a - 2b\ \delta b + a\ \delta c \geq 0 \tag{15}$$

$$\Leftrightarrow\ z\ \delta z - x\ \delta x - y\ \delta y \geq 0. \tag{16}$$

Similarly, the inequality $(\mathrm{tr}(\Sigma^{-1} X))^2 - \mu\,\mathrm{tr}(\Sigma^{-1} X \Sigma^{-1} X) \geq 0$ is given by

$$\begin{aligned} 2(x\ \delta x + y\ \delta y - z\ \delta z)^2 - \mu\,\big[&(z^2 + x^2 - y^2)\delta x^2 + (z^2 - x^2 - y^2)\delta y^2 \\ &+ (x^2 + y^2 + z^2)\delta z^2 + 4xy\ \delta x \delta y - 4xz\ \delta x \delta z - 4yz\ \delta y \delta z\big] \geq 0, \end{aligned} \tag{17}$$

where $(\delta x, \delta y, \delta z) \in T_{(x,y,z)}K \cong T_\Sigma S_2^+$. In the case $\mu = 1$, this reduces to $(\frac{2}{\mu} - 1)\delta z^2 - \delta x^2 - \delta y^2 \geq 0$. That is, for $\mu = 1$ the quadratic cone field generated by affine-invariance coincides with the corresponding translation-invariant cone field. Generally, however, affine-invariant and translation-invariant cone fields do not agree, as depicted in Figure 2. Each of the different cone fields in Figure 2 induces a distinct partial order on S_n^+.

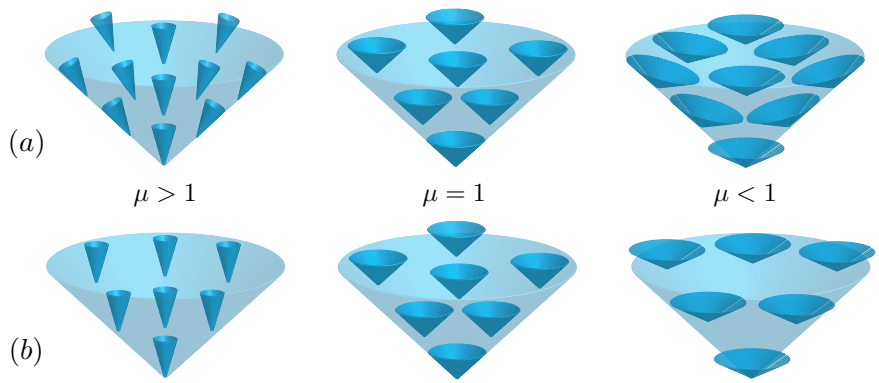

Fig. 2. Cone fields on S_2^+: (a) Quadratic affine-invariant cone fields for different choices of the parameter $\mu \in (0, 2)$. (b) The corresponding translation-invariant cone fields.

3.3 The Löwner order

The Löwner order is the partial order $\geq_L$ on S_n^+ defined by

$$A \geq_L B \quad \Leftrightarrow \quad A - B \geq_L O, \tag{18}$$

where the inequality on the right denotes that $A - B$ is positive semi-definite [2]. The definition in (18) is based on translations and the 'flat' geometry of S_n^+. It is clear that the Löwner order is translation invariant in the sense that $A \geq_L B$ implies that $A + C \geq_L B + C$ for all $A, B, C \in S_n^+$. From the perspective of conal orders, the Löwner order is the partial order induced by the cone field generated by translations of the cone of positive semi-definite matrices at $T_I S_n^+$.

In the previous section, we gave an explicit construction showing that the cone field generated through translations of the cone of positive semi-definite matrices at $T_I S_n^+$ coincides with the cone field generated through affine-invariance in the $n = 2$ case. We will now show that this is a general result which holds for all n. First note that the cone at $T_I S_n^+$ can be expressed as

$$\mathcal{K}(I) = \{X \in T_I S_n^+ : u^T X u \geq 0 \ \forall u \in \mathbb{R}^n, \ u^T X u = 0 \Rightarrow u = 0\}, \tag{19}$$

and the resulting translation-invariant cone field is simply given by

$$\mathcal{K}_T(\Sigma) = \{X \in T_\Sigma S_n^+ : u^T X u \geq 0 \ \forall u \in \mathbb{R}^n, \ u^T X u = 0 \Rightarrow u = 0\}. \tag{20}$$

The corresponding affine-invariant cone field is given by

$$\mathcal{K}_A(\Sigma) = \{X \in T_\Sigma S_n^+ : u^T \Sigma^{-1/2} X \Sigma^{-1/2} u \geq 0 \ \forall u \in \mathbb{R}^n, \\ u^T \Sigma^{-1/2} X \Sigma^{-1/2} u = 0 \Rightarrow u = 0\}, \quad (21)$$

which is seen to be equal to $\mathcal{K}_T$ by introducing the invertible transformation $\bar{u} = \Sigma^{-1/2} u$ in (21). Thus we see that the Löwner order enjoys the special status of being both affine-invariant and translation-invariant, even though its classical definition is based on the 'flat' or translational geometry on S_n^+.

4 Monotonicity on S_n^+

Let f be a map of S_n^+ into itself. We say that f is *monotone* with respect to a partial order $\geq$ on S_n^+ if $f(\Sigma_1) \geq f(\Sigma_2)$ whenever $\Sigma_1 \geq \Sigma_2$. Such functions were introduced by Löwner in his seminal paper [9] on operator monotone functions. Since then operator monotone functions have been studied extensively and found applications to many fields including electrical engineering, network theory, and quantum information theory [3, 10]. One of the most fundamental results in operator theory is the Löwner-Heinz theorem [9, 6] stated below.

Theorem 1 (Löwner-Heinz) *If $\Sigma_1 \geq_L \Sigma_2$ in S_n^+ and $r \in [0,1]$, then*

$$\Sigma_1^r \geq_L \Sigma_2^r. \quad (22)$$

Furthermore, if $n \geq 2$ and $r > 1$, then $\Sigma_1 \geq_L \Sigma_2 \not\Rightarrow \Sigma_1^r \geq_L \Sigma_2^r$.

There are several different proofs of the Löwner-Heinz theorem. See [2, 11, 9, 6], for instance. Most of these proofs are based on analytic methods, such as integral representations from complex analysis. Instead we employ a geometric approach to study monotonicity based on a differential analysis of the system. One of the advantages of such an approach is that it is immediately applicable to all of the conal orders considered in this paper, while providing a deeper geometric insight into the behavior of the map under consideration. Recall that a smooth map $f : S_n^+ \to S_n^+$ is said to be differentially positive with respect to a cone field $\mathcal{K}$ on S_n^+ if

$$\delta\Sigma \in \mathcal{K}(\Sigma) \quad \Rightarrow \quad df|_\Sigma(\delta\Sigma) \in \mathcal{K}(f(\Sigma)), \quad (23)$$

where $df|_\Sigma : T_\Sigma S_n^+ \to T_{f(\Sigma)} S_n^+$ denotes the differential of f at Σ. Assuming that $\geq_\mathcal{K}$ is a partial order induced by $\mathcal{K}$, then f is monotone with respect to $\geq_\mathcal{K}$ if and only if it is differentially positive with respect to $\mathcal{K}$. Applying this to the family of affine-invariant cone fields in (12), we arrive at the following extension to the Löwner-Heinz theorem.

Theorem 2 (Generalized Löwner-Heinz) *For any of the quadratic affine-invariant orders (12) parameterized by μ, and $r \in [0,1]$, the map $f(\Sigma) = \Sigma^r$ is monotone on S_n^+.*

This result suggests that the monotonicity of the map $f : \Sigma \mapsto \Sigma^r$ for $r \in (0, 1)$ is intimately connected to the affine-invariant geometry of S_n^+ and not its translational geometry. The proof of Theorem 2 has been omitted from this abstract due to length limitations. A more detailed treatment of the topics discussed here, alongside new results and a proof of Theorem 2 will be provided in a subsequent journal paper.

5 Conclusion

The choice of partial order is a key part of studying monotonicity of functions that is often taken for granted. Invariant cone fields provide a geometric approach to systematically construct 'natural' orders by connecting the geometry of the state space to the search for orders. Coupled with differential positivity, invariant cone fields provide an insightful and powerful method for studying monotonicity, as shown in the case of S_n^+.

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