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Werner M. Seiler

Involution

The Formal Theory of Differential Equations and its Applications in Computer Algebra

With 18 Figures and 2 Tables

Springer

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To Marion

involution: anything internally complex or intricate

Oxford Advanced Learner’s Dictionary of Current English

inuoluere, ML *involvere*, to envelop, roll up, wrap up, whence ‘to *involve*’; pp *inuolutus*, ML *involutus*, yields both the tech adj, Geom n, *involute* and the L *inuolutio*, ML o/s *involution* whence E *involution*

Eric Partridge: *Origins: an etymological dictionary of modern English*

Preface

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at rapid pace towards perfection

Joseph L. Lagrange

The theory of differential equations is one of the largest fields within mathematics and probably most graduates in mathematics have attended at least one course on differential equations. But differential equations are also of fundamental importance in most applied sciences; whenever a continuous process is modelled mathematically, chances are high that differential equations appear. So it does not surprise that many textbooks exist on both ordinary and partial differential equations. But the huge majority of these books makes an implicit assumption on the structure of the equations: either one deals with scalar equations or with *normal* systems, i. e. with systems in Cauchy–Kovalevskaya form. The main topic of this book is what happens, if this popular assumption is dropped.

This is not just an academic exercise; non-normal systems are ubiquitous in applications. Classical examples include the incompressible Navier–Stokes equations of fluid dynamics, Maxwell's equations of electrodynamics, the Yang–Mills equations of the fundamental gauge theories in modern particle physics or Einstein's equations of general relativity. But also the simulation and control of multibody systems, electrical circuits or chemical reactions lead to non-normal systems of ordinary differential equations, often called differential algebraic equations. In fact, most of the differential equations nowadays encountered by engineers and scientists are probably not normal.

In view of this great importance of non-normal systems, the relative lack of literature on their general theory is all the more surprising. Specific (classes of) systems like the Navier–Stokes equations have been studied in great depth, but the existence of general approaches to non-normal systems seems to be hardly known, although some of them were developed about a century ago! In fact, again and again new attempts have been started for such general theories, in particular for ordinary differential equations where the situation is comparatively straightforward. Classical examples are the Dirac theory of mechanical systems with constraints and the currently fairly popular numerical analysis of differential algebraic equations. However, in both cases researchers have had to learn (sometimes the hard way) that the generalisation to partial differential equations is far from trivial, as new phenomena emerge requiring new techniques.

There are probably many reasons for this rather unfortunate situation. One is surely that the treatment of general systems requires fairly sophisticated tools from differential geometry and algebra which at least practitioners in differential equations are usually not familiar with. Another serious problem is the available literature. Many of the (few) authors in this domain seem to be determined to found a "school." They go to great length to reinvent any piece of mathematics they use and thus effectively create their own language with the result that their works are more or less unreadable for most mathematicians. Furthermore, as in many other fields of science, there is a certain tendency to "religious wars" between the different schools.

While writing this book, I tried to refrain from such ambitions and not to take sides (although experts will easily notice a strong influence by the work of Pommaret which is not surprising, as I made my first steps in this domain under the guidance of his books and lectures). I believe that all approaches contain useful ideas and on closer inspection I have found that the similarities between the various theories are usually much greater than the differences. One of my central goals was to build a coherent framework using only standard notations and terminology in order to make this book as accessible as possible. The general theory of differential equations requires sufficiently much digestion time even without being hidden behind bizarre non-standard constructions.

Another important goal of mine was to strike a reasonable balance between elucidating the underlying theoretical structures and the development of effective algorithms for concrete computations. More applied mathematicians may find some chapters overly abstract. However, over the years I have made the experience that it is highly rewarding and fruitful to study even purely computational problems from a higher theoretical level. One gains sometimes surprising insights opening completely new solution paths.

The main topic of this book is usually called the *formal theory of differential equations*. It combines geometric and algebraic structures into a powerful framework useful not only for theoretical analysis but also for concrete applications. In this context, the adjective "formal" has at least two different meanings. A simple one is that the origin of the theory was the problem of determining formal power series solutions for general systems of differential equations, i. e. of proving the formal integrability. In a broader sense, the word "formal" refers to the fact that we nowhere work with explicit solutions; all analysis is solely based on manipulations of the equations themselves.

Oversimplifying, one could say that most of the works on differential equations up to the early 20th century are of a more formal nature, whereas currently the functional analytic approach dominates the literature. It is often overlooked that in fact formal and functional analytic methods are complementary to each other and that an in-depth treatment of a general system of differential equations will need both. Typically, one starts with a formal analysis, in particular one will first assert the existence of formal solutions, as otherwise there is no point in proceeding any further. Only for normal systems one can skip this step, as it is trivial.

An important part of the formal analysis consists of completing the system to an involutive one. This includes in particular the addition of all hidden integrability conditions. Only after the completion one has a well defined starting point for functional analytic or numerical methods; thus one may consider the formal analysis as a "preconditioning" for the latter. As the title clearly indicates, the notion of involution takes a central place in the book. Indeed, in my opinion *involution is the central principle in the theory of under- or overdetermined systems*.

One may compare the completion to involution of a differential equation with rendering a linear system of equations triangular, say by Gaussian elimination. Most properties of the system, like for example its solvability and the dimension of its solution space, become apparent only after such a transformation. The same is true for differential equations: only after a completion one can decide about the consistency of the system and make statements about the size of the formal solution space.

Purely formal methods will not lead much further; under certain assumptions one can proceed to prove the convergence of the formal solutions and thus obtains an analytic solution theory. However, an applied mathematician will immediately object that such a theory is not nearly sufficient for any practical purposes. Thus the addition of other methods, in particular from functional analysis, is required for further progress. Nevertheless, in this book we will almost exclusively be concerned with the formal analysis of differential equations. One reason is that this alone represents already a fairly demanding task. Another point is the fact that the connection of formal and functional analytic or numerical methods has not been much studied yet. Even many fairly fundamental questions are still unanswered so that aspiring graduate students can find here a wide open field.

As already mentioned, the formal theory relies mainly on a combination of algebraic and geometric techniques. A geometric formalism, the jet bundles, form the foundation. It leads in a natural manner to questions in commutative (not differential!) algebra. Actually, a deeper understanding of involutive systems can be only obtained by taking a closer look at the arising algebraic structures. While the importance of abstract algebra has been evident at the latest since the seminal work of Spencer in 1960s, the formal theory has developed completely independently of commutative algebra. For example, the almost trivial fact that the notion of (the degree of) involution of a differential equation is equivalent to the algebraic notion of Castelnuovo–Mumford regularity of an associated polynomial module has been overlooked until fairly recently.

It is another goal of this book to clarify these relations between differential equations and commutative algebra. For this reason the book contains several chapters dealing exclusively with commutative algebra. These chapters should also be of independent interest to (computer) algebraists, as some of ideas originally developed for the formal analysis of differential equations are quite powerful and considerably simplify many computations with arbitrary polynomial modules.

Unfortunately, the term "involution" is used in different approaches with somewhat different meanings. In fact, even within this book I will use it sometimes in a broader and sometimes in a narrower meaning. In the algebraic theory a fairly general notion of involutive bases of polynomial ideals will be introduced; but later only the special case of Pommaret bases will be used. In the context of differential equations involution is often confused with formal integrability, i. e. with the mere absence of integrability conditions. But while formal integrability is a necessary condition for involution, the latter requires more and I will stress at a number of occasions the importance of this "more".

The book is divided into ten chapters the contents of which I now briefly describe. The first chapter is a short general introduction into the kind of problems treated in the book. It demonstrates the need for treating more general types of differential equations and explains the relationship to some classical algebraic problems.

The second chapter is concerned with the geometry behind the formal theory. It introduces the jet bundle formalism as the geometric foundation of differential equations. In fact, it contains two different introductions of jet bundles. The "pedestrian" one is based on Taylor series and puts more emphasis on local coordinates. The "intrinsic" one uses an abstract construction stressing the affine structure of jet bundles; it does not often appear in the literature but turns out to be rather natural for the formal theory. Then differential equations are intrinsically defined as fibred submanifolds of jet bundles and the basic geometric operations with them, prolongation and projection, are introduced. This leads to an intrinsic picture of integrability conditions and the notion of formal integrability.

While formal integrability is a very natural and intuitive concept, it turns out that for many purposes it does not suffice. The first problem is already to decide effectively whether or not a given differential equation is formally integrable. But it will become evident at several places that it has further shortcomings; for example even in the analytic category an existence *and* uniqueness theorem can be proven only for involutive equations.

In order to resolve these difficulties one must resort to algebraic methods. So the third chapter introduces our main theme—involution—in a purely algebraic framework which, at first sight, is not at all related to differential equations. Involutive bases are introduced for the Abelian monoid $\mathbb{N}_0^n$ and then extended to a fairly general class of rings: the polynomial algebras of solvable type. This approach has the advantage that it does not only allow us to extract the simple combinatorial idea underlying involution in a very clear form, it may also be applied without modifications to many different situations: the classical polynomial ring, rings of differential operators, universal enveloping algebras of Lie algebras, quantum algebras,... Involutive bases are a special form of the familiar Gröbner bases which have become a central algorithmic tool in computer algebra and therefore much of their theory is modelled on the theory of Gröbner bases.

As a kind of interlude, the fourth chapter discusses the computational side of involutive bases. It presents concrete algorithms for their determination. As any involutive basis is simultaneously a Gröbner basis, we obtain here alternatives to the famous Buchberger algorithm and its variants. Benchmarks have shown that the involutive approach is often highly competitive in terms of the computation times. Compared with the classical Gröbner basis theory, the question of termination becomes much more subtle. This is in particular true for the bases of greatest interest to us, the Pommaret bases which will later be used for a constructive definition of involution for differential equations.

This problem of the existence of a Pommaret basis (or more generally of effectively deciding involution) is known under the name δ-regularity and appears in many different disguises in any coordinate based approach to involution. It is often considered as a purely technical nuisance and thus a reason to use other types of involutive bases instead of Pommaret bases. However, later results will show that the "problem" of δ-regularity is in fact a highly useful *feature* of Pommaret bases and, for example, related to Noether normalisation or characteristics in the context of differential equations. Furthermore, a simple and effective solution based on a cheap criterion for δ-singular coordinates exists and can be easily incorporated into completion algorithms.

Instead of an immediate application of the developed algebraic tools to differential equations, the fifth chapter probes deeper into the theory of involutive bases and shows that the fundamental idea behind them consists of the determination of combinatorial decompositions for polynomial modules. Involutive bases unite these decompositions with Gröbner bases. Although quite obvious, this point of view is still fairly new (in commutative algebra, for differential equations it was already exploited by Riquier and Janet at the beginning of the 20th century).

Pommaret bases are not only important for differential equations, but also define a special type of decomposition, a Rees decomposition. The main topic of the fifth chapter is to show that this fact makes them a very powerful tool for computational algebraic geometry. Most of these applications exploit that Pommaret bases possess a highly interesting syzygy theory. For example, they allow for directly reading off the depth, the projective dimension and the Castelnuovo–Mumford regularity of a module and thus for simple constructive proofs of both Hilbert's Syzygy Theorem and the Auslander–Buchsbaum Formula. In addition, they provide a simple realisation of Hironaka's criterion for Cohen-Macaulay modules.

Of course, these results makes one wonder why Pommaret bases are so special compared with other involutive bases. It seems that the answer lies in homological algebra. All invariants that are easy to read off a Pommaret basis are of a homological origin. Hence it is not surprising that Pommaret bases possess a homological interpretation which is the theme of the sixth chapter. It starts with studying the Spencer cohomology and the dual Koszul homology. Then algebraic versions of the classical Cartan test for involution from the theory of exterior differential systems are given. Finally, the relationship between Pommaret bases and these homological constructions is studied in detail.

The seventh chapter returns to differential equations and applies the developed algebraic theory to the analysis of symbols. The (geometric) symbol induces at each point of the differential equation a polynomial (co)module and one may now use either the Spencer cohomology or, in local coordinates, Pommaret bases for defining involutive differential equations. An important structural property of such equations is the existence of a Cartan normal form for local representations. As a first simple application of involution, a rigorous definition of under- and overdetermined equations is given with the help of the principal symbol. This classification makes sense only for involutive equations and it turns out that the classical counting rules comparing the number of equations and unknowns may be misleading.

Rather surprisingly, it seems to be difficult to find a satisfactory treatment of this elementary and natural question in the literature.

We also discuss now the question of rendering an arbitrary differential equation involutive. For ordinary differential equations it is answered by a simple geometric procedure which has been rediscovered many times in different fields; for partial differential equation an answer is provided by the Cartan–Kuranishi completion. In principle, we cover in both cases equations with arbitrary nonlinearities. However, in practice certain steps are hard to perform effectively. In the case of polynomial nonlinearities one can always resort to Gröbner bases techniques, though possibly at the cost of a prohibitively high complexity. Furthermore, in nonlinear equations one must always expect the emergence of singularities, a problem which requires expensive case distinctions and which we will mostly ignore.

The eighth chapter is devoted to some "combinatorial games": abstract measures for the size of the formal solution space like the Hilbert polynomial or the Cartan characters. One may argue about their usefulness, but some classical notions like the number of degrees of freedom of a physical system are in fact based on such "games." The concept of a differential relation between two differential equations generalising Bäcklund transformations is introduced and the induced relation between the respective Hilbert polynomials is studied. For applications in physics, it is often of interest to formally subtract the effect of gauge symmetries. Within the framework of the formal theory, a fairly simple pseudogroup approach to this question exists. Einstein introduced for similar purposes the strength of a differential equation which can be easily related to the formal theory.

A central question in the theory of differential equations is of course the existence and uniqueness of solutions. Since jets may be understood as an intrinsic version of Taylor series, it does not surprise that the simplest results are obtained for analytic solutions. The ninth chapter starts by recalling the Cauchy–Kovalevskaya Theorem as the most general (with respect to the structure of the equation) existence and uniqueness result for analytic solutions of analytic normal systems.

One could try to directly extend its proof to involutive systems. Indeed, this approach is taken in the Janet–Riquier Theory leading to Riquier's theorem on the existence and uniqueness of analytic solutions of involutive[1] systems (here involution is understood in the broader sense of involutive bases). A more flexible approach leads to the Cartan–Kähler Theorem for arbitrary involutive differential equations. Its proof is based on considering a sequence of normal systems to which the Cauchy–Kovalevskaya Theorem is applied; thus power series appear only implicitly in contrast to the proof of Riquier's theorem. The proof also clearly demonstrates why formal integrability is not sufficient for the analysis of differential equations but the crucial tool of the Cartan normal form of an involutive equation is necessary.

Because of the use of the Cauchy–Kovalevskaya Theorem, the Cartan–Kähler Theorem inherits the restriction to the analytic category and this fact seriously limits its practical value. However, if more is known about the structure of the equation, then the presented technique of proof can sometimes be used for obtaining

[1] In the Janet–Riquier Theory one usually speaks of *passive* systems instead of involutive ones.

stronger existence and/or uniqueness results in function spaces of relevance for applied mathematics. This will be the case, if it is possible to substitute the Cauchy–Kovalevskaya Theorem by other results from the extensive literature on existence and uniqueness for normal systems. Two examples of such generalisations concerning linear systems of differential equations will appear in the tenth chapter.

Vessiot developed a dual version of Cartan's theory of exterior differential systems based on a distribution defined over the differential equation. Unfortunately, it has been largely ignored in the literature so far, although it appears to be highly useful for the geometric analysis of differential equations. The remainder of the ninth chapter is devoted to a new presentation of his approach. First of all, it is better embedded into the geometry of jet bundles (in particular the contact structure) which leads to some simplifications. Then it is shown that Vessiot's construction succeeds only for involutive equations (again formal integrability is not sufficient).

The tenth chapter specialises to linear systems of differential equations. It starts with some elementary geometric aspects of the linearity. After studying an extension of the Holmgren Theorem to arbitrary involutive equations and overdetermined elliptic and hyperbolic systems, we consider linear systems from a more algebraic point of view. Firstly, an introduction to basic algebraic analysis is given. Then it is shown how the Cartan–Kuranishi and the involutive completion algorithm may be merged into a new algebraic algorithm that is able to deliver intrinsic geometric results. Compared with straightforward realisations of the Cartan–Kuranishi completion, it is much faster, as due to a clever book-keeping much less prolongations must be computed. Finally, the inverse syzygy problem and the integration of linear systems with constant coefficients of finite type are studied.

As one can see, the book jumps between geometric and algebraic approaches treating quite diverse topics. Somebody reading the whole text will therefore need a certain familiarity with jet bundles and differential geometry on one side and commutative and homological algebra on the other side. Not all readers will have the necessary background or actually be interested in all these topics. This fact has been taken into account in a two-fold way.

Two fairly long appendices collect some basic results from algebra and differential geometry that are used in the main text; in particular, an introduction into the theory of Gröbner bases is given. These appendices try to make the book to a reasonable degree self-contained, but of course they cannot substitute real introductions into the subjects involved. Therefore additional references to standard textbooks are given. I have also included proofs for some of the mentioned results—either because the used techniques also appear elsewhere in the main text or for a comparison with alternative approaches developed in this book.

Many chapters and sections are rather independent of each other. Depending on the interests of a reader many parts may be safely ignored. For example, somebody interested exclusively in the algebraic theory of involutive bases needs only Chapters 3–6. On the other hand, somebody who does not like commutative algebra may ignore precisely these parts, as the most important results needed for the analysis of differential equations reappear in Chapter 7 in an alternative formulation. For the benefit of readers not so familiar with differential geometry most geometric constructions are described both in an intrinsic manner and in local coordinates. Furthermore, many concrete examples are included to illustrate the theoretical results.

Each chapter ends with a brief section called *Notes*. It usually gives some pointers to the literature and to alternative approaches. I have also tried to trace a bit the history of the main ideas. But as I am not a historian of science, the results should be more taken as rough indications than as a definite answer. Furthermore, in several places I rely on other sources and I have not always checked the correctness of the references. A valuable source of information on the state of the art of the theory of differential equations at the end of the 19th century is the article of von Weber [475] in the *Enzyklopädie der Mathematischen Wissenschaften*.

This book evolved out of my Habilitation thesis [406] admitted by Universität Mannheim in 2002. It is, however, not identical with it; in fact, the book has about twice the size of the thesis. Once I had started with some "minor" changes for the publication of the thesis, I could not resist the temptation to include all the things I left out of the thesis for lack of time (on the other hand I excluded the chapter on numerical analysis, as I am not yet satisfied with the status of this part of the theory). In particular, the material on the Vessiot theory and on the homological theory of involutive bases are completely new. In the end, these "minor" changes lead to a delay of several years in the publication and I am very glad that Springer–Verlag, in particular in the person of Ruth Allewelt and Martin Peters, accepted this delay without complaints.

Finally, I would like to thank some people who either helped me to get a deeper understanding of involution and its applications or who read some version of the manuscript and gave me valuable comments. This list includes in particular (in alphabetic order) Ernst Binz, Dirk Fesser, Vladimir P. Gerdt, Ulrich Oberst, Peter J. Olver, Jean-François Pommaret, Alban Quadrat, Julio Rubio, Robin W. Tucker, Jukka Tuomela, Peter J. Vassiliou and Eva Zerz. Furthermore, the editors of the *Algorithms and Computation in Mathematics* series gave me useful feedback. Special thanks are deserved by Marcus Hausdorf who has been my collaborator on most of the topics covered here and who has implemented large parts of the MuPAD code used in concrete computations. Concerning this last point it is also a pleasure to acknowledge the cooperation of Benno Fuchssteiner and the whole MuPAD group in Paderborn over many years.

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Karlsruhe/Kassel, June 2009 *Werner M. Seiler*

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