# Algorithms for Secretary Problems on Graphs and Hypergraphs

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#### **Abstract**

We examine several online matching problems, with applications to Internet advertising reservation systems. Consider an edge-weighted bipartite graph G, with partite sets L, R. We develop an 8-competitive algorithm for the following secretary problem: Initially given R, and the size of L, the algorithm receives the vertices of L sequentially, in a random order. When a vertex  $l \in L$  is seen, all edges incident to l are revealed, together with their weights. The algorithm must immediately either match l to an available vertex of R, or decide that l will remain unmatched.

In [4], the authors show a 16-competitive algorithm for the transversal matroid secretary problem, which is the special case with weights on vertices, not edges. (Equivalently, one may assume that for each  $l \in L$ , the weights on all edges incident to l are identical.) We use a similar algorithm, but simplify and improve the analysis to obtain a better competitive ratio for the more general problem. Perhaps of more interest is the fact that our analysis is easily extended to obtain competitive algorithms for similar problems, such as to find disjoint sets of edges in hypergraphs where edges arrive online. We also introduce secretary problems with adversarially chosen *groups*.

Finally, we give a 2e-competitive algorithm for the secretary problem on graphic matroids, where, with edges appearing online, the goal is to find a maximum-weight acyclic subgraph of a given graph.

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### 1 Introduction

Many optimization problems of interest can be phrased as picking a maximum-weight independent subset from a ground set of elements, for a suitable definition of independence. A well-known example is the (Maximum-weight) Independent Set problem on graphs, where we wish to find a set of vertices, no two of which are adjacent. A more tractable problem in this setting is the Maximum-weight Matching problem, in which we wish to find a set of edges such that no two edges share an endpoint. This notion of independence can be naturally extended to hypergraphs, where a set of hyperedges is considered independent if no two hyperedges share a vertex.

In the previous examples, independent sets are characterized by forbidding certain pairs of elements from the ground set. A somewhat related, but different notion of independence comes from the independent sets of a matroid. For example, in the uniform matroid of rank k, any set of at most k elements is independent. For graphic matroids, a set of edges in an undirected graph is independent if and only if it does not contain a cycle; the optimization goal is to find a maximum-weight acyclic subgraph of a graph G. In transversal matroids, a set of left-vertices of a bipartite graph is independent if and only if there is a matching that matches each vertex in this set to some right-vertex.

In many applications, the elements of the ground set and their weights are not known in advance, but arrive online one at a time. When an item arrives, we must immediately decide to either irrevocably accept it into the final solution, or reject it and never be able to go back to it again. We will be interested in competitive analysis, that is, comparing the performance of an online algorithm to an optimal offline algorithm which is given the whole input in advance. In this setting, even simple problems like selecting a maximum-weight element become difficult, because we do not know if elements that come in the future will have weight significantly higher or lower than the element currently under consideration. If we make no assumptions about the input, any algorithm can be fooled into performing arbitrarily poorly by ofering it a medium-weight item, followed by a high-weight item if it accepts, and a low-weight item if it rejects. To solve such problems, which frequently arise in practice, various assumptions are made. For instance, one might assume that weights are all drawn from a known distribution, or (if independent sets may contain several elements) that the weight of any single element is small compared to the weight of the best independent set.

One useful assumption that can be made is that the elements of the ground set appear in a random order. The basic problem in which the goal is to select the maximum-weight element is well known as the *Secretary Problem*. It was first published by Martin Gardner in [6], though it appears to have arisen as folklore a decade previously [5]. An optimal solution is to observe the first n/e elements, and select the first element from the rest with weight greater than the heaviest element seen in the first set; this algorithm gives a 1/e probability of finding the heaviest element, and has been attributed to several authors (see [5]).

Motivated by this simple observation, several results have appeared for more complex problems in this random permutation model; these are often called secretary-type problems. Typically, given a random permutation of elements appearing in an online fashion, the goal is to find a maximum-weight independent set. For example, Kleinberg [7] gives a  $1 + O(1/\sqrt{k})$ -competitive algorithm for the problem of selecting at most k elements from the set to maximize their sum. Babaioff  $et\ al.$  [2] give a constant-competitive algorithm for the more general Knapsack secretary problem, in which each element has a size and weight, and the goal is to find a maximum-weight set of elements whose total size is at most a given integer B.

Babaioff et al. [1] had earlier introduced the so-called matroid secretary problem, and gave an  $O(\log k)$ -competitive algorithm to find the max-weight independent set of elements, where k is the rank of the underlying matroid. A 16-competitive algorithm was also given in [1] for the special case of graphic matroids; this was based on their 4d-competitive algorithm algorithm for the important case of transversal matroids, where d is the maximum degree of any left-vertex. Recently, Dimitrov and Plaxton [4] improved the latter to a ratio of 16 for all transversal matroids. A significant open question is whether there exists a O(1)-competitive algorithm for general matroids, or for other secretary problems with non-matroid constraints.

These secretary-type problems arise in many practical situations where decisions must be made in real-time without knowledge of the future, or with very limited knowledge. For example, a factory needs to decide which orders to fulfil, without knowing whether more valuable orders will be placed later. Buyers and sellers of houses must decide whether to go through with a transaction, though they may receive a better offer in a week or a month. Below, we give an example from online advertising systems, which we use as a recurring motivation through the paper.

Internet-based systems are now being used to sell advertising space in other media, such as newspapers, radio and television broadcasts, etc. Advertisers in these media typically plan advertising campaigns and reserve slots well in advance to coincide with product launches, peak shopping seasons, or other events. In such situations, it is unreasonable to run an auction immediately before the event to determine which ads are shown, as is done for sponsored search and other online advertising.

Consider an automatic advertising reservation system, in which the seller controls a number of *slots*, each representing a position in which an advertisement (hereafter *ad*) can be published. Advertisers/Bidders appear periodically, and report which slots they would like to place an ad in, and how much they are willing to pay for each slot. When an advertiser reports a bid, the system must immediately decide whether or not to accept it; if a bid is accepted, the ad *must* be placed in the corresponding slot, and if not, the ad is permanently rejected. Note that in disallowing the removal of an accepted ad, our model differs significantly from that of [3], in which the seller can subsequently remove an accepted ad if he makes a compensatory payment to the advertiser.

We model this system as as an online edge-weighted matching problem on a bipartite graph  $G(L \cup R, E)$ : the vertices of set R correspond to the set of slots, and those of set L to the ads. For each vertex  $l \in L$ , its neighbors in R correspond to the slots in which ad l can appear, and the weight of edge (l,r) is the amount the advertiser is willing to pay if l appears in slot r. Initially, the seller knows the set of slots R; vertices of L appear sequentially in a random order, as advertisers bid on slots. When a vertex  $l \in L$  is seen, all the edges from l to R are revealed, together with their weights; the seller must immediately decide whether to accept ad l, and if so, which of the relevant slots to place it in. The seller's goal, obviously, is to maximize his revenue. Subsequently, we refer to this problem as Bipartite Vertex-at-a-time Matching (BVM). We describe our results for BVM and other problems below.

#### 1.1 Results and Outline

Recall that the elements of a transversal matroid are one partite set L (subsequently referred to as the *left vertices*) of a bipartite graph, and a set of vertices  $S \subseteq L$  is independent if the graph constains a perfect matching from S to the other partite set. That is, the transversal matroid secretary problem is equivalent to the special case of BVM in which all edges incident to each  $l \in L$  have the same weight. (Equivalently, the weights are on vertices of L instead of edges.) In Section 2, we give a simpler and tighter analysis for an algorithm essentially similar to that of Dimitrov and Plaxton [4] for transversal matroids; this allows us to improve the competitive ratio from 16 to 8, even for the more general BVM problem.

In addition to an improved ratio, our methods are of interest as they appear robust to changes in the model and can be naturally applied to more general problems. We illustrate this in Section 3 by extending our algorithms to hypergraph problems, with applications to more complex advertising systems in which advertisers desire *bundles* of slots, as opposed to a single slot. In particular, we obtain constant-competitive algorithms for finding independent edge sets in hypergraphs of constant edge-size.

We also introduce secretary problems with *groups*, to model applications in which we do not see a truly random permutation of elements. We assume that an adversary can group the elements arbitrarily, but once the groups are constructed, they appear in random order. When a group appears, the algorithm can see all the elements in the group. We discuss this idea further in Section 4.

Finally, in Section 5, we obtain a simple 2*e*-competitive algorithm for the problem of finding independent edge-sets in graphic matroids, improving the ratio of 16 from [1].

The majority of our algorithms follow the "sample-and-price" method common to many solutions to secretary problems. That is, we look at a random sample of elements containing a constant fraction of the input, and use the values observed to determine prices or thresholds. In the second half, we accept an element if its weight/value is above the given price. For instance, in the optimal solution to the original secretary problem, the price is set to be the highest value seen in the first 1/e fraction of the input, and we accept any element from the remaining set with value greater than this price.

### 2 The Bipartite Vertex-at-a-time Matching Problem

Recall that in the BVM problem, the algorithm is initially given one partite set R of a bipartite graph  $G(L \cup R, E)$ , together with the size of the other partite set L. The algorithm sees the vertices of L sequentially, in a random order. When a vertex  $l \in L$  is seen, all edges incident to l are revealed, together with their weights. The algorithm must immediately either match l to an available vertex of R, or decide that l will remain permanently unmatched. In this section, we show that an algorithm based on that of [4] gives a competitive ratio of 8 for this problem. Before presenting the algorithm for BVM, we describe a closely related algorithm SIMULATE that is easier to analyze, and then show that our final algorithm does at least as well as SIMULATE.

Let Greedy denote the following greedy algorithm for the offline Edge-weighted bipartite matching problem:

Let w(F) denote the weight of a set of edges F, and OPT denote the weight of an optimum (max-weight) matching on G. It is easy to see the following proposition, that GREEDY is a 2-approximation.

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Proposition 2.1 w(M) \geq OPT/2.
```

We now describe the algorithm SIMULATE, which we use purely to analyze our final algorithm for BVM.

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\begin{array}{l} \underline{\text{SIMULATE}} \colon \\ \text{Sort edges of } G(L \cup R, E) \text{ in decreasing order of weight.} \\ M_1, M_2 \leftarrow \emptyset \\ \text{Mark each vertex } l \in L \text{ as unassigned.} \\ \text{For each edge } e = (l, r) \in E, \text{ in sorted order} \\ \text{If } l \text{ is unassigned } \mathbf{AND} \ M_1 \cup e \text{ is a matching:} \\ \text{Mark } l \text{ as assigned} \\ \text{Flip a coin with probability } p \text{ of heads} \\ \text{If heads, } M_1 \leftarrow M_1 \cup e \\ \text{Else } M_2 \leftarrow M_2 \cup e \\ \\ M_3 \leftarrow M_2 \\ \text{For each vertex } r \in R \\ \text{If } r \text{ has degree} > 1 \text{ in } M_3 \\ \text{Delete all edges incident to } r \text{ from } M_3. \\ \end{array}
```

Say that an edge e is *considered* by SIMULATE if we flip a coin and assign e to either  $M_1$  or  $M_2$ . We make two observations about SIMULATE: Once any edge incident to a vertex  $l \in L$  has been considered, no other

edge incident to l will be considered later. Second, once an edge incident to  $r \in R$  has been added to  $M_1$ , no subsequent edge incident to r will be considered. (Note that multiple edges incident to r might be considered until one of these edges is added to  $M_1$ .)

Observe that from our description of SIMULATE,  $M_1$  is a matching, but  $M_2$  may not be, as a vertex  $r \in R$  may be incident to multiple edges of  $M_2$ . Hence, we have a final pruning step in case there are multiple edges incident to the same vertex of R; this gives us a matching  $M_3$ . We now prove three statements about SIMULATE, and later show that the matching returned by our online algorithm is at least as good as  $M_3$ .

### **Proposition 2.2** $\mathbb{E}[w(M_1)] \geq p \text{OPT}/2$ .

**Proof:** SIMULATE tosses a coin (at most) once for each vertex in L;  $M_1$  is precisely the matching one would obtain from running GREEDY on  $L' \cup R$ , where L' denotes the vertices which came up heads. (If the coin for a vertex comes up tails, this vertex has no effect on  $M_1$ .) If OPT' denotes the weight of an optimum matching on  $L' \cup R$ , it is easy to see that  $\mathbb{E}[\mathsf{OPT'}] \geq p\mathsf{OPT}$ , and hence that  $\mathbb{E}[w(M_1)] \geq p\mathsf{OPT}/2$ .

**Lemma 2.3** 
$$\mathbb{E}[w(M_2)] \geq (1-p)\text{OPT}/2.$$

**Proof:** Consider any history of coin tosses in which an arbitrary edge e is being considered, and we are about to flip a coin to determine whether e is added to  $M_1$  or  $M_2$ . Its expected contribution to  $M_1$  is pw(e), and to  $M_2$ , is (1-p)w(e). This holds for each edge e and any history in which e can contribute to the weight of  $M_1$  or  $M_2$ ; hence  $\mathbb{E}[w(M_2)] = \frac{(1-p)}{p}\mathbb{E}[w(M_1)]$ , completing the proof.

# **Lemma 2.4** $\mathbb{E}[w(M_3)] \ge \frac{p^2(1-p)}{2} \text{OPT}.$

Similarly, for each vertex  $v \in R$ , let  $Revenue_3(v)$  denote the revenue earned by vertex v in  $M_3$ , which is the weight of the (at most one) edge incident to v in  $M_3$ . Let  $\mathbb{E}[Revenue_3(v)|e]$  denote the expected revenue earned by v, conditioned on e being the first edge incident to v added to  $M_2$ . With probability p, the next considered edge incident to v is added to  $M_1$ , and hence v has degree 1 in  $M_2$ . Therefore,  $\mathbb{E}[Revenue_3(v)|e] \geq pw(e)$ , and so  $\mathbb{E}[Revenue_3(v)|e] \geq p^2\mathbb{E}[Revenue_2(v)|e]$ ; it follows that  $\mathbb{E}[w(M_3)] \geq p^2\mathbb{E}[w(M_2)] = \frac{p^2(1-p)}{2}\text{OPT}$ .

Before describing our final algorithm for EBP, we show that the matching returned by an intermediate algorithm SampleandPermute is at least as good as  $M_3$ , which implies that we have a  $\frac{2}{(1-p)p^2}$ -competitive algorithm: setting p=2/3, we get a 13.5-competitive algorithm. However, our pruning step allows us to take an edge for  $M_3$  only if its right endpoint has degree 1; a more careful pruning step allows more edges in the matching. We use this fact to give a tighter analysis for the next algorithm, obtaining a competitive ratio of 8.

Note that the matching  $M_1$  in SampleandPermute is precisely the same as  $M_1$  from Simulate; intuitively, in the former, we toss all the coins at once and run Greedy, while in the latter, we toss coins while constructing the Greedy Matching. (More precisely, the two algorithms to generate the matchings are equivalent.) Similarly, the "matching"  $M_2$  in this algorithm is essentially  $M_2$  from Simulate. The difference between the two algorithms is in the pruning step: To construct  $M_3$  in Simulate, we delete all edges incident

to any vertex  $r \in R$  with degree greater than 1; in SAMPLEANDPERMUTE, we add to M the first such edge seen in our permutation of L-L'. It follows immediately from Lemma 2.4 that  $\mathbb{E}[w(M)] \geq p^2(1-p)\mathrm{OPT}/2$ , but accounting for the difference in pruning allows the following tighter statement, which we prove in the appendix.

```
Lemma 2.5 \mathbb{E}[w(M)] \ge \frac{p(1-p)}{2} \text{OPT}.
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We now present our final algorithm, a trivial modification of SAMPLEANDPERMUTE for the online BVM problem.

```
\frac{\mathsf{SAMPLEANDPRICE}(|L|,R)}{k \leftarrow Binom(|L|,p)} Let L' be the first k vertices of L. M_1 \leftarrow \mathsf{GREEDY}(G[L' \cup R]). For each r \in R: \mathsf{Set}\ price(r)\ \mathsf{to}\ \mathsf{be}\ \mathsf{the}\ \mathsf{weight}\ \mathsf{of}\ \mathsf{the}\ \mathsf{edge}\ \mathsf{incident}\ \mathsf{to}\ r\ \mathsf{in}\ M_1. M \leftarrow \emptyset For each subsequent l \in L - L', : Let e = (l,r)\ \mathsf{be}\ \mathsf{the}\ \mathsf{highest-weight}\ \mathsf{edge}\ \mathsf{such}\ \mathsf{that}\ w(e) \geq price(r) If M \cup e is a matching, accept e for M.
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As the input to SAMPLEANDPRICE is a random permutation, L' is a subset of L in which each vertex of L is selected with probability p; it is easy to see that this algorithm is equivalent to SAMPLEANDPERMUTE. Therefore,  $\mathbb{E}[w(M)] \geq \frac{p(1-p)}{2} \text{OPT}$ ; setting p = 1/2 implies that the expected competitive ratio is 8.

# 3 Independent Edge Sets in Hypergraphs

In the Hypergraph Edge-at-a-time Matching (HEM) problem, we are initally given the vertex set of a hypergraph; subsequently, hyperedges appear in a random order. When an edge (together with its weight) is revealed, the algorithm must immediately decide whether or not to accept it; as before, the goal is for the algorithm to select a maximum-weight set of disjoint edges. For arbitrary hypergraphs, one can observe that even the offline version of this problem is NP-Complete (and also hard to approximate) via an easy reduction from the Independent Set problem. However, the difficulty is related to the size of the hyperedges; if all edges contain only 2 vertices, for instance, then we are simply trying to find a matching in a (possibly non-bipartite) graph. (Even in this special case, the problem is of interest in an online setting.) Let d denote the maximum size of an edge in the hypergraph.

We provide an  $O(d^2)$ -competitive algorithm for the HEM problem by solving the more general Hypergraph Vertex-at-a-time Matching (HVM) problem, described as follows: We are initally given a subset R of the vertex set of a hypergraph. The remaining vertices L arrive online; each edge of the hypergraph is constrained to contain exactly one vertex of L, together with some vertices of R. The vertices of L appear online in a random order; when  $l \in L$  is revealed, the algorithm also sees all edges incident to l, together with their weight. At this point, the algorithm must immediately decide whether or not to accept some edge containing l, and if so, which edge; again, the goal is for the algorithm to select a maximum-weight set of disjoint edges. Here, let d denote the maximum number of vertices of R contained in a single edge (so the largest edge has d+1 vertices). First, we observe that the HEM problem with edge size d reduces to the HVM problem with edge size d+1: Let R be the vertex set of the original hypergraph, and add one vertex to L for each original edge. An edge of the new hypergraph consists of an old edge, together with the corresponding vertex of L. Clearly, observing a random permutation of L together with the incident edges is equivalent to a random permutation of the edge set of the original hypergraph. Also, notice that the BVM problem of Section 2 is simply the special case of HVM when d=1. (See Figure 1 at the end of this section.)

These hypergraph problems capture the notion of demand bundles. For instance, in ad reservation systems, advertisers rarely make reservations for a single ad at a time; they are more likely to plan advertising campaigns involving multiple individual ads. In many campaigns, advertisers create various ads which are related to and complement or reinforce each other; these advertisers might be interested in acquiring a bundle or set of slots for this campaign. They submit to the reservation system the bundles they are interested in, together with the price they are willing to pay; the system must either accept a request for an entire bundle or reject it, as it does not receive revenue for providing the advertiser with a part of the bundle. If each advertiser submits a request for a single bundle, we obtain the HEM problem with vertex set corresponding to the set of slots. More generally, an advertiser may submit a request for *one of* a set of bundles, together with a price for each bundle. (For example, an advertiser might want an ad to appear in any three out of four local newspapers.) This leads to the HVM problem, with vertex set L corresponding to the set of advertisers, and set R to the set of slots: We receive a random permutation of advertisers, and each advertiser informs us of the bundles she is interested in, together with a price for each bundle.

Let Greedy denote the offline algorithm for HVM that sorts edges in decreasing order of weight, and selects an edge if it is disjoint from all previously selected edges. For ease of exposition, we subsequently assume that the hypergraph is (d+1)-uniform; that is, that each edge contains exactly d vertices of R together with one vertex of L.

**Proposition 3.1** Greedy returns a (d+1)-approximation to the maximum-weight disjoint edge set.

We again define an algorithm SIMULATE, as in Section 2:

```
Sort edges of E in decreasing order of weight.

Mark each vertex l \in L as unassigned.

M_1, M_2 \leftarrow \emptyset

For each edge e \in E in sorted order:

Let l be the vertex of L in e

If l is unassigned AND e is disjoint from M_1:

Mark l as assigned.

Flip a coin with probability p of heads

If heads, add e to M_1

If tails, add e to M_2

M_3 \leftarrow \emptyset

For each e \in M_2:

Add e to M_3 if e is disjoint from the rest of M_2.
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As before, we let w(F) denote the weight of an edge set F. The proofs of the following two propositions are exactly analogous to Proposition 2.2 and Lemma 2.3.

**Proposition 3.2** 
$$\mathbb{E}[w(M_1)] \geq p \cdot \text{OPT}/(d+1)$$
.

**Proposition 3.3** 
$$\mathbb{E}[w(M_2)] \ge (1-p) \text{OPT}/(d+1)$$
.

It is now slightly more complex to bound the weight of  $M_3$  than it was for the BVM problem; for BVM, the set of edges in  $M_2$  incident to  $v \in R$  interfere only with each other, but in the hypergraph version, edges  $e_1$  and  $e_2$  might not intersect, though they may both intersect  $e_3$ , and hence all of  $e_1, e_2, e_3$  will have to be deleted. However, we can use a similar intuition: In BVM, we charge all edges of  $M_2$  incident to v to the heaviest such edge; in expectation, each edge is charged a constant number of times. For the HVM problem, we charge all the edges in a "connected component" to the heaviest edge in the component, and argue that (with a suitable choice of p) the average size of the components is small. More formally, we prove the following lemma:

**Lemma 3.4** Setting 
$$p = 1 - 1/2d$$
,  $\mathbb{E}[w(M_3)] \ge \frac{\text{OPT}}{12d(d+1)}$ .

**Proof:** Construct an auxiliary directed graph F as follows: For each  $e \in M_2$ , add a corresponding vertex  $v_e$  to F. If e' is the heaviest edge in  $M_2$  that intersects e, add a directed arc from  $v_e$  to  $v_{e'}$  to F. (If e itself is this heaviest edge,  $v_e$  has no out-neighbors.) Note that the graph F is obviously a forest. For each  $e \in M_2$ , if  $v_e$  is not the root of its tree in F, we define  $Revenue_2(e)$  to be 0, and if it is the root, we set  $Revenue_2(e)$  to be the weight of all edges of  $M_2$  in the tree. Clearly,  $\sum_e Revenue_2(e) = w(M_2)$ .

We define  $Revenue_3(e)$  to be equal to the weight of e iff e is an edge in  $M_2$  that does not intersect any other such edge. (In which case, it follows that  $v_e$  is the root of its tree.) We prove that  $\mathbb{E}[Revenue_3(e)] \geq \frac{\mathbb{E}[Revenue_2(e)]}{6}$ , which proves the lemma, since  $\sum_e Revenue_3(e) = w(M_3)$ .

First, note that the probability that any edge e added to  $M_2$  intersects an edge added later is at most 1/2: For each vertex u of R contained in e, the probability that e intersects a later edge because of u is at most 1/2d, as with probability 1-1/2d, the next edge containing u considered by SIMULATE will be added to  $M_1$ . As e contains only d vertices in R, the desired probability is at most 1/2. (Every vertex of L is incident to at most one edge in  $M_2$ , and so e cannot intersect any other edge through its vertex in L.) It follows that the probability that any  $v_e \in F$  has a child is at most 1/2. We also count the expected number of children of  $v_e$ ; the edge corresponding to each child of  $v_e$  must share some vertex with e, and the expected number of children through a particular vertex is at most  $\sum_{i=1}^{\infty} ip(1-p)^i = (1-p)/p$ . As e contains d vertices of R, the expected number of children of  $v_e$  is at most d(1-p)/p = 1/(2-1/d); since  $d \ge 2$ , the expected number of children is at most 2/3. It follows that the expected size of a subtree rooted at  $v_e$  is at most 3.

Note that  $Revenue_2(e)$  and  $Revenue_3(e)$  are both 0 if  $v_e$  is not the root of its tree in F. Conditioned on  $v_e$  being a root,  $\mathbb{E}[Revenue_2(e)] \leq 3w(e)$ , as e is the heaviest edge in its tree, and the expected size of the tree is at most 3.  $\mathbb{E}[Revenue_3(e)]$  is at least w(e)/2, as e intersects no previously added edges, and with probability at least 1/2, it intersects no edge added to  $M_2$  later. Therefore, the ratio of these expectations is at most 6, completing the proof.

Now, we define our final algorithm SAMPLEANDPRICE for the HVM problem:

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\frac{\mathsf{SAMPLEANDPRICE}(|L|,R)}{k \leftarrow Binom(|L|,1-\frac{1}{2d})} Let L' be the first k elements of L. M_1 \leftarrow \mathsf{GREEDY}(G(L',R)). For each v \in V: Set price(v) to be the weight of the edge incident to v in M_1. M \leftarrow \emptyset For each subsequent l \in L-L': Let e be the highest-weight edge containing l such that for each v \in e, w(e) \geq price(v) If e is disjoint from M, add e to M.
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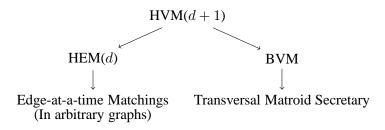


Figure 1: Relationships between the HVM problem and various special cases.

As before, since the input is a random permutation of L, L' is a subset of L in which every vertex is selected independently with probability 1-1/2d, and the matching M is at least as good as  $M_3$  from SIMULATE. Therefore, we have proved the following theorem:

**Theorem 3.5** SampleAndPrice is an  $O(d^2)$ -competitive algorithm for the HVM secretary problem.

Note that M may also contain extra edges that occur earlier in the permutation than edges they intersect; for the BVM problem, this was the difference between Lemma 2.4 and the stronger bound 2.5. We do not provide a tighter analysis similar to Lemma 2.5 for the HVM problem in this extended abstract, nor make an attempt to optimize the constants of Lemma 3.4. In particular, for the HEM problem with d=2 (finding an online matching in a non-bipartite graph G(V,E), given a random permutatation of E), we have a constant bound on the competitive ratio; a smaller constant can easily be obtained.

### 4 Secretary Problems with Groups

Consider a secretary-type problem in which, instead of receiving a random permutation of the elements, elements can be grouped by an adversary. The algorithm receives the number of groups in advance, instead of the number of elements. However, once the groups have been constructed, they arrive in random order; when a group arrives, the algorithm can see all its elements at once. Note that the groups are fixed in advance; the adversary cannot construct groups in response to the algorithm's choices or the set of groups seen so far. The effect of such grouping on the difficulty of the problem is not immediately clear: The adversary can ensure that some permutations of the element set never occur, which might make the problem more difficult. On the other hand, as the algorithm is allowed to see several elements at once, it may be easier to compute a good solution.

For instance, consider the classical secretary problem with groups. An optimal algorithm will never hire any but the best secretary from a group, and it is easy to obtain an e-competitive algorithm: Ignore all but the best secretary from each group, and run the standard secretary algorithm on these. That is, observe a constant (1/e) fraction of the groups, and note the value/price of the best secretary seen so far. From the rest of the input, hire the best secretary from the first group with a secretary to beat this price. Perhaps a reason this problem is as easy as the original version is that only one element is to be selected.

By way of contrast, consider the following matching problem, even restricted to bipartite graphs: The algorithm is initially given the vertex set of a bipartite graph, and an adversary groups the edges arbitrarily. The groups arrive in random order; when a group arrives, the algorithm sees the weights of all edges it contains. The goal is to find a maximum-weight matching; note that as a special case of HEM with d=2, we have an O(1)-competitive algorithm for this problem without edge grouping. A natural Sample-And-Price algorithm for this problem is as follows: Look at a constant fraction of the input, and construct a matching with these edges (either the optimal matching, or the greedy matchings we used in the previous sections). Use the weights of edges in the matching to set vertex prices, and in the remainder of the input, select an edge if its weight is at least the price of each of its endpoints, and if it does not conflict with edges already selected. Unfortunately, this algorithm does not work: Consider a bipartite graph  $G(L \cup R)$ , with  $L = \{l_1, l_2, \ldots, l_n\}$  and  $R = \{r_1, r_2, \ldots, r_n\}$ . We have

two groups of edges:  $E_1 = \{(l_i, r_i) | 1 \le i \le n\}$ , with  $w((l_i, r_i)) = 1 + 2i\varepsilon$ , and  $E_2 = \{(l_i, r_{i+1}) | 1 \le i < n\}$ , with  $w((l_i, r_{i+1})) = 1 + (2i + 1)\varepsilon$ . Assuming  $\varepsilon \ll 1/n^2$ ,  $E_1$  corresponds to an optimal matching, with weight  $\approx n$ . If  $E_1$  arrives first, the price of each  $r_i$  is  $1 + 2i\varepsilon$ . Subsequently, when  $E_2$  arrives,  $w((l_{i-1}, r_i)) = 1 + (2i - 1)\varepsilon$ , and hence no edge of  $E_2$  beats the price of its right endpoint. If  $E_2$  arrives first, the price of each  $l_i$  is  $1 + (2i + 1)\varepsilon$ . Subsequently, when  $E_1$  arrives,  $w((l_i, r_i)) = 1 + 2i\varepsilon$ , and so no edge except  $(l_n, r_n)$  beats the price of its left endpoint, for a total revenue of  $\approx 1$ .

We believe, therefore, that the introduction of groups affects these secretary-type problems in non-trivial ways, and these problems are likely to be of theoretical interest; in addition, they have applications to problems where groups occur naturally, and we do not receive a random permutation of the entire element set. To take another example from the advertising world, when a merchant plans a campaign, she may submit to the reservation system multiple ads, together with the slots in which each ad can be placed, and a price for each ad-slot combination. Even if the merchants arrive in a random order, this does not correspond to a random permutation of ads, and hence our previous analysis is not directly applicable. We model this (as in BVM) as an edge-weighted matching problem on a bipartite graph  $G(L \cup R, E)$  in which vertices of L may be grouped; here, the groups correspond to the set of ads for a given advertiser. The algorithm initially receives R (the set of slots), and the number of advertisers/groups; the adversary can construct groups from L arbitrarily. Once the groups have been fixed, a random permutation of the groups is seen, and when a group arrives, the algorithm must decide which ads to accept, and where to place them; as always, decisions are irrevocable. We refer to this as the BVM problem with groups.

#### **Theorem 4.1** There is an $O(\log n)$ -competitive algorithm for the BVM problem with groups.

It is easy to prove this theorem using standard techniques: Sample the first half of the vertices, and let w denote the weight of the heaviest edge seen so far. Pick an integer j uniformly at random in  $[0, 1 + \lceil \log_2 n \rceil]$ , and set a threshold of  $w/2^j$ . In the second half, greedily construct a matching using edges with weight above the threshold. (See, for instance, Theorem 3.2 of [1] for analysis of an essentially similar algorithm.) For completeness, we give a proof of Theorem 4.1 in Section A.2 of the appendix.

A natural question is whether one can find a constant-competitive algorithm for BVM with groups. Note that one must be careful about using Sample-And-Price algorithms: First, as the example above shows, the natural algorithm with groups of edges instead of vertices does not work. Second, one might sample a constant fraction of groups, construct a matching  $M_1$  on the sampled groups, and then use  $M_1$  to set prices. However, once prices have been set in this way, the edges assigned to a group g may not be the same as the edges that would have been assigned to g in  $M_1$  if g had been sampled. This was not the case for the basic BVM problem: If an edge (l,r) is in  $M_2$ , then by construction – fixing all other coin flips – if the coin for l had come up heads instead of tails, (l,r) would be in  $M_1$ . As the example in Figure 2 shows, this desirable property no longer holds once groups are introduced.

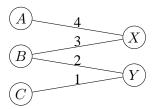


Figure 2: Example for BVM with groups. Vertices A,C are in group 1, and vertex B is in group 2. Using the SAMPLEANDPRICE algorithm, if group 2 is sampled and group 1 is not, both edges incident to A and C beat their prices, and hence are added to  $M_2$ . If both groups are sampled, A will be matched to X and B to Y in  $M_1$ , while C will remain unmatched.

We conjecture that the following algorithm SAMPLEWITHGROUPS is constant-competitive for BVM with Groups. Here,  $\mathcal{G}$  denotes the set of groups:

```
\begin{array}{c} {\color{red} {\bf SAMPLEWITHGROUPS}(|\mathcal{G}|,R)} \\ {\color{red} {\bf Sample each group with probability } p.} \\ {\color{red} {\bf Construct a greedy matching } M_1 \ {\rm on the set of sampled groups } \mathcal{G}'.} \\ {\color{red} {M_2} \leftarrow \emptyset.} \\ {\color{red} {\bf For each group } g \ {\rm in } \mathcal{G} - \mathcal{G}':} \\ {\color{red} {\bf Let } E' \ {\rm denote the edges assigned to vertices of } g \ {\rm in the greedy matching on } \mathcal{G}' \cup g.} \\ {\color{red} {M_2} \leftarrow M_2 \cup E'.} \\ {\color{red} {M_3} \leftarrow M_2} \\ {\color{red} {\bf For each } r \in R:} \\ {\color{red} {\bf If } r \ {\rm has \ degree} > 1 \ {\rm in } M_3:} \\ {\color{red} {\bf Delete \ all \ edges \ incident \ to } r \ {\rm from } M_3.} \\ \end{array}
```

It is easy to see that  $\mathbb{E}[w(M_1)] \geq p\text{OPT}/2$ . By construction, the edges assigned to g in  $M_2$  are precisely those that would have been assigned to g in  $M_1$  if g had been sampled. (Hence, this algorithm differs from the natural SAMPLEANDPRICE.) Therefore, it follows that the probability an edge contributes to  $M_2$  is (1-p)/p times the probability it contributes to  $M_1$ . If p=1/2, it follows that  $\mathbb{E}[w(M_2)] = \mathbb{E}[w(M_1)]$ , and further, that the expected degree of  $r \in R$  in  $M_2$  is equal to its expected degree in  $M_1$ , which is at most 1 since  $M_1$  is a matching. This does not suffice to give a lower bound on the expected weight of  $M_3$ , but we conjecture that the expected weight of  $M_3$  is at most a constant factor lower than that of  $M_2$ .

**Conjecture 1** SAMPLEWITHGROUPS is constant-competitive for the BVM problem with groups.

## 5 Graphic Matroids

In this section, we describe a 2e-competitive algorithm for the Graphic Matroid Secretary problem. Here, we are initially given the set of vertices V of an undirected edge-weighted graph G=(V,E) together with the size of its edge set |E|. The edges of the graph appear in a random order, and the goal is to accept a maximum-weight subset of edges F that does not contain any cycles. As always, the decision to accept an edge must be made upon its arrival, and cannot be revoked.

This problem is equivalent to finding the maximum-weight spanning tree (assuming G is connected) and is also equivalent to finding the maximum-weight independent set in the graphic matroid defined by the graph G. Babaioff et al. [1] give a 16-competitive algorithm for the secretary version of this problem based on a related algorithm for transversal matroids. We give a simple reduction to the classical secretary problem, losing a factor of 2 in the reduction. In this way, we obtain a  $2e \approx 5.436$ -competitive algorithm for the Graphic Matroid Secretary problem.

Fix an ordering  $v_1, v_2, \ldots, v_n$  on the vertices of G. Consider two directed graphs: graph  $G_0$  is obtained by orienting every edge of G from higher numbered to lower numbered vertex, and graph  $G_1$  by orienting every edge from lower to higher numbered vertex.

Our online algorithm initially flips a fair coin  $X \in \{0,1\}$ . For each vertex v independently, it runs a secretary algorithm to find the maximum-weight edge leaving v in  $G_X$ . The output of the algorithm is F', the union of all edges accepted by the individual secretary algorithms. Since the graph  $G_X$  is acyclic and each vertex has at most one outgoing edge, the set of edges F' must be acyclic even in the undirected sense.

It remains to show a lower bound on the weight of F'. For each vertex v, let  $h_X(v)$  be the heaviest edge leaving vertex v in  $G_X$ . Let  $F_X = \{h_X(v) \mid v \in V\}$ . Let  $F^*$  be a maximum-weight acyclic subgraph of G.

**Proposition 5.1** 
$$\sum_{v \in V} w(h_0(v)) + w(h_1(v)) \ge \sum_{e \in F^*} w(e)$$
.

Conditioned on the coin flip X, each secretary algorithm recovers at least 1/e fraction of the weight of the heaviest edge leaving its vertex. Hence  $E[w(F') \mid X = x] = \frac{1}{e}w(F_x)$  for x = 0, 1. Using Proposition 5.1,

 $E[w(F')] = \frac{1}{e} \left( \frac{1}{2} E[w(F') \mid X = 0] + \frac{1}{2} E[w(F') \mid X = 1] \right) \ge \frac{1}{2e} w(F^*)$ . Therefore, we obtain the following theorem:

**Theorem 5.2** There is a 2e-competitive algorithm for the graphic matroid secretary problem.

## 6 Conclusions and Open Problems

We list several problems that remain to be solved:

- An improved understanding of groups and their contribution to the difficulty of secretary-type problems

   is likely to be of interest. In particular, it may be possible to find a constant-competitive algorithm for the BVM problem with groups.
- Few lower bounds for these problems are known beyond 1/e for the original secretary problem; obtaining such bounds may require new techniques.
- In the basic BVM problem, we lose a factor of 2 by constructing greedy matchings. If, instead, we modified our algorithm to set prices using an optimal matching  $M_1$  on the sampled vertices, is the resulting algorithm 4-competitive? Is it even O(1)-competitive?
- Finally, obtaining an O(1)-competitive algorithm for the general matroid secretary problem is still open, though the competitive ratios for important special cases such as transversal and graphic matroids have been reduced to small constants.

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### **A** Omitted Proofs

### A.1 Proof of Lemma 2.5

We prove Lemma 2.5 below, showing that SAMPLEANDPRICE is 8-competitive for the BVM problem.

For each  $v \in R$ , we let  $Revenue_2(v)$  be the revenue earned by v in  $M_2$ , which is the total weight of edges in  $M_2$  incident to v. Similarly,  $Revenue_3(v)$  denotes the weight of the (at most one) edge of  $M_3$  incident to v. Let  $P_i$  be the probability that v is incident to i edges in  $M_2$ . Finally, we let  $\mathbb{E}[Revenue_2(v)|i]$  and  $\mathbb{E}[Revenue_3(v)|i]$  be the expected revenue earned by v in  $M_2$  and  $M_3$  respectively, conditioned on v being incident to i edges in  $M_2$ .

First, we note that  $\mathbb{E}[Revenue_3(v)|i] = \frac{\mathbb{E}[Revenue_2(v)|i]}{i}$ , as for each set of coin flips in which v has degree i in  $M_2$ , we may see any of the i edges incident to v first in the random permutation; on average, then, we receive a 1/i fraction of  $Revenue_2(v)$ . We then have the following equations:

$$\mathbb{E}[Revenue_2(v)] = \sum_{i=1}^{\infty} P_i \cdot \mathbb{E}[Revenue_2(v)|i]. \tag{1}$$

$$\mathbb{E}[Revenue_3(v)] = \sum_{i=1}^{\infty} P_i \cdot \frac{\mathbb{E}[Revenue_2(v)|i]}{i}.$$
 (2)

For ease of notation below, we use  $w_i$  to denote  $\mathbb{E}[Revenue_2(v)|i]$ . We wish to bound  $\mathbb{E}[Revenue_3(v)]$  in terms of  $\mathbb{E}[Revenue_2(v)]$ , and we do this as follows: First, we show that  $P_i \leq (1-p)P_{i-1}$ , and  $w_i \leq \frac{i}{i-1}w_{i-1}$ . Next, we prove that subject to these constraints, the worst-case ratio of these two expectations occurs when all the constraints hold with equality. We can then evaluate the sums, and show that  $\mathbb{E}[Revenue_3(v)] \geq p\mathbb{E}[Revenue_2(v)]$ , completing our proof.

It is easy to see that  $w_i \leq \frac{i}{i-1}w_{i-1}$ ; consider any partial history of SIMULATE in which i-1 edges incident to v have been added to  $M_2$  so far; as we process edges in decreasing order of weight, the ith edge must be the lightest of those seen so far. As this is true for each (partial) history, it holds in expectation, and so  $w_i \leq w_{i-1} + \frac{w_{i-1}}{i-1}$ . Similarly, to see that  $P_i \leq (1-p)P_{i-1}$ , consider a partial history until the (i-1)st edge has just been added:  $M_2$  will have i-1 edges incident to v if the coin for the next edge incident to v considered by SIMULATE comes up "heads", with probability p.  $M_2$  will have i edges incident to v if the coin for the next edge incident to v comes up "tails", and that for the following edge comes up heads, with probability  $(1-p) \cdot p$ . Again, as this holds for each history, we have  $P_i \leq (1-p)P_{i-1}$ .

To see that the worst-case ratio occurs when all these constraints hold with equality, notice that the ratio between successive terms of Equations (1) and (2) is increasing: The ratio between the *i*th terms is simply i. Let  $\alpha$  denote the worst-case ratio of the expectations; from Lemma 2.4, we already know that  $\alpha \leq p^2$ . If  $j = \lfloor 1/\alpha \rfloor$ , for  $1 \leq i \leq j$ , the ratio between the *i*th term of the two sums is at most  $\alpha$ , while for i > j, the ratio is greater than  $\alpha$ . Consider a choice of  $w_i$ 's and  $P_i$ 's such that the ratio between (1) and (2) be as large as possible, and suppose the constraints on  $P_i$  and  $w_i$  do not all hold with equality. Let k be an index such that  $P_k < (1-p)P_{k-1}$  or  $w_k < \frac{k}{k-1}w_{k-1}$ . If k > j, then by increasing  $P_k$  or  $w_k$ , we do not violate any constraint, and the increase in (1) is greater than  $\alpha$  times the increase in (2). Similarly, if  $k \leq j$ , by decreasing  $P_{k-1}$  or  $w_{k-1}$  to achieve equality, and also decreasing  $P_1 \dots P_{k-2}$  or  $w_1 \dots w_{k-2}$  to maintain feasibility, the decrease in (1) is less than  $\alpha$  times the decrease in (2). In either of these situations, we increase the ratio between the two sums, contradicting our initial setting of  $w_i$ ,  $P_i$ .

<sup>&</sup>lt;sup>1</sup>It is possible that there is only one more edge incident to v, in which case v will have i edges with probability (1-p). However, this only helps the analysis. Alternatively, one can assume the existence of a large number of "zero-weight" edges incident to v.

Finally, we can now evaluate this worst case ratio. Setting  $w_i = \frac{i}{i-1}w_i$  and  $P_i = (1-p)P_{i-1}$ , we find:

$$\mathbb{E}[Revenue_{2}(v)] = \sum_{i=1}^{\infty} iw_{1}P_{1}(1-p)^{i-1} = w_{1}P_{1}/p^{2}.$$

$$\mathbb{E}[Revenue_{3}(v)] = \sum_{i=1}^{\infty} w_{1}P_{1}(1-p)^{i-1} = w_{1}P_{1}/p = p\mathbb{E}[Revenue_{2}(v)]$$

As  $\sum_{v} \mathbb{E}[Revenue_2(v)] = \mathbb{E}[w(M_2)] \ge (1-p)\text{OPT}/2$ , we have  $\mathbb{E}[w(M_3)] \ge p(1-p)\text{OPT}/2$ , completing the proof of Lemma 2.5.

#### A.2 Other Proofs

**Proof of Theorem 4.1.** We show that the algorithm of Theorem 4.1 is  $O(\log n)$ -competitive for BVM with groups, closely following the analysis of [1] for an  $O(\log k)$ -competitive algorithm for general matroids. Recall that the algorithm observes the first half of the vertices, and picks a random integer  $j \in [0, 1 + \lceil \log n \rceil]$ . If w is the weight of the heaviest edge seen so far, the algorithm sets a threshold of  $w/2^j$ , and in the second half, greedily constructs a matching using edges of weight greater than this threshold.

Let OPT be an optimal matching; we also abuse notation and use OPT to refer to the weight of this matching, though the meaning will be clear from context. Let  $w_1, w_2, \ldots w_k$  denote the weights of edges in OPT, such that  $w_i \geq w_{i+1}$  for  $1 \leq i < k$ . Let q denote the largest index in [1,k] such that  $w_q \geq w_1/n$ . Clearly,  $\sum_{i=1}^q w_i > \text{OPT}/2$ , as the remaining edges all have weight less than  $w_1/n$ , and there are fewer than n of them. For any set of edges F, we use  $n_i(F)$  to denote the number of edges in F with weight at least  $w_i$ , and  $m_i(F)$  to denote the number of edges in F with weight at least  $w_i/2$ . Now, we have:

$$\sum_{i=1}^{q} w_i = \left(\sum_{i=1}^{q-1} n_i(OPT)(w_i - w_{i+1})\right) + n_q(OPT)w_q$$

Let M be the matching returned by our algorithm. We lower bound the weight of M as follows:

$$w(M) \ge \frac{1}{2} \left( \sum_{i=1}^{q-1} m_i(M)(w_i - w_{i+1}) \right) + \frac{m_q(M)w_q}{2}$$

In order to obtain an  $O(\log n)$ -competitive algorithm, it suffices to show that for each  $1 \le i \le q$ ,  $\mathbb{E}[m_i(M)] \ge n_i(\mathsf{OPT})/O(\log n)$ . First, consider the case of i=1:  $n_1(\mathsf{OPT})=1$ , and we argue that  $\mathbb{E}[m_1(M)] \ge 1/4(\lceil \log n \rceil + 1)$ . With probability 1/4, the vertex v incident to the heaviest edge appears in the second half, and the heaviest edge not incident to any vertex of v's group appears in the first half. If this occurs, and the algorithm picks j=0 (which happens with probability  $1/(\lceil \log n \rceil + 1)$ ), then the only edges with weight above the threshold are those incident to vertices in v's group. Thereore, the greedy algorithm will select the heaviest edge with probability  $1/4(\lceil \log n \rceil + 1)$ , and hence  $\mathbb{E}[m_1(M)] \ge 1/4(\lceil \log n \rceil + 1)$ .

We now complete the argument for each i > 1. Let v be the vertex incident to the heaviest edge. We consider two cases: First, that at least half the edges of OPT with weight at least  $w_i$  are incident to vertices not in the same group as v, and second, that more than half these edges are incident to vertices of v's group.

In the former case, suppose that v is seen in the first half. Let w be the weight of this heaviest edge, and let i' be the smallest integer in  $[0, 1 + \lceil \log n \rceil]$  such that  $w/2^{i'} \le w_i$ . With probability  $\frac{1}{(\lceil \log n \rceil + 1)}$ , the algorithm picks j = i', and the threshold is set to be  $w/2^{i'} > w_i/2$ . Let X denote the event that the threshold is set to be

<sup>&</sup>lt;sup>2</sup>Note that w may be greater than  $w_1$ , as the heaviest edge may not be in opt. However, it is easy to see that  $w \le 2w_1$ , and since  $w_i \ge w_1/n$ , there always exists such an index i'.

 $w/2^{i'}$ ; as we have seen,  $\Pr[X] \ge 1/2(\lceil \log n \rceil + 1)$ . We show that conditioned on X,  $\mathbb{E}[m_i(M)]$  is sufficiently large.

Recall that OPT contains a matching of size i using edges of weight at least  $w_i$ ; it follows that in expectation, using edges of this weight, there is a matching in the second half of size at least i/4. (This is because at least half of these i edges are in other groups; even conditioned on v appearing in the first half, each of the remaining  $\geq i/2$  edges could appear in either half.) Since we construct a greedy matching using edges of weight at least  $w_i/2$ , the expected size of this matching is at least i/8. Hence, with probability at least  $\frac{1}{2(\lceil \log n \rceil + 1)}$ ,  $\mathbb{E}[m_i(M)] \geq i/8$ . That is,  $\mathbb{E}[m_i(M)] \geq i/16(\lceil \log n \rceil + 1)$ .

We now consider the second case, when more than half the edges of OPT with weight at least  $w_i$  are in the same group as v. Let u be the vertex outside this group incident to the heaviest-weight edge. Suppose u's group appears in the first half, and v's group in the second. Let w be the weight of the heaviest edge incident to u; if  $w \le w_i$  and we pick j = 0, the *only* edges above the threshold will be vertices in v's group. Since we construct the greedy matching using only the group of v, and there exists a matching in this group with more than i/2 edges of weight  $w_i$ , the matching we construct has at least i/4 edges of weight at least  $w_i$ . If  $w > w_i$ , then with probability  $1/\lceil \log n \rceil + 1$ , we pick an index j such that  $w_i \ge w/2^j > w_i/2$ . Again, we will find a matching in which at least i/4 edges have weight at least  $w_i/2$ . Therefore, with probability at least  $\frac{1}{4(\lceil \log n + 1 \rceil)}$ , we find a matching of size at least i/4. Therefore,  $\mathbb{E}[m_i(M)] \ge i/16(\lceil \log n \rceil + 1)$ .

Therefore, we have 
$$\mathbb{E}[w(M)] \geq \frac{1}{2} \frac{1}{16(\lceil \log n \rceil + 1)} \sum_{i=1}^{q} w_i \geq \text{OPT}/64(\lceil \log n \rceil + 1).$$

**Proof of Proposition 5.1.** Let h(v) denote the heaviest edge incident to v; clearly  $\sum_v w(h_0(v)) + w(h_1(v)) \ge \sum_v w(h(v))$ . It remains to show that this latter sum is at least  $\sum_{e \in F^*} w(e)$ . To see this, consider the tree  $F^*$ , and root it arbitrarily. For each edge  $e = (u, v) \in F^*$ , the weight of e is at most h(v), where v is the vertex further from the root. Each vertex v is charged by at most one edge, and so  $\sum_v w(h(v)) \ge \sum_{e \in F^*} w(e)$ .  $\square$