Dynamic Load Balancing Strategies for Hierarchical p-FEM Solvers

Ralf-Peter Mundani¹, Alexander Düster¹, Jovana Knežević², Andreas Niggl³, and Ernst Rank¹

TU München, Chair for Computation in Engineering, 80333 München, Germany
University of Belgrade, Department of Mathematics, 11000 Belgrade, Serbia
SOFiSTiK AG, 85764 Oberschleissheim, Germany

Abstract. Equation systems resulting from a *p*-version FEM discretisation typically require a special treatment as iterative solvers are not very efficient here. Applying hierarchical concepts based on a nested dissection approach allow for both the design of sophisticated solvers as well as for advanced parallelisation strategies. To fully exploit the underlying computing power of parallel systems, dynamic load balancing strategies become an essential component.

1 Introduction

Within computational engineering, structure dynamics are one main source for challenging numerical computations. Here, the p-version of finite element methods is a very prominent technique allowing to increase accuracy without increasing the amount of elements, too. Nevertheless, the resulting equation systems fail to be efficiently solved with iterative methods such as CG or multigrid and, hence, need to be processed via expensive direct solvers (Gauss and relatives) that also entail limited potential for the parallelisation. Applying hierarchical methods based on the nested dissection concept opens the door to sophisticated solvers but also to new problems concerning the scalability of parallelisation strategies related to tree structures. Hence, an optimisation of both run time and parallel efficiency arises the necessity of dynamic load balancing strategies that are able to exploit the underlying hierarchy and, thus, to leverage parallelisation on all levels ranging from multithreading to distributed computing.

In this paper, we will show the hierarchical organisation of the *p*-version using octrees and their advantages for the solution of equation systems and parallelisation. Furthermore, we will show a dynamic load balancing strategy that allows to tackle the scalability problem which is essential — for instance — within interactive computational steering applications in order to achieve small run times and, thus, high update frequencies. As this paper describes work in progress, we will mainly highlight concepts and their benefits for parallelisation instead of concrete benchmark results which are subject to future research.

2 Structural Analysis of Thin-Walled Structures

The development of accurate and efficient element formulations for thin-walled structures has been in the focus of research in Computational Mechanics since the advent of the finite element method. At a very early stage it seemed to be clear that an investigation of plate or shell problems with tetrahedral or hexahedral elements is not feasible for practical problems, as a sufficient accuracy could only be obtained by a prohibitively large amount of degrees of freedom and computational effort. The major reasons for this observation are the mapping requirements of isoparametric, low order elements. Accurate solutions can only be obtained if the ratio aspect of an element is close to one, resulting in an enormous amount of elements, even if only one or a few layers are used over the thickness of the structure. A natural consequence of this observation was to use dimensionally reduced models like Reissner-Mindlin plates or Naghdi shells, and to build element formulations based on these theories. However, it turned out that pure displacement type elements for these models lead to notorious numerical problems like locking, giving rise to the development of numerous improvements, like mixed elements, for example.

The approach presented in this paper is different from concepts usually applied when low order elements are chosen. The idea is to construct a hierarchic family of high order elements for both thin and thick-walled structures to make it possible to control the model error inherent in every plate or shell theory by simply increasing the polynomial degree of the trial or Ansatz functions in thickness direction. The high order finite element approach for three-dimensional thin and thick-walled structures is based on a hexahedral element, applying hierarchic shape functions [8, 1, 9, 2]. The present implementation not only allows the polynomial degree to be varied for the three different local directions but also a different degree to be chosen for each primary variable, reducing the numerical effort significantly.

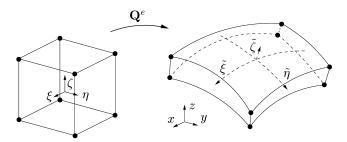


Fig. 1. Curved shell-like solid element of high order

Figure 1 depicts a hexahedral element, discretising a part of a thin-walled structure. Since the blending function method is used (see [8,1,9], e.g.) the geometry of the discretised domain may be quite complex. The shell-like solid may, for instance, be doubly curved with a non-constant thickness. When thin-walled structures like the one depicted in Figure 1 are to be discretised, it is

important to treat the in-plane direction (ξ, η) and the thickness direction (ζ) differently. This can be accounted for by using anisotropic Ansatz functions for the three-dimensional displacement field $\mathbf{u} = [u_x, u_y, u_z]^T$. In some situations it may be sufficient to restrict the polynomial degree of the Ansatz in thickness direction (ζ) to a certain degree, for example, q = 3 whereas the Ansatz chosen for the in-plane direction (ξ, η) is to be of order $p \geq 3$.

Since the p-version is less prone to locking effects [8, 9], a pure, strictly threedimensional displacement formulation can be applied. The numerical effort related to the computation of thin as well as thick-walled structures based on this formulation has two major sources: the computation of the element stiffness matrices and the solution of the resulting linear equation system. Comparing classical dimensionally reduced low-order finite elements for plates or shells with the proposed high-order formulation it turned out in many benchmark computations [1,9] that the high-order approach needs much less degrees of freedoms for the same accuracy and shifts the computational work from the global level (solution of equation system) to the local level, i.e. the computation of element matrices which is due to the numerical integration of the element matrices numerically quite demanding. Considering the parallelisation of this approach this can be regarded as an important advantage since the computation of the element matrices does not necessitate any communication and can, thus, be parallelised very efficiently, see [7]. Whereas the parallelisation of the computation of the element stiffness matrices is straightforward, the solution of the resulting linear equation system is more involved. This is due to the fact that the parallel solution of the equation system can not be carried out without any communication between the parallel processes. Furthermore, we are restricted in the choice of the solver since we apply a strict three-dimensional element formulation which results in a equation system with a poor condition number when discretising thin-walled structures. Therefore, iterative procedures, such as the preconditioned conjugate gradient method, for example, turn out to be not efficient. This drawback is even more pronounced when nonlinear problems of structural mechanics (hyperelasticity, elastoplasticity, etc.) are considered, worsening the condition of the equation system. We therefore prefer to apply a direct solver which will be described in the next section.

3 Hierarchical Organisation of the p-Version

As described in the previous section, iterative solvers are not very efficient due to the poor condition number of the equation system. Even direct solvers are advantageous here, they nevertheless suffer from a high computational complexity and they typically entail extensive parallelisation strategies in order to exploit the underlying computing power. Hence, different approaches are necessary to cover the aforementioned problems. Well-known from the field of domain decomposition is the nested dissection method that was first introduced by J.A. George [3]. The basic idea of nested dissection (ND) is to recursively subdivide the computational domain and to set up a local equation system on each subdomain

to be solved in a bottom-up step by successively eliminating local influences (computing the so-called Schur complement).

For our p-version this means to start from the computed element stiffness matrices and to organise those in a hierarchical way in order to apply ND. Therefore, octrees are used as their underlying principle of spatial partitioning is advantageous for the hierarchical organisation of the p-version. Element stiffness matrices are stored to the octree's leaf nodes — each leaf node stores at most one element or stays empty — while degrees of freedom (DOF) are stored to leaf nodes and inner nodes. The corresponding node for storing a DOF can easily be determined by finding the common parent node of all leaf nodes, i.e. element stiffness matrices, that share this DOF. It's obvious that the closer some DOF is stored to the tree root node the later it will be eliminated and the more processing time has to be invested [6]. The position of a DOF in the tree highly depends on the spatial partitioning of the domain. As octrees always halve each spatial dimension in every step, structures with huge dimensions in length and small dimensions in width and height, for instance, suffer from too many DOFs being concentrated in the root node. Here, a two-stage approach will help that halves only in one spatial dimension as long as the size of the subdomain is larger than some threshold value [10].

The results achieved with ND so far are very promising, especially ND allows to vastly reduce the computational complexity in case the underlying structure changes. As only those tree branches that contain a modification (material parameters, e.g.) have to be re-computed, all others stay untouched and the Schur complements that have been computed in a previous run can be re-used [5]. Nevertheless, the computational effort for complex scenarios is too high for retrieving results in real time — as necessary within interactive computational steering applications, for instance — thus, parallelisation is inevitable. Efficient strategies for the dynamic parallelisation of ND are subject of the next section.

4 Parallelisation Strategies

For the parallelisation of our ND approach we want to address both multi-threading and distributed computing. This allows us to easily incorporate latest developments in hardware such as multi- or manycore CPUs as well as to provide the necessary flexibility towards a unique workload distribution which — as we will see — is quite difficult to achieve and moreover plays a dominant role concerning fair speed-up and scalability values. Therefore, starting from a classical tree parallelisation we will then advance to a sophisticated dynamic load balancing strategy.

4.1 Problem Analysis

One main advantage due to the hierarchical organisation of ND via octress is the pure vertical communication between parent and children nodes (upwards for sending Schur complements and downwards for receiving the solution). Hence, cutting the tree at some level L—defining L=0 for the root level—leads to 8^L independent sub-trees which could be processed in parallel. Assuming now a full and balanced tree with N leaf nodes, i. e. N element stiffness matrices, one would achieve the highest parallelism for cutting this tree at level $L = \lceil \log_8 N \rceil - 1$. Nevertheless, speed-up and efficiency are verly limited, as this approach is similar to the problem that Minsky et al. posed in [4] concerning the parallel summation of 2N numbers on N processors. As the amount of active processes decreases in our case by a factor of 8 in each ND level, the possible values for speed-up and efficiency are slightly better, nevertheless far away from being a satisfying result due to the huge amount of inactive processes and the bad scalability inherent to this approach.

In order to achieve good results for both, i. e. speed-up and scalability, some efficient load balancing strategy is inevitable. A master-slave approach will serve as starting point here, nevertheless arising the necessesity for being adopted to the underlying problem. First, when dealing with a large amount of slaves one single master might become the bottleneck due to a huge communication advent, hence, a multi-level concept is required. Furthermore, independent tasks have to be identified and according to their dependencies on the results of other tasks then administrated by those masters. Tasks per se are processed by the slaves in parallel and should incorporate (simultaneous) multithreading to speed-up local computations. As this strategy also covers distributed computing, topics such as distributed storage are of high relevance but not part of our work in the current stage.

4.2 Task Management

Before single tasks might be administrated by some master, an equal distribution of tasks among all masters has to be initiated. Therefore, we choose a 2-level hierarchy with one master on the first level and several masters on the second. To distinguish between those masters, the one on the first level is called main master and the rest are called traders. In case of more than two levels, only the masters on the lowest level are traders, the rest are main master, second masters, third masters and so on. The difference between masters and traders is that only the masters are serving requests from the slaves, delegating these requests to the corresponding traders which then take care about the real data exchange.

To initiate a work load distribution, the main master starts to analyse the octree structure and estimates the amount of work load, i.e. the amount of necessary elimination steps for computing the Schur complement, in each node. Based on these values he is able to predict the total amount of work and, thus, to decide how many traders and how many slaves should be spawned. Furthermore, the estimated work load in each node allows the master to distribute the octree among the traders more equally even in case of very imbalanced trees. This is not the case when just cutting an imbalanced tree at some level L and assigning the resulting sub-trees to the traders. Nevertheless, to achieve an equal distribution the tree might be cut into much more parts than traders, thus, one trader has to administrate several sub-trees which do not always preserve neighbourhood

relations. This might entail further complexity due to more communication but could not be observed so far for the current implementation.

Once sub-trees have been assigned to a trader, the trader himself analyses the respective tree structures in order to determine all tasks, i.e. the equation systems in all nodes, together with their dependencies on child-nodes concerning the input data (Schur complements). The tasks that have been identified together with their dependencies are then stored to a priority queue which is updated by the trader each time some slave returns the results of its computation. That means, tasks in the queue are checked if any of their dependencies are fulfilled. Tasks without further dependencies are ready to be processed by a slave and therefore get a higher priority while tasks that cannot be processed yet have a priority $p=\infty$. The trader picks one task among those ready for processing and sends the corresponding task ID to the master for being advertised to the slaves. The master stores tuples consisting of task ID and trader ID — one tuple per trader — in order to serve requests from the slaves. Hence, the master is not involved into heavy communications as he just has to tell the slaves which trader to contact for which task.

4.3 Processing Tasks

Slaves always contact the master to request new tasks. If tasks are available, the master picks one and sends the tuple task ID and trader ID as answer back to the slave before he continues serving the next request. Receiving such a tuple, a slave can now contact the trader and request the corresponding task for a local processing. Depending on the type of task the trader initiates a data transfer, consisting of an element stiffness matrix or several Schur complements. Again, within the current implementation we do not cover distributed storage. In this case, the trader would send a mixture of locally stored data and keys to remote repositories where the slave can retrieve the rest of the information. When the data transfer has been finished, the slave holds all subsequent data to start its computation without any further communication to the master or the trader.

If a slave receives an element stiffness matrix it can immediately start with the static condensation of local DOFs in order to compute the Schur complement. In case it received several Schur complements (as result from static condensation in the respective child-levels) it first has to assemble its local equation system $K \cdot u = d$. Therefore, all Schur complements K_i are build together as $K = \sum_i K_i$ by summing up corresponding matrix entries. Using several threads for the assembly step might entail heavy synchronisation in case parallelisation takes places over the K_i as no two threads are allowed to update the same matrix element k_{ij} in parallel. This can easily be solved deploying several critical sections (one per column, row or some block of K) or by using all threads for processing just one K_i instead. While the latter one comes without the need for synchronisation it entails lower parallelism due to the serial processing of all K_i . Concerning the static condensation of local DOFs, a partial Gaussian elimination is performed. Here, a multithreaded approach concerning the middle of the three nested loops from Gaussian elimination is advantageous, as threads profit

both from the independent loop iterations and the shared memory. Obviously, this results in perfect, i. e. linear speed-up values on (SMP) architectures with 2, 4, and 8 cores as been tested. When finished, the slave contacts the same trader again to return its Schur complement for the next level tasks.

In the current stage, slaves are implemented as memoryless processes, deleting all subsequent information when the task has been finished. Keeping the regarding data even after the static condensation saves communication time in the final solution step, as these parts do not have to be transmitted again. Nevertheless, it implies a rigid order which slave has to process which task what might lead to bottlenecks in some cases.

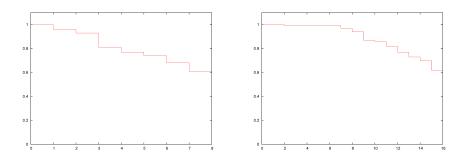


Fig. 2. Normalised working index for 8 and 16 slaves processing a problem with 4171 initial independent tasks on leaf level

A much more interesting questions is related to the working index of slaves. Slaves are either active, i.e. processing a task, or idle, i.e. waiting to be served. Summing up all phases a slave i is active and dividing it by the slave's total processing time results in a normalised working index $\omega_i \in [0,1]$. Plotting all ω_i in decreasing order provides a step function that allows to estimate about the mean time slaves are busy. In the ideal case, all ω_i are close to 1, but due to the problem of decreasing activities (i.e. independent tasks) in each higher ND level, this will be the rare case. Figure 2 shows the working index for 8 and 16 slaves processing a problem with 4171 elements, i.e. inital independent tasks, served by 4 and 8 traders, resp. It can be observed, that the working indices vary between 0.6 and 1 which is a promising result especially when compared to the theoretical values according to Minsky et al. Nevertheless, only about half of the processes are active more than 90% of the total processing time, which is somewhat clear as the task size increases on each ND level when approaching the tree root and, thus, inactive processes need to wait longer before being served a new task. Here, further optimisation is possible if tasks larger than some threshold size are distributed among several processes in order to be processed in parallel on both process and block (via threads) level. This approach is closely coupled with the question at which point distributed parallel processing is more efficient than multhithreading—a question that cannot be answered easily regarding latest trends in multi- and manycore architectures and which is still part of our researches.

5 Conclusion

In this paper, we have proposed a dynamic load balancing strategy for linear equation solvers based on the nested dissection approach in order to tackle known problems related to tree parallelisations. Applied to the p-version of finite element methods, first results sound very promising, bringing us one step closer to the long-term objective of structure dynamics in real time as needed for interactive computational steering scenarios.

6 Acknowledgements

Parts of this work have been carried out with the financial support of the International Graduate School of Science and Engineering (IGSSE) at Technische Universität München.

References

- Düster, A., Bröker, H., Rank, E.: The p-version of the finite element method for three-dimensional curved thin walled structures. Int. J. for Numer. Meth. in Eng., 52:673-703 (2001)
- 2. Düster, A., Scholz, D., Rank, E.: pq-Adaptive solid finite elements for three-dimensional plates and shells. Comp. Meth. Appl. Mech. Eng., 197:243–254 (2007)
- George, J.A.: Nested dissection of a regular finite element mesh. SIAM J. on Numer. Analysis, 10:345–363 (1973)
- 4. Minsky, M., Papert, S.: On some associative, parallel and analog computations. Associative Information Technologies, Elsevier North Holland (1971)
- 5. Mundani, R.-P.: Hierarchische Geometriemodelle zur Einbettung verteilter Simulationsaufgaben. Shaker Verlag (2006)
- 6. Mundani, R.-P., Bungartz, H.-J., Rank, E., Niggl, A., Romberg, R.: Extending the p-version of finite elements by an octree-based hierarchy. Proc. of the 16th Int. Conf. on Domain Decomp. Methods, LNCSE Vol. 55, Springer, 699–706 (2006)
- 7. Rank, E., Rücker, M., Düster, A., Bröker, H.: The efficiency of the *p*-version finite element method in a distributed computing environment. Int. J. for Numer. Meth. in Eng., **52**:589–604 (2001)
- 8. Szabó, B.A., Babuška, I.: Finite Element Analysis. John Wiley & Sons (1991)
- Szabó, B.A., Düster, A., Rank, E.: The p-version of the Finite Element Method. Encyclopedia of Computational Mechanics, John Wiley & Sons, Volume 1, Chapter 5, pp 119-139 (2004)
- Trummer, T.: Implementierung eines hochperformanten Lösungskonzeptes einer Simulations- und Steering-Umgebung im Bereich der Medizintechnik. Bachelor's thesis, Institut für Informatik, TU München (2008)