Counting Paths and Packings in Halves

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Abstract. It is shown that one can count k-edge paths in an n-vertex graph and m-set k-packings on an n-element universe, respectively, in time $\binom{n}{k/2}$ and $\binom{n}{mk/2}$, up to a factor polynomial in n, k, and m; in polynomial space, the bounds hold if multiplied by $3^{k/2}$ or $5^{mk/2}$, respectively. These are implications of a more general result: given two set families on an n-element universe, one can count the disjoint pairs of sets in the Cartesian product of the two families with $O(n\ell)$ basic operations, where ℓ is the number of members in the two families and their subsets.

1 Introduction

Some combinatorial structures can be viewed as two halves that meet in the middle. For example, a k-edge path is a combination of two k/2-edge paths. Bidirectional search [10, 25] finds such structures by searching the two halves simultaneously until the two search frontiers meet. In instantiations of this idea, it is crucial to efficiently join the two frontiers to obtain a valid or optimal solution. For instance, the meet-in-the-middle algorithm for the Subset Sum problem, by Horowitz and Sahni [15], implements the join operation via a clever pass through two sorted lists of subset sums.

In the present paper, we take the meet-in-the-middle approach to counting problems, in particular, to counting paths and packings. Here, the join operation amounts to consideration of pairs of disjoint subsets of a finite universe, each subset weighted by the number of structures that span the subset. We begin in Sect. 2 by formalizing this as the Disjoint Sum problem and providing an algorithm for it based on inclusion-exclusion techniques [5-7,17,18,20]. In Sect. 3 we apply the method to count paths of k edges in a given n-vertex graph in time $O^*(\binom{n}{k/2})$; throughout the paper, O^* suppresses a factor polynomial in the

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mentioned parameters (here, n and k). In Sect. 4 we give another application, to count k-packings in a given family of m-element subsets of an n-element universe in time $O^*(\binom{n}{mk/2})$. For both problems we also present slightly slower algorithms that require only polynomial space.

We note that an earlier report on this work under a different title [8] already introduces a somewhat more general technique and an application to counting paths. The report has been cited in some recent papers [1, 26], which we, among other related previous work, discuss below.

1.1 Related Work and Discussion

Deciding whether a given n-vertex graph contains a Hamiltonian path, that is, a simple path of n-1 edges, is well known to be NP-hard. The fastest known algorithms, due to Bellman [3, 4] and, independently, Held and Karp [14], are based on dynamic programming across the vertex subsets and run in time $O^*(2^n)$. Equally fast polynomial-space variants that actually count all Hamiltonian paths via inclusion—exclusion were discovered later by Kohn, Gottlieb, and Kohn [20], and independently, Karp [17]. Our algorithm (cf. Theorem 2, for k = n - 1), too, runs in time $O^*(2^n)$, if allowing exponential space.

In this light, it is intriguing that the parameterized problem of counting paths of k edges seems harder than the corresponding decision problem; this is the present understanding that has emerged from a series of works, starting perhaps in Papadimitriou and Yannakakis's [24] conjecture that for $k = O(\log n)$ the decision problem can be solved in polynomial time. The conjecture was proved by Alon, Yuster, and Zwick's [2] color-coding technique that gave a randomized algorithm with expected running time $O^*(5.44^k)$ and a derandomized variant with running time $O^*(c^k)$ for a large constant c. With a more efficient color-coding scheme, Chen, Lu, Sze, and Zhang [9] improved the latter bound to $O^*(12.8^k)$; see also Kneis, Mölle, Richter, and Rossmanith [19]. Using completely different techniques, Koutis [22], followed by Williams [27], developed a randomized algorithm that runs in expected time $O^*(2^k)$. Unfortunately, it is unlikely that the randomization based techniques extend to counting. For instance, very recently Alon and Gutner [1] showed that color-coding is doomed to fail as every "balanced" family of hash functions from a k-set to an n-set is of size at least $c(k)n^{\lfloor k/2 \rfloor}$ for some function c. Flum and Grohe [12] proved another negative result, namely that the counting problem is #W[1]-hard with respect to the parameter k. From a positive side, a very recent result of Vassilevska and Williams [26] implies that k-edge paths can be counted in time $O^*(2^k(k/2)!\binom{n}{k/2})$ in polynomial space; our polynomial-space algorithm (cf. Theorem 3) is faster still, by a factor of $(4/3)^{k/2}(k/2)!$.

Concerning set packings the situation is analogous, albeit the research has been somewhat less extensive. Deciding whether a given family of f subsets of an n-element universe contains a k-packing is known to be W[1]-hard [11], and thus it is unlikely that the problem is fixed parameter tractable, that is, solvable in time $c(k) f^d$ for some function c and constant d. If f is fairly large,

say exponential in n, the fastest known algorithms actually count the packings by employing the inclusion–exclusion machinery [5, 6] and run in time $O^*(2^n)$. This bound holds also for the presented algorithm (cf. Theorem 4).

Again, it is interesting that there is a natural parameterization under which counting k-packing seems harder than the corresponding decision problem. Indeed, Jia, Zhang, and Chen [16] showed that the decision problem is fixed parameter tractable with respect to the total size mk of the packing, assuming each member is of size m. Koutis [21], followed by Chen, Lu, Sze, and Zhang [9], gave faster algorithms with running time $O^*(c^{mk})$ for some constant c; we note that here the running time also grows about linearly in the input size f, which can be as large as $\binom{n}{m}$. For counting m-set k-packings, previous techniques [5, 6] alone only give a running time bound of $O^*(\binom{n}{mk})$ if $mk \leq n/2$ and $O^*(2^n)$ otherwise. Besides the present work, we are aware of two recent improvements: For the special case of counting t-matchings, that is 2-set t/2-packings, ⁴ Vassilevska and Williams [26] give a time bound of $O^*(2^{t+c(t)}\binom{n}{t/2})$, where c(t) is of the order o(t); our polynomial-space algorithm (cf. Corollary 1) turns out to be slightly faster, by a factor of about $(4/3)^{t/2}$. For the general case, Koutis and Williams [23] give a time bound of $O^*(n^{mk/2})$; our bounds (Theorem 4) appear to be superior, e.g., when mk grows linearly in n.

The presented meet-in-the-middle approach resembles the randomized divideand-conquer technique by Chen, Lu, Sze, and Zhang [9] and the similar divideand-color method by Kneis, Mölle, Richter, and Rossmanith [19], designed for parameterized decision problems. These can, in turn, be viewed as extensions of the recursive partitioning technique of Gurevich and Shelah [13] for the Hamiltonian Path problem. That said, our contribution is rather in the observation that, in the counting context, the join operation can be done efficiently using the inclusion–exclusion machinery. While our formalization of the problem as the Disjoint Sum problem is new, the solution itself can, in essence, already be found in Kennes [18], even though in terms of possibility calculus and without the idea of "trimming," that is, restricting the computations to small subsets. Kennes's results were rediscovered in a dual form and extended to accommodate trimming in the authors' recent works [5–7].

2 The Disjoint Sum Problem

Given two set families \mathcal{A} and \mathcal{B} , and functions α and β that associate each member of \mathcal{A} and \mathcal{B} , respectively, an element from a ring R, the *Disjoint Sum* problem is to find the sum of the products $\alpha(A)\beta(B)$ over all *disjoint* pairs of subsets (A,B) in the Cartesian product $\mathcal{A} \times \mathcal{B}$; denote the sum by $\alpha \boxtimes \beta$. In applications, the ring R is typically the set of integers equipped with the usual addition and multiplication operation. Note that, had the condition of disjointness removed, the problem could be easily solved using about $|\mathcal{A}| + |\mathcal{B}|$

⁴ Whether counting t-matchings is fixed-parameter tractable remains a major open question in parameterized complexity.

additions and one multiplication. However, to respect the disjointness condition, the straightforward algorithm appears to require about $|\mathcal{A}||\mathcal{B}|$ ring operations and tests of disjointness.

In many cases, we fortunately can do better by applying the principle of inclusion and exclusion. The basic idea is to compute the sum over pairs (A, B) with $A \cap B = \emptyset$ by subtracting the sum over pairs with $A \cap B = X \neq \emptyset$ from the sum over pairs with no constraints. For a precise treatment, it is handy to denote by N the union of all the members in the families \mathcal{A} and \mathcal{B} , and extend the functions α and β to all subsets of N by letting them evaluate to 0 outside \mathcal{A} and \mathcal{B} , respectively. We also use the Iverson bracket notation: [P] = 1 if P is true, and [P] = 0 otherwise. Now, by elementary manipulation,

$$\alpha \boxtimes \beta = \sum_{A} \sum_{B} [A \cap B = \emptyset] \alpha(A) \beta(B)$$

$$= \sum_{A} \sum_{B} \sum_{X} (-1)^{|X|} [X \subseteq A \cap B] \alpha(A) \beta(B)$$

$$= \sum_{X} (-1)^{|X|} \sum_{A} \sum_{B} [X \subseteq A] [X \subseteq B] \alpha(A) \beta(B)$$

$$= \sum_{X} (-1)^{|X|} \left(\sum_{A \supset X} \alpha(A) \right) \left(\sum_{B \supset X} \beta(B) \right). \tag{1}$$

Here we understand that A, B, and X run through all subsets of N unless otherwise specified. Note also that the second equality holds because every *nonempty* set has exactly as many subsets of even size as subsets of odd size.

To analyze the complexity of evaluating the inclusion–exclusion expression (1), we define the *lower set* of a set family \mathcal{F} , denoted by $\downarrow \mathcal{F}$, as the family consisting of all the sets in \mathcal{F} and their subsets. We first observe that in (1) it suffices to let X run over the intersection of $\downarrow \mathcal{A}$ and $\downarrow \mathcal{B}$, for any other X has no supersets in \mathcal{A} or in \mathcal{B} . Second, we observe that the values

$$\hat{\alpha}(X) \doteq \sum_{A\supset X} \alpha(A) ,$$

for all $X \in \ \downarrow A$, can be computed in a total of $|\ \downarrow A|$ n ring and set operations, as follows. Let a_1, a_2, \ldots, a_n be the n elements of N. For any $i = 0, 1, \ldots, n$ and $X \in \ \downarrow A$ define $\hat{\alpha}_i(X)$ as the sum of the $\alpha(A)$ over all sets $A \in \ \downarrow A$ with $A \supseteq X$ and $A \cap \{a_1, a_2, \ldots, a_i\} = X \cap \{a_1, a_2, \ldots, a_i\}$. In particular, $\hat{\alpha}_n(X) = \alpha(X)$ and $\hat{\alpha}_0(X) = \hat{\alpha}(X)$. Furthermore, by induction on i one can prove the recurrence

$$\hat{\alpha}_{i-1}(X) = [a_i \notin X] \, \hat{\alpha}_i(X) + [X \cup \{a_i\} \in \mathcal{A}] \, \hat{\alpha}_i(X \cup \{a_i\}) ;$$

for details, see closely related recent work on trimmed zeta transform and Moebius inversion [6, 7]. Thus, for each i, the values $\hat{\alpha}_i(X)$ for all $X \in \mathcal{A}$ can be computed with $|\mathcal{A}|$ ring and set operations.

We have shown the following.

Theorem 1. The Disjoint Sum problem can be solved with $O(n(|\downarrow A| + |\downarrow B|))$ ring and set operations, and with a storage for $O(|\downarrow A| + |\downarrow B|)$ ring elements, where n is the number of distinct elements covered by the members of A and B.

3 Paths

Consider paths in an undirected graph with vertex set V and edge set E. Define a k-edge path as a sequence of k+1 distinct vertices $v_0v_1\cdots v_k$ such that the adjacent vertices v_{i-1} and v_i are connected by an edge $v_{i-1}v_i$ in E, for $i=1,2,\ldots,k$. We call the set $\{v_0,v_1,\ldots,v_k\}$ the support of the path and v_0 and v_k the ends of the path. For any vertex v and a subset of j vertices $S\subseteq V$, let $p_j(S,v)$ denote the number of j-edge paths with an end v and support $S\cup\{v\}$. Clearly, the values can be computed by dynamic programming using the recurrence

$$p_0(S, v) = [S = \emptyset], \quad p_j(S, v) = \sum_{u \in S} p_{j-1}(S \setminus \{u\}, u) [uv \in E] \text{ for } j > 0.$$

Alternatively, one may use the inclusion–exclusion formula [17, 20]

$$p_j(S, v) = \sum_{Y \subset S} (-1)^{|S \setminus Y|} w_j(Y, v) ,$$

where $w_j(Y, v)$ is the number of j-edge walks starting from v and visiting some vertices of Y, that is, sequences $u_0u_1\cdots u_j$ with $u_0=v$, each $u_{i-1}u_i\in E$, and $u_1,u_2,\ldots,u_j\in Y$. Note that for any given Y, v, and j, the term $w_j(Y,v)$ can be computed in time polynomial in n. Using either of the above two formulas, the values $p_j(S,v)$ for all $v\in V$ and sets $S\subseteq V\setminus\{v\}$ of size j, can be computed in time $O^*(\binom{n}{\lfloor j})$; here and henceforth, $\binom{q}{\rfloor r}$ denotes the sum of the binomial coefficients $\binom{q}{0}+\binom{q}{1}+\cdots+\binom{q}{r}$. In particular, the number of k-edge paths in the graph is obtained as the sum of $p_k(S,v)$ over all $v\in V$ and $S\subseteq V\setminus\{v\}$ of size k, in time $O^*(\binom{n}{\lfloor k})$.

However, meet-in-the-middle yields a much faster algorithm. Assuming for simplicity that k is even, the path has a mid-vertex, $v_{k/2}$, at which the path uniquely decomposes into two k/2-edge paths, namely $v_0v_1\cdots v_{k/2}$ and $v_{k/2}v_{k/2+1}\cdots v_k$, with almost disjoint supports. Thus, the number of k-edge paths is obtained as the sum of the products

$$p_{k/2}(S,v) p_{k/2}(T,v) / 2$$

over all vertices $v \in V$ and disjoint pairs of subsets $S, T \subseteq V \setminus \{v\}$ of size k/2. Applying Theorem 1, once for each $v \in V$, with $\mathcal{A} \doteq \mathcal{B} \doteq \{S \subseteq V \setminus \{v\} : |S| = k/2\}$ and $\alpha \doteq \beta \doteq p_{k/2}$ gives the following.

Theorem 2. The k-edge paths in a given graph on n vertices can be counted in time $O^*(\binom{n}{k/2})$.

In the remainder of this section we present a polynomial-space variant of the above described algorithm. Let the mid-vertex v be fixed. Then the task is to compute, for each $X \subseteq V \setminus \{v\}$ of size at most k/2, the sum

$$\sum_{S\supset X} p_{k/2}(S,v) = \sum_{S\supset X} \sum_{Y\subset S} (-1)^{|S\backslash Y|} \, w_{k/2}(Y,v)$$

in space polynomial in n and k. If done in a straightforward manner, the running time, ignoring polynomial factors, becomes proportional to the number of triplets (X, S, Y) with $X, Y \subseteq S \subseteq V \setminus \{v\}$ and |S| = k/2. This number is $\binom{n-1}{k/2} 2^k$ because there are $\binom{n-1}{k/2}$ choices for S and for any fixed S, there are $2^{k/2}$ choices for X and $2^{k/2}$ choices for Y.

A faster algorithm is obtained by reversing the order of summation:

$$\begin{split} \sum_{S\supseteq X} p_{k/2}(S,v) &= \sum_Y w_{k/2}(Y,v) \sum_S (-1)^{|S\backslash Y|} \left[X,Y\subseteq S\right] \\ &= \sum_Y w_{k/2}(Y,v) \left(-1\right)^{k/2-|Y|} \binom{n-|X\cup Y|}{k/2-|X\cup Y|} \,; \end{split}$$

here Y and S run through all subsets of $V \setminus \{v\}$ of size at most k/2 and exactly k/2, respectively. The latter equality holds because S is of size k/2 and contains $X \cup Y$. It remains to find in how many ways one can choose the sets X and Y such that the union $U \doteq X \cup Y$ is of size at most k/2. This number is

$$\sum_{s=0}^{k/2} {n-1 \choose s} 3^s \le \frac{3}{2} {n-1 \choose k/2} 3^{k/2} ,$$

because there are $\binom{n-1}{s}$ ways to choose U of size s, and one can put each element in U either to X or Y or both.

Theorem 3. The k-edge paths in a given graph on n vertices can be counted in time $O^*(3^{k/2}\binom{n}{k/2})$ in space polynomial in n and k.

4 Set Packing

Next, consider packings in a set family \mathcal{F} consisting of subsets of a universe N. We will assume that each member of \mathcal{F} is of size m. A k-packing in \mathcal{F} is a set of k mutually disjoint members of \mathcal{F} . The members F_1, F_2, \ldots, F_k of a k-packing can be ordered in k! different ways to an ordered k-packing $F_1F_2\cdots F_k$. Define the support of the ordered k-packing as the union of its members. For any $S\subseteq N$, let $\pi_j(S)$ denote the number of ordered j-packings in \mathcal{F} with support S. The values can be computing by dynamic programming using the recurrence

$$\pi_0(S) = [S = \emptyset] , \quad \pi_j(S) = \sum_{F \subseteq S} \pi_{j-1}(S \setminus F) [F \in \mathcal{F}] \quad \text{for } j > 0 .$$

Alternatively, one may use the inclusion–exclusion formula

$$\pi_j(S) = \sum_{Y \subseteq S} (-1)^{|S \setminus Y|} \bigg(\sum_{F \subseteq Y} [F \in \mathcal{F}] \bigg)^j$$

(here we use the assumption that every member of \mathcal{F} is of size m) [5, 6]. Using the inclusion–exclusion formula, the values $\pi_j(S)$ for all $S \subseteq N$ of size mj can be computed in time $O^*(\binom{n}{\lfloor mj})$, where n is the cardinality of N; a straightforward implementation of the dynamic programming algorithm yields the same bound, provided that m is a constant. In particular, the number of k-packings in \mathcal{F} is obtained as the sum of $\pi_k(S)/k!$ over all $S \subseteq N$ of size mk, in time $O^*(\binom{n}{\lfloor mk \rfloor})$.

Again, meet-in-the-middle gives a much faster algorithm. Assuming for simplicity that k is even, we observe that the ordered k-packing decomposes uniquely into two ordered k/2-packings $F_1F_2\cdots F_{k/2}$ and $F_{k/2+1}F_{k/2+2}\cdots F_k$ with disjoint supports. Thus the number of ordered k-packings in $\mathcal F$ is obtained as the sum of the products

$$\pi_{k/2}(S) \, \pi_{k/2}(T) / 2$$

over all disjoint pairs of subsets $S, T \subseteq N$ of size mk/2. Applying Theorem 1 with $A \doteq \mathcal{B} \doteq \{S \subseteq N : |S| = mk/2\}$ and $\alpha \doteq \beta \doteq \pi_{k/2}$ gives the following.

Theorem 4. The k-packings in a given family of m-element subsets of an n-element set can be counted in time $O^*(\binom{n}{mk/2})$.

We next present a polynomial-space variant. The task is, in essence, to compute for each $X \subseteq N$ of size at most mk/2 the sum

$$\sum_{S\supseteq X} \pi_{k/2}(S) = \sum_{S\supseteq X} \sum_{Y\subseteq S} (-1)^{|S\backslash Y|} \bigg(\sum_{F\subseteq Y} [F\in\mathcal{F}]\bigg)^{k/2}$$

in space polynomial in n, k, and m.

As with counting paths in the previous section, a faster than the straightforward algorithm is obtained by reversing the order of summation:

$$\begin{split} \sum_{S\supseteq X} \pi_{k/2}(S) &= \sum_{Y} \bigg(\sum_{F\subseteq Y} [F\in\mathcal{F}]\bigg)^{k/2} \sum_{S} (-1)^{|S\backslash Y|} [X,Y\subseteq S] \\ &= \sum_{Y} \bigg(\sum_{F\subseteq Y} [F\in\mathcal{F}]\bigg)^{k/2} (-1)^{k/2-|Y|} \binom{n-|X\cup Y|}{mk/2-|X\cup Y|}; \end{split}$$

here Y and S run through all subsets of N of size at most mk/2 and exactly mk/2, respectively. It remains to find the number of triplets (X,Y,F) satisfying $|X \cup Y| \le mk/2$, |F| = m, and $F \subseteq Y$. This number is

$$\sum_{s=m}^{mk/2} \binom{n}{s} \binom{s}{m} 2^m 3^{s-m} < \frac{3}{2} \binom{n}{mk/2} \binom{mk/2}{m} 2^m 3^{mk/2-m}$$

$$\leq \frac{3}{2} \binom{n}{mk/2} 5^{mk/2},$$
(2)

because there are $\binom{n}{mk/2}$ choices for the union $U \doteq X \cup Y$ of size s, within which there are $\binom{s}{m}$ choices for F; the elements in F can be put to only Y or to both X and Y, whereas each of the remaining s-m elements in U is put to either X or Y or both.

Theorem 5. The k-packings in a given family of m-element subsets of an n-element set can be counted in time $O^*(5^{mk/2}\binom{n}{mk/2})$ in space polynomial in n, k. and m.

We remark that the upper bound (2) is rather crude for small values of m. In particular, provided that m is a constant, we can replace the constant 5 by 3.

Corollary 1. The k-packings in a given family of 2-element subsets of an n-element set can be counted in time $O^*(3^k\binom{n}{k})$ in space polynomial in n and k.

References

- 1. Alon, N., Gutner, S.: Balanced hashing, color coding and approximate counting. Electronic Colloquium on Computational Complexity, Report TR09-12 (2009)
- Alon, N., Yuster, R., Zwick, U.: Color-coding. J. Assoc. Comput. Mach. 42, 844–856 (1995)
- Bellman, R.: Combinatorial processes and dynamic programming. In: Bellman, R., Hall, M., Jr. (eds.) Combinatorial Analysis, Proceedings of Symposia in Applied Mathematics 10, pp. 217–249. American Mathematical Society (1960)
- Bellman, R.: Dynamic programming treatment of the travelling Salesman Problem.
 J. Assoc. Comput. Mach. 9, 61–63 (1962)
- Björklund, A., Husfeldt, T., Koivisto, M.: Set partitioning via inclusion-exclusion. SIAM J. Comput., Special Issue for FOCS 2006, to appear
- Björklund, A., Husfeldt, T., Kaski, P., Koivisto, M.: Fourier meets Möbius: fast subset convolution. In: 39th ACM Symposium on Theory of Computing (STOC 2007), pp. 67–74. ACM Press (2007)
- 7. Björklund, A., Husfeldt, T., Kaski, P., Koivisto, M.: Trimmed Moebius inversion and graphs of bounded degree. In: 25th International Symposium on Theoretical Aspects of Computer Science (STACS 2008). Dagstuhl Seminar Proceedings 08001, pp. 85–96. IBFI Schloss Dagstuhl (2008)
- 8. Björklund, A., Husfeldt, T., Kaski, P., Koivisto, M.: The fast intersection transform with applications to counting paths. CoRR, abs/0809.2489 (2008)
- 9. Chen, J., Lu, S., Sze, S.-H., Zhang, F.: Improved algorithms for path, matching, and packing problems. In: 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2007), Philadelphia, PA, USA, pp. 298–307. Society for Industrial and Applied Mathematics (2007)
- 10. Danzig, G.: Linear Programming and Extensions. Princeton University Press (1963)
- 11. Downey, R.G., Fellows, M.R.: Parameterized Complexity. Springer, Berlin (1999)
- 12. Flum, J., Grohe, M.: The parameterized complexity of counting problems. SIAM J. Comput. 33, 892–922 (2004)
- 13. Gurevich, Y., Shelah, S.: Expected computation time for Hamiltonian path problem. SIAM J. Comput. 16, 486–502 (1987)

- Held, M., Karp, R.M.: A dynamic programming approach to sequencing problems.
 J. Soc. Indust. Appl. Math. 10, 196–210 (1962)
- 15. Horowitz, E., Sahni, S.: Computing partitions with applications to the knapsack Problem. J. Assoc. Comput. Mach. 21, 277–292 (1974)
- Jia, W., Zhang, C., Chen. J.: An efficient parameterized algorithm for m-set packing. J. Algorithms 50, 106–117 (2004)
- 17. Karp, R.M.: Dynamic programming meets the principle of inclusion and exclusion. Oper. Res. Lett. 1, 49–51 (1982)
- 18. Kennes, R.: Computational aspects of the Moebius transform on a graph. IEEE Transactions on System, Man, and Cybernetics 22, 201–223 (1991)
- Kneis, J., Mölle, D., Richter, S., Rossmanith, P.: Divide-and-color. 32nd International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2006), LNCS 4271, pp. 58–67. Springer (2008)
- Kohn, S., Gottlieb, A., Kohn, M.: A generating function approach to the traveling salesman problem. In: ACM Annual Conference (ACM 1977), pp. 294–300. ACM Press (1977)
- Koutis, I.: A faster parameterized algorithm for set packing. Information Processing Letters 94, 4–7 (2005)
- 22. Koutis, I.: Faster algebraic algorithms for path and packing problems. In: Aceto, L. et al. (eds.) 35th International Colloquium on Automata, Languages and Programming (ICALP 2008), Part I, LNCS 5125, pp. 575–586. Springer (2008)
- 23. Koutis, I., Williams, R.: Limitations and applications of group algebras for parameterized problems. In: 35th International Colloquium on Automata, Languages and Programming (ICALP 2009), to appear
- Papadimitriou, C.H., Yannakakis, M.: On limited nondeterminism and the complexity of the V-C dimension. J. Comput. Syst. Sci. 53, 161–170 (1996)
- 25. Pohl, I.: Bi-directional and heuristic search in path problems. PhD thesis, Report SLAC-104, Stanford University (1969)
- Vassilevska, V., Williams, R.: Finding, minimizing, and counting weighted subgraphs. In: 41st ACM Symposium on Theory of Computing (STOC 2009), to appear
- 27. Williams, R.: Finding paths of length k in $O^*(2^k)$ time. Information Processing Letters 109, 315–318 (2009)