# MEASURING THE INFLUENCE OF THE $k$ TH LARGEST VARIABLE ON FUNCTIONS OVER THE UNIT HYPERCUBE 

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#### Abstract

By considering a least squares approximation of a given square integrable function $f:[0,1]^{n} \rightarrow \mathbb{R}$ by a shifted $L$-statistic function (a shifted linear combination of order statistics), we define an index which measures the global influence of the $k$ th largest variable on $f$. We show that this influence index has appealing properties and we interpret it as an average value of the difference quotient of $f$ in the direction of the $k$ th largest variable or, under certain natural conditions on $f$, as an average value of the derivative of $f$ in the direction of the $k$ th largest variable. We also discuss a few applications of this index in statistics and aggregation theory.


## 1. Introduction

Consider a real-valued function $f$ of $n$ variables $x_{1}, \ldots, x_{n}$ and suppose we want to measure a global influence degree of every variable $x_{i}$ on $f$. A reasonable way to define such an influence degree consists in considering the coefficient of $x_{i}$ in the best least squares approximation of $f$ by affine functions of the form

$$
g\left(x_{1}, \ldots, x_{n}\right)=c_{0}+\sum_{i=1}^{n} c_{i} x_{i} .
$$

This approach was considered in [6, 10] for pseudo-Boolean functions] $f:\{0,1\}^{n} \rightarrow$ $\mathbb{R}$ and in [9 for square integrable functions $f:[0,1]^{n} \rightarrow \mathbb{R}$. It turns out that, in both cases, the influence index of $x_{i}$ on $f$ is given by an average "derivative" of $f$ with respect to $x_{i}$.

Now, it is also natural to consider and measure a global influence degree of the smallest variable, or the largest variable, or even the $k$ th largest variable for some $k \in\{1, \ldots, n\}$. As an application, suppose we are to choose an appropriate aggregation function $f:[0,1]^{n} \rightarrow \mathbb{R}$ to compute an average value of $[0,1]$-valued grades obtained by a student. If, for instance, we use the arithmetic mean function, we might expect that both the smallest and the largest variables are equally influent. However, if we use the geometric mean function, for which the value 0 (the left endpoint of the scale) is multiplicatively absorbent, we might anticipate that the smallest variable is more influent than the largest one.

[^0]Similarly to the previous problem, to define the influence of the $k$ th largest variable on $f$ it is natural to consider the coefficient of $x_{(k)}$ in the best least squares approximation of $f$ by symmetric functions of the form

$$
g\left(x_{1}, \ldots, x_{n}\right)=a_{0}+\sum_{i=1}^{n} a_{i} x_{(i)}
$$

where $x_{(1)}, \ldots, x_{(n)}$ are the order statistics obtained by rearranging the variables in ascending order of magnitude.

In this paper we solve this problem for square integrable functions $f:[0,1]^{n} \rightarrow \mathbb{R}$. More precisely, we completely describe the least squares approximation problem above and derive an explicit expression for the corresponding influence index (§2). We also show that this index has several natural properties, such as linearity and continuity, and we give an interpretation of it as an average value of the difference quotient of $f$ in the direction of the $k$ th largest variable. Under certain natural conditions on $f$, we also interpret the index as an average value of the derivative of $f$ in the direction of the $k$ th largest variable (§3). We then provide some alternative formulas for the index to possibly simplify its computation (§4) and we consider some examples including the case when $f$ is the Lovász extension of a pseudoBoolean function (§5). Finally, we discuss a few applications of the index (§6).

We employ the following notation throughout the paper. Let $\mathbb{I}^{n}$ denote the $n$ dimensional unit cube $[0,1]^{n}$. We denote by $L^{2}\left(\mathbb{I}^{n}\right)$ the class of square integrable functions $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ modulo equality almost everywhere. For any $S \subseteq[n]=$ $\{1, \ldots, n\}$, we denote by $\mathbf{1}_{S}$ the characteristic vector of $S$ in $\{0,1\}^{n}$ (with the particular case $\mathbf{0}=\mathbf{1}_{\varnothing}$ ).

Recall that if the $\mathbb{I}$-valued variables $x_{1}, \ldots, x_{n}$ are rearranged in ascending order of magnitude $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$, then $x_{(k)}$ is called the $k$ th order statistic and the function $\mathrm{os}_{k}: \mathbb{I}^{n} \rightarrow \mathbb{R}$, defined as $\mathrm{os}_{k}(\mathbf{x})=x_{(k)}$, is the $k$ th order statistic function. As a matter of convenience, we also formally define $\mathrm{os}_{0} \equiv 0$ and $\mathrm{os}_{n+1} \equiv 1$. To stress on the arity of the function, we can replace the symbols $x_{(k)}$ and $\mathrm{os}_{k}$ with $x_{k: n}$ and $\mathrm{os}_{k: n}$, respectively. For general background on order statistics, see for instance [1, 4].

Finally, we use the lattice notation $\wedge$ and $\vee$ to denote the minimum and maximum functions, respectively.

## 2. Influence index for the $k$ Th Largest variable

An $L$-statistic function is a linear combination of the functions $\mathrm{os}_{1}, \ldots, \mathrm{os}_{n}$. A shifted $L$-statistic function is a constant plus an $L$-statistic function. Denote by $V_{L}$ the set of shifted $L$-statistic functions. Clearly, $V_{L}$ is spanned by the linearly independent set

$$
\begin{equation*}
B=\left\{\mathrm{os}_{1}, \ldots, \mathrm{os}_{n}, \mathrm{os}_{n+1}\right\} \tag{1}
\end{equation*}
$$

and thus is a linear subspace of $L^{2}\left(\mathbb{I}^{n}\right)$ of dimension $n+1$. For a given function $f \in L^{2}\left(\mathbb{I}^{n}\right)$, we define the best shifted L-statistic approximation of $f$ as the function $f_{L} \in V_{L}$ that minimizes the distance

$$
\|f-g\|^{2}=\int_{\mathbb{I}^{n}}(f(\mathbf{x})-g(\mathbf{x}))^{2} d \mathbf{x}
$$

among all $g \in V_{L}$, where $\|\cdot\|$ is the norm in $L^{2}\left(\mathbb{I}^{n}\right)$ associated with the inner product $\langle f, g\rangle=\int_{\mathbb{I}^{n}} f(\mathbf{x}) g(\mathbf{x}) d \mathbf{x}$. Using the general theory of Hilbert spaces, we immediately see that the solution of this approximation problem exists and is uniquely determined by the orthogonal projection of $f$ onto $V_{L}$. This projection is given by

$$
\begin{equation*}
f_{L}=\sum_{j=1}^{n+1} a_{j} \mathrm{os}_{j} \tag{2}
\end{equation*}
$$

where the coefficients $a_{j}$ (for $j \in[n+1]$ ) are characterized by the conditions

$$
\begin{equation*}
\left\langle f-f_{L}, \mathrm{os}_{i}\right\rangle=0 \quad \text { for all } \quad i \in[n+1] \tag{3}
\end{equation*}
$$

Consider the matrix representing the inner product in the basis (11), that is, the square matrix $M$ of order $n+1$ defined by $(M)_{i j}=\left\langle\mathrm{os}_{i}\right.$, os $\left._{j}\right\rangle$ for all $i, j \in[n+1]$. Denote also by $\mathbf{b}$ the $(n+1) \times 1$ column matrix defined by $(\mathbf{b})_{i}=\left\langle f, \mathrm{os}_{i}\right\rangle$ for all $i \in[n+1]$ and by a the $(n+1) \times 1$ column matrix defined by $(\mathbf{a})_{j}=a_{j}$ for all $j \in[n+1]$. Using this notation, the unique solution of the approximation problem defined in (2) and (3) is simply given by

$$
\begin{equation*}
\mathbf{a}=M^{-1} \mathbf{b} \tag{4}
\end{equation*}
$$

To give an explicit expression of this solution, we shall make use of the following formula (see [3]). For any integers $1 \leqslant k_{1}<\cdots<k_{m} \leqslant n$ and any nonnegative integers $c_{1}, \ldots, c_{m}$, we have

$$
\begin{equation*}
\int_{\mathbb{I}^{n}} \prod_{j=1}^{m} x_{k_{j}: n}^{c_{j}} d \mathbf{x}=\frac{n!}{\left(n+\sum_{j=1}^{m} c_{j}\right)!} \prod_{j=1}^{m} \frac{\left(k_{j}-1+\sum_{i=1}^{j} c_{i}\right)!}{\left(k_{j}-1+\sum_{i=1}^{j-1} c_{i}\right)!} \tag{5}
\end{equation*}
$$

Lemma 1. For every $i, j \in[n+1]$, we have

$$
\begin{equation*}
(M)_{i j}=\frac{\min (i, j)(\max (i, j)+1)}{(n+1)(n+2)} \tag{6}
\end{equation*}
$$

and

$$
\frac{\left(M^{-1}\right)_{i j}}{(n+1)(n+2)}= \begin{cases}2, & \text { if } i=j<n+1  \tag{7}\\ \frac{n+1}{n+2}, & \text { if } i=j=n+1 \\ -1, & \text { if }|i-j|=1 \\ 0, & \text { otherwise }\end{cases}
$$

Proof. The formula for $(M)_{i j}=\left\langle\mathrm{os}_{i}, \mathrm{os}_{j}\right\rangle$ immediately follows from (15). The formula for $\left(M^{-1}\right)_{i j}$ can be checked easily.

Recall that the central second difference operator is defined for any real sequence $\left(z_{k}\right)_{k \geqslant 1}$ as $\delta_{k}^{2} z_{k}=z_{k+1}-2 z_{k}+z_{k-1}$. For every $k \in[n]$, define the function $g_{k} \in L^{2}\left(\mathbb{I}^{n}\right)$ as

$$
\begin{equation*}
g_{k}=-(n+1)(n+2) \delta_{k}^{2} \mathrm{os}_{k} \tag{8}
\end{equation*}
$$

Using (4) and (17), we immediately obtain the following explicit forms for the components of $f_{L}$ in the basis (11).
Proposition 2. The best shifted L-statistic approximation $f_{L}$ of a function $f \in$ $L^{2}\left(\mathbb{I}^{n}\right)$ is given by (2), where

$$
a_{k}= \begin{cases}\left\langle f, g_{k}\right\rangle, & \text { if } k \in[n],  \tag{9}\\ (n+1)^{2}\langle f, 1\rangle-(n+1)(n+2)\left\langle f, \mathrm{os}_{n}\right\rangle, & \text { if } k=n+1\end{cases}
$$

Now, to measure the global influence of the $k$ th largest variable $x_{(k)}$ on an arbitrary function $f \in L^{2}\left(\mathbb{I}^{n}\right)$, we naturally define an index $I: L^{2}\left(\mathbb{I}^{n}\right) \times[n] \rightarrow \mathbb{R}$ as $I(f, k)=a_{k}$, where $a_{k}$ is obtained from $f$ by (9). We will see in the next section that this index indeed measures an influence degree.
Definition 3. Let $I: L^{2}\left(\mathbb{I}^{n}\right) \times[n] \rightarrow \mathbb{R}$ be defined as $I(f, k)=\left\langle f, g_{k}\right\rangle$, that is

$$
\begin{equation*}
I(f, k)=-(n+1)(n+2) \int_{\mathbb{I}^{n}} f(\mathbf{x}) \delta_{k}^{2} x_{(k)} d \mathbf{x} \tag{10}
\end{equation*}
$$

Remark 1. By combining (5) and (10), we see that the index $I(f, k)$ can be easily computed when $f$ is any polynomial function of order statistics.

Thus we have defined an influence index from an elementary approximation (projection) problem. Conversely, the following result shows that the best shifted $L$-statistic approximation of $f \in L^{2}\left(\mathbb{I}^{n}\right)$ is the unique function of $V_{L}$ that preserves the average value and the influence index. To this extent, we observe that letting $i=n+1$ in (33) leads to $\left\langle f_{L}, 1\right\rangle=\langle f, 1\rangle$, that is,

$$
\begin{equation*}
\frac{1}{n+1} \sum_{k=1}^{n+1} k a_{k}=\langle f, 1\rangle \tag{11}
\end{equation*}
$$

Proposition 4. A function $g \in V_{L}$ is the best shifted L-statistic approximation of $f \in L^{2}\left(\mathbb{I}^{n}\right)$ if and only if $\int_{\mathbb{I}^{n}} f(\mathbf{x}) d \mathbf{x}=\int_{\mathbb{I}^{n}} g(\mathbf{x}) d \mathbf{x}$ and $I(f, k)=I(g, k)$ for all $k \in[n]$.
Proof. We formally extend $I(f, \cdot)$ to $[n+1]$ by defining $I(f, n+1)=a_{n+1}$. By (3), the function $g \in V_{L}$ is the best shifted $L$-statistic approximation of $f \in L^{2}\left(\mathbb{I}^{n}\right)$ if and only if $\left\langle f, \mathrm{os}_{i}\right\rangle=\left\langle g, \mathrm{os}_{i}\right\rangle$ for all $i \in[n+1]$. By (4), this condition is equivalent to $I(f, k)=I(g, k)$ for all $k \in[n+1]$. We then conclude by (11).

Remark 2. Combining (22) with (11), we can rewrite the best shifted $L$-statistic approximation of $f$ as $f_{L}=\langle f, 1\rangle+\sum_{k=1}^{n} I(f, k)\left(x_{(k)}-\frac{k}{n+1}\right)$.

## 3. Properties and interpretations

In this section we present various properties and interpretations of the index $I(f, k)$. The first result follows immediately from Definition 3

Proposition 5. For every $k \in[n]$, the mapping $f \mapsto I(f, k)$ is linear and continuous.

We now present an interpretation of $I(f, k)$ as a covariance. Considering the unit cube $\mathbb{I}^{n}$ as a probability space with respect to the Lebesgue measure, we see that, for any $k \in[n]$, the index $I(f, k)$ is the covariance of the random variables $f$ and $g_{k}$. Indeed, we have $I(f, k)=E\left(f g_{k}\right)=\operatorname{cov}\left(f, g_{k}\right)+E(f) E\left(g_{k}\right)$, where $E\left(g_{k}\right)=\left\langle 1, g_{k}\right\rangle=I(1, k)=0$. From the usual interpretation of the concept of covariance, we see that $I(f, k)$ is positive whenever the values of $f-E(f)$ and $g_{k}-E\left(g_{k}\right)=g_{k}$ have the same sign. Note that $g_{k}(\mathbf{x})$ is positive whenever $x_{(k)}$ is greater than $\frac{1}{2}\left(x_{(k+1)}+x_{(k-1)}\right)$, which is the midpoint of the range of $x_{(k)}$ when the other order statistics are fixed at $\mathbf{x}$.

We now provide an interpretation of $I(f, k)$ as an expected value of the derivative of $f$ in the direction of the $k$ th largest variable (see Proposition 7).

Let $S_{n}$ denote the symmetric group on $[n]$. Recall that the unit cube $\mathbb{I}^{n}$ can be partitioned almost everywhere into the open standard simplexes

$$
\mathbb{I}_{\pi}^{n}=\left\{\mathbf{x} \in \mathbb{I}^{n}: x_{\pi(1)}<\cdots<x_{\pi(n)}\right\} \quad\left(\pi \in S_{n}\right)
$$

Definition 6. Given $k \in[n]$, let $f: \cup_{\pi \in S_{n}} \mathbb{I}_{\pi}^{n} \rightarrow \mathbb{R}$ be a function such that the partial derivative $\left.D_{\pi(k)} f\right|_{\mathbb{I}_{\pi}^{n}}$ exists for every $\pi \in S_{n}$. The derivative of $f$ in the direction $(k)$ is the function $D_{(k)} f: \cup_{\pi \in S_{n}} \mathbb{I}_{\pi}^{n} \rightarrow \mathbb{R}$ defined as

$$
D_{(k)} f(\mathbf{x})=D_{\pi(k)} f(\mathbf{x}) \quad \text { for all } \quad \mathbf{x} \in \mathbb{I}_{\pi}^{n}
$$

Remark 3. By considering the chain rule in $\cup_{\pi \in S_{n}} \mathbb{I}_{\pi}^{n}$ with the usual assumptions, we immediately obtain the formula

$$
D_{(k)} f\left(g_{1}(\mathbf{x}), \ldots, g_{n}(\mathbf{x})\right)=\sum_{i=1}^{n}\left(D_{i} f\right)\left(g_{1}(\mathbf{x}), \ldots, g_{n}(\mathbf{x})\right) D_{(k)} g_{i}(\mathbf{x})
$$

Now, for every $k \in[n]$, consider the function $h_{k} \in L^{2}\left(\mathbb{I}^{n}\right)$ defined as

$$
h_{k}=(n+1)(n+2)\left(\mathrm{os}_{k+1}-\mathrm{os}_{k}\right)\left(\mathrm{os}_{k}-\mathrm{os}_{k-1}\right)
$$

It is immediate to see that $h_{k}$ is nonnegative and continuous and that $D_{(k)} h_{k}=-g_{k}$, where $g_{k}$ is defined in (8). Moreover, using (5) or (6), we easily see that $h_{k}$ is a probability density function on $\mathbb{I}^{n}$. This fact can also be derived by choosing $f=\operatorname{os}_{k}$ in the following result.

Proposition 7. For every $k \in[n]$ and every $f \in L^{2}\left(\mathbb{I}^{n}\right)$ such that $D_{(k)} f$ is continuous and integrable on $\cup_{\pi \in S_{n}} \mathbb{I}_{\pi}^{n}$, we have

$$
\begin{equation*}
I(f, k)=\int_{\mathbb{I}^{n}} h_{k}(\mathbf{x}) D_{(k)} f(\mathbf{x}) d \mathbf{x} \tag{12}
\end{equation*}
$$

Proof. Fix $k \in[n]$. Using the product rule, we obtain

$$
h_{k}(\mathbf{x}) D_{(k)} f(\mathbf{x})=D_{(k)}\left(h_{k}(\mathbf{x}) f(\mathbf{x})\right)+g_{k}(\mathbf{x}) f(\mathbf{x}),
$$

and hence we only need to show that

$$
\begin{equation*}
\int_{\mathbb{I}^{n}} D_{(k)}\left(h_{k}(\mathbf{x}) f(\mathbf{x})\right) d \mathbf{x}=0 \tag{13}
\end{equation*}
$$

But the left-hand side of (13) can be rewritten as

$$
\begin{aligned}
& \sum_{\pi \in S_{n}} \int_{\mathbb{I}_{\pi}^{n}} D_{\pi(k)}\left(h_{k}(\mathbf{x}) f(\mathbf{x})\right) d \mathbf{x} \\
& =\sum_{\pi \in S_{n}} \int_{0}^{1} \int_{0}^{x_{\pi(n)}} \cdots \int_{0}^{x_{\pi(3)}} \int_{0}^{x_{\pi(2)}} D_{\pi(k)}\left(h_{k}(\mathbf{x}) f(\mathbf{x})\right) d x_{\pi(1)} d x_{\pi(2)} \cdots d x_{\pi(n)},
\end{aligned}
$$

that is, if we permute the integrals so that we integrate first with respect to $x_{\pi(k)}$,

$$
\sum_{\pi \in S_{n}} \int_{0}^{1} \cdots \int_{x_{\pi(k-1)}}^{x_{\pi(k+1)}} D_{\pi(k)}\left(h_{k}(\mathbf{x}) f(\mathbf{x})\right) d x_{\pi(k)} \cdots d x_{\pi(1)}
$$

which is zero since so is the inner integral.
Remark 4. Under the assumptions of Proposition 7, if $D_{(k)} f=0$ (resp. $\geqslant 0, \leqslant 0$ ) almost everywhere, then $I(f, k)=0$ (resp. $\geqslant 0, \leqslant 0$ ).

We now give an alternative interpretation of $I(f, k)$ as an expected value, which does not require the additional assumptions of Proposition 7. In this more general framework, we naturally replace the derivative with a difference quotient. To this extent, we introduce some further notation. As usual, we denote by $\mathbf{e}_{i}$ the $i$ th vector of the standard basis for $\mathbb{R}^{n}$. For every $k \in[n]$ and every $h \in[0,1]$, we define the $(k)$-difference (or discrete $(k)$-derivative) operator $\Delta_{(k), h}$ over the set of real functions on $\mathbb{I}^{n}$ by

$$
\Delta_{(k), h} f(\mathbf{x})=f\left(\mathbf{x}+h \mathbf{e}_{\pi(k)}\right)-f(\mathbf{x})
$$

for every $\mathbf{x} \in \mathbb{I}_{\pi}^{n}$ such that $\mathbf{x}+h \mathbf{e}_{\pi(k)} \in \mathbb{I}_{\pi}^{n}$. Thus defined, the value $\Delta_{(k), h} f(\mathbf{x})$ can be interpreted as the marginal contribution of $x_{(k)}$ on $f$ at $\mathbf{x}$ with respect to the increase $h$. For instance, we have $\Delta_{(k), h} x_{(k)}=h$.

Similarly, we define the $(k)$-difference quotient operator $Q_{(k), h}$ over the set of real functions on $\mathbb{I}^{n}$ by $Q_{(k), h} f(\mathbf{x})=\frac{1}{h} \Delta_{(k), h} f(\mathbf{x})$.
Theorem 8. For every $k \in[n]$ and every $f \in L^{2}\left(\mathbb{I}^{n}\right)$, we have

$$
\begin{equation*}
I(f, k)=(n+1)(n+2) \int_{\mathbb{I}^{n}} \int_{x_{(k)}}^{x_{(k+1)}} \Delta_{(k), y-x_{(k)}} f(\mathbf{x}) d y d \mathbf{x} . \tag{14}
\end{equation*}
$$

Proof. The right-hand side of (14) can be rewritten as

$$
\begin{equation*}
(n+1)(n+2) \sum_{\pi \in S_{n}} \int_{\mathbb{I}_{\pi}^{n}} \int_{x_{\pi(k)}}^{x_{\pi(k+1)}}\left(f\left(\mathbf{x}+\left(y-x_{\pi(k)}\right) \mathbf{e}_{\pi(k)}\right)-f(\mathbf{x})\right) d y d \mathbf{x} \tag{15}
\end{equation*}
$$

On the one hand, we have

$$
\begin{equation*}
\int_{\mathbb{I n}_{\pi}^{n}} \int_{x_{\pi(k)}}^{x_{\pi(k+1)}} f(\mathbf{x}) d y d \mathbf{x}=\int_{\mathbb{I}_{\pi}^{n}}\left(x_{\pi(k+1)}-x_{\pi(k)}\right) f(\mathbf{x}) d \mathbf{x} \tag{16}
\end{equation*}
$$

On the other hand, by permuting the integrals exactly as in the proof of Proposition 7 we obtain

$$
\begin{aligned}
\int_{\mathbb{I}_{\pi}^{n}} \int_{x_{\pi(k)}}^{x_{\pi(k+1)}} & f\left(\mathbf{x}+\left(y-x_{\pi(k)}\right) \mathbf{e}_{\pi(k)}\right) d y d \mathbf{x} \\
\quad= & \int_{0}^{1} \cdots \int_{x_{\pi(k-1)}}^{x_{\pi(k+1)}} \int_{x_{\pi(k)}}^{x_{\pi(k+1)}} f\left(\mathbf{x}+\left(y-x_{\pi(k)}\right) \mathbf{e}_{\pi(k)}\right) d y d x_{\pi(k)} \cdots d x_{\pi(1)}
\end{aligned}
$$

which, by permuting the two inner integrals, becomes

$$
\begin{aligned}
& \int_{0}^{1} \cdots \int_{x_{\pi(k-1)}}^{x_{\pi(k+1)}} \int_{x_{\pi(k-1)}}^{y} f\left(\mathbf{x}+\left(y-x_{\pi(k)}\right) \mathbf{e}_{\pi(k)}\right) d x_{\pi(k)} d y \cdots d x_{\pi(1)} \\
& \quad=\int_{0}^{1} \cdots \int_{x_{\pi(k-1)}}^{x_{\pi(k+1)}}\left(y-x_{\pi(k-1)}\right) f\left(\mathbf{x}+\left(y-x_{\pi(k)}\right) \mathbf{e}_{\pi(k)}\right) d y \cdots d x_{\pi(1)}
\end{aligned}
$$

By renaming $y$ as $x_{\pi(k)}$, we finally obtain

$$
\begin{equation*}
\int_{\mathbb{I}_{\pi}^{n}} \int_{x_{\pi(k)}}^{x_{\pi(k+1)}} f\left(\mathbf{x}+\left(y-x_{\pi(k)}\right) \mathbf{e}_{\pi(k)}\right) d y d \mathbf{x}=\int_{\mathbb{I}_{\pi}^{n}}\left(x_{\pi(k)}-x_{\pi(k-1)}\right) f(\mathbf{x}) d \mathbf{x} \tag{17}
\end{equation*}
$$

By substituting (16) and (17) in (15), we finally obtain $I(f, k)$.
As an immediate consequence of Theorem8, we have the following interpretation of the index $I(f, k)$ as an expected value of a difference quotient with respect to some distribution.

Corollary 9. For every $k \in[n]$ and every $f \in L^{2}\left(\mathbb{I}^{n}\right)$, we have

$$
I(f, k)=\int_{\mathbb{I}^{n}} \int_{x_{(k)}}^{x_{(k+1)}} p_{k}(\mathbf{x}, y) Q_{(k), y-x_{(k)}} f(\mathbf{x}) d y d \mathbf{x}
$$

where $p_{k}(\mathbf{x}, y)=(n+1)(n+2)\left(y-x_{(k)}\right)$ defines a probability density function on the set $\left\{(\mathbf{x}, y): \mathbf{x} \in \mathbb{I}^{n}, y \in\left[x_{(k)}, x_{(k+1)}\right]\right\}$.

Another important feature of the index is its invariance under the action of permutations. Recall that a permutation $\pi \in S_{n}$ acts on a function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ by $\pi(f)\left(x_{1}, \ldots, x_{n}\right)=f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)$. By the change of variables theorem, we immediately see that every $\pi \in S_{n}$ is an isometry of $L^{2}\left(\mathbb{I}^{n}\right)$, that is, $\langle\pi(f), \pi(g)\rangle=$ $\langle f, g\rangle$. From this fact, we derive the following result.
Proposition 10. For every $f \in L^{2}\left(\mathbb{I}^{n}\right)$ and every $\pi \in S_{n}$, both functions $f$ and $\pi(f)$ have the same best shifted L-statistic approximation $f_{L}$. Moreover, we have $\left\|\pi(f)-f_{L}\right\|=\left\|f-f_{L}\right\|$.
Proof. Let $f \in L^{2}\left(\mathbb{I}^{n}\right), g \in V_{L}$, and $\pi \in S_{n}$. Since $\pi$ is an isometry of $L^{2}\left(\mathbb{I}^{n}\right)$ and $g$ is symmetric, by (3) we have $\langle\pi(f), g\rangle=\langle f, g\rangle=\left\langle f_{L}, g\right\rangle$, which shows that $\pi(f)_{L}=f_{L}$. Using similar arguments, we obtain

$$
\begin{aligned}
\left\|\pi(f)-f_{L}\right\|^{2} & =\left\langle\pi(f)-f_{L}, \pi(f)-f_{L}\right\rangle=\left\langle f-f_{L}, f-f_{L}\right\rangle \\
& =\left\|f-f_{L}\right\|^{2}
\end{aligned}
$$

which completes the proof.
With any function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ we can associate the following symmetric function

$$
\operatorname{Sym}(f)=\frac{1}{n!} \sum_{\pi \in S_{n}} \pi(f)
$$

It follows immediately from Propositions 5 and 10 that both functions $f$ and $\operatorname{Sym}(f)$ have the same best shifted $L$-statistic approximation $f_{L}$. Combining this observation with Proposition 10 we derive immediately the following corollary.

Corollary 11. For every $k \in[n]$, every $f \in L^{2}\left(\mathbb{I}^{n}\right)$, and every $\pi \in S_{n}$, we have $I(f, k)=I(\pi(f), k)=I(\operatorname{Sym}(f), k)$.
Remark 5. Corollary 11 shows that, to compute $I(f, k)$, we can replace $f$ with $\operatorname{Sym}(f)$. For instance, if $f(\mathbf{x})=x_{i}$ for some $i \in[n]$ then $\operatorname{Sym}(f)=\frac{1}{n} \sum_{i=1}^{n} x_{i}=$ $\frac{1}{n} \sum_{i=1}^{n} x_{(i)}$ and hence, using Proposition 7, we obtain $I(f, k)=\frac{1}{n}$.

We say that two functions $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ and $g: \mathbb{I}^{n} \rightarrow \mathbb{R}$ are symmetrically equivalent (and we write $f \sim g$ ) if $\operatorname{Sym}(f)=\operatorname{Sym}(g)$. By Corollary 11, for any $f, g \in L^{2}\left(\mathbb{I}^{n}\right)$ such that $f \sim g$, we have $I(f, k)=I(g, k)$.

We end this section by analyzing the behavior of the influence index $I(f, k)$ on some special classes of functions.

Given $k \in[n]$, we say that the order statistic $x_{(k)}$ is ineffective almost everywhere for a function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ if $\Delta_{(k), y-x_{(k)}} f(\mathbf{x})=0$ for almost all $\mathbf{x} \in \cup_{\pi \in S_{n}} \mathbb{I}_{\pi}^{n}$ and almost all $y \in] x_{(k-1)}, x_{(k+1)}\left[\right.$. For instance, given unary functions $f_{1}, f_{2} \in L^{2}(\mathbb{I})$, the order statistic $x_{(1)}$ is ineffective almost everywhere for the function $f: \mathbb{I}^{2} \rightarrow \mathbb{R}$ such that

$$
f\left(x_{1}, x_{2}\right)= \begin{cases}f_{1}\left(x_{1}\right), & \text { if } x_{1}>x_{2} \\ f_{2}\left(x_{2}\right), & \text { if } x_{1}<x_{2}\end{cases}
$$

The following result immediately follows from Theorem 8 ,
Proposition 12. Let $k \in[n]$ and $f \in L^{2}\left(\mathbb{I}^{n}\right)$. If $x_{(k)}$ is ineffective almost everywhere for $f$, then $I(f, k)=0$.

The dual of a function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ is the function $f^{d}: \mathbb{I}^{n} \rightarrow \mathbb{R}$ defined by $f^{d}(\mathbf{x})=$ $1-f\left(\mathbf{1}_{[n]}-\mathbf{x}\right)$. A function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ is said to be self-dual if $f^{d}=f$. By using the change of variables theorem, we immediately derive the following result.

Proposition 13. For every $f \in L^{2}\left(\mathbb{I}^{n}\right)$ and every $k \in[n]$, we have $I\left(f^{d}, k\right)=$ $I(f, n-k+1)$. In particular, if $f$ is self-dual, then $I(f, k)=I(f, n-k+1)$.

## 4. Alternative expressions for the index

The computation of the index $I(f, k)$ by means of (10) or (12) might be not very convenient due to the presence of the order statistic functions. To make those integrals either more tractable or easier to evaluate numerically, we provide in this section some alternative expressions for the index $I(f, k)$ that do not involve any order statistic.

We first derive useful formulas for the computation of the integral $\left\langle f, \operatorname{os}_{k}\right\rangle$ (Proposition (17). To this extent, we consider the following direct generalization of order statistic functions.

Definition 14. For every nonempty $S=\left\{i_{1}, \ldots, i_{s}\right\} \subseteq[n], s=|S|$, and every $k \in[s]$, we define the function $\mathrm{os}_{k: S}: \mathbb{I}^{n} \rightarrow \mathbb{R}$ as $\mathrm{os}_{k: S}(\mathbf{x})=\mathrm{os}_{k: s}\left(x_{i_{1}}, \ldots, x_{i_{s}}\right)$.

To simplify the notation, we will write $x_{k: S}$ for $\operatorname{os}_{k: S}(\mathbf{x})$. Thus $x_{k: S}$ is the $k$ th order statistic of the variables in $S$.

Lemma 15. For every $s \in[n]$ and every $k \in[s]$, we have

$$
\begin{equation*}
\sum_{\substack{S \subseteq[n] \\|S|=s}} x_{k: S}=\sum_{j=k}^{n}\binom{j-1}{k-1}\binom{n-j}{s-k} x_{j: n} \tag{18}
\end{equation*}
$$

Proof. Since both sides of (18) are symmetric and continuous functions on $\mathbb{I}^{n}$, we can assume $x_{1}<\cdots<x_{n}$. Then, for every $j \in[n]$, we have $x_{k: S}=x_{j}$ if and only if $S \ni j$ and $|S \cap[j-1]|=k-1$. The result then follows by counting those sets $S$ of cardinality $s$ and having these two properties.

Lemma 16. For every $k \in[n]$, we have

$$
\begin{align*}
& x_{k: n}=\sum_{\substack{S \subseteq[n] \\
|S| \geqslant k}}(-1)^{|S|-k}\binom{|S|-1}{k-1} x_{|S|: S}  \tag{19}\\
& x_{k: n}=\sum_{\substack{S \subseteq[n] \\
|S| \geqslant n-k+1}}(-1)^{|S|-n+k-1}\binom{|S|-1}{n-k} x_{1: S} \tag{20}
\end{align*}
$$

Proof. By using (18), we can rewrite the right-hand side of (19) as

$$
\begin{aligned}
\sum_{s=k}^{n}(-1)^{s-k}\binom{s-1}{k-1} \sum_{\substack{S \subseteq[n] \\
|S|=s}} x_{s: S} & =\sum_{s=k}^{n}(-1)^{s-k}\binom{s-1}{k-1} \sum_{j=s}^{n}\binom{j-1}{s-1} x_{j: n} \\
& =\sum_{j=k}^{n}\binom{j-1}{k-1} x_{j: n} \sum_{s=k}^{j}(-1)^{s-k}\binom{j-k}{s-k}
\end{aligned}
$$

where the inner sum equals $(1-1)^{j-k}$. This proves (19). Identity (20) then follows by dualization.

We now provide four formulas for the computation of the integral $\int_{\mathbb{I}^{n}} f(\mathbf{x}) x_{(k)} d \mathbf{x}$. From these formulas we will easily derive alternative expressions for the index $I(f, k)$.

Proposition 17. For every function $f \in L^{2}\left(\mathbb{I}^{n}\right)$ and every $k \in[n]$, the integral $J_{k: n}=\int_{\mathbb{I}^{n}} f(\mathbf{x}) x_{(k)} d \mathbf{x}$ is given by each of the following expressions:

$$
\begin{gather*}
\int_{\mathbb{I}^{n}} f(\mathbf{x}) d \mathbf{x}-\sum_{S \subseteq[n]:|S| \geqslant k}(-1)^{|S|-k}\binom{|S|-1}{k-1} \int_{0}^{1} \int_{[0, y]^{S}} \int_{[0,1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y  \tag{21}\\
\sum_{S \subseteq[n]:|S| \geqslant n-k+1}(-1)^{|S|-n+k-1}\binom{|S|-1}{n-k} \int_{0}^{1} \int_{[y, 1]^{S}} \int_{[0,1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y  \tag{22}\\
\int_{\mathbb{I}^{n}} f(\mathbf{x}) d \mathbf{x}-\sum_{S \subseteq[n]:|S| \geqslant k} \int_{0}^{1} \int_{[0, y]^{S}} \int_{[y, 1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y  \tag{23}\\
\sum_{S \subseteq[n]:|S|<k} \int_{0}^{1} \int_{[0, y]^{S}} \int_{[y, 1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y \tag{24}
\end{gather*}
$$

Proof. By linearity of the integrals, we can assume that $f$ has nonnegative values. Then, we define the measure $\mu_{f}$ as $\mu_{f}(A)=\int_{A} f(\mathbf{x}) d \mathbf{x}$ for every Borel subset $A$ of $\mathbb{I}^{n}$. To compute integral $J_{k: n}$, we can use Lemma 16 and compute only the integrals $J_{|S|: S}=\int_{\mathbb{I}^{n}} f(\mathbf{x}) \vee_{i \in S} x_{i} d \mathbf{x}$ and $J_{1: S}=\int_{\mathbb{I}^{n}} f(\mathbf{x}) \wedge_{i \in S} x_{i} d \mathbf{x}$. To this extent, we define

$$
F_{|S|: S}(y)=\mu_{f}\left(\left\{\mathbf{x} \in \mathbb{I}^{n}: \vee_{i \in S} x_{i} \leqslant y\right\}\right)=\int_{[0, y]^{S}} \int_{[0,1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x}
$$

and

$$
\begin{aligned}
F_{1: S}(y) & =\mu_{f}\left(\left\{\mathbf{x} \in \mathbb{I}^{n}: \wedge_{i \in S} x_{i} \leqslant y\right\}\right)=\mu_{f}\left(\mathbb{I}^{n}\right)-\mu_{f}\left(\left\{\mathbf{x} \in \mathbb{I}^{n}: \wedge_{i \in S} x_{i}>y\right\}\right) \\
& =\int_{\mathbb{I}^{n}} f(\mathbf{x}) d \mathbf{x}-\int_{[y, 1]^{S}} \int_{[0,1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x}
\end{aligned}
$$

We then have

$$
J_{|S|: S}=\int_{\mathbb{I}^{n}} \vee_{i \in S} x_{i} d \mu_{f}=\int_{0}^{1} y d F_{|S|: S}(y)=\lim _{y \rightarrow 1} F_{|S|: S}(y)-\int_{0}^{1} F_{|S|: S}(y) d y
$$

and similarly for $J_{1: S}$. This proves (21) and (22).
We prove (23) similarly by considering

$$
\begin{aligned}
F_{k: n}(y) & =\mu_{f}\left(\left\{\mathbf{x} \in \mathbb{I}^{n}: x_{(k)} \leqslant y\right\}\right) \\
& =\mu_{f}\left(\cup_{|S| \geqslant k}\left\{\mathbf{x} \in \mathbb{I}^{n}: \vee_{i \in S} x_{i} \leqslant y<\wedge_{i \in[n] \backslash S} x_{i}\right\}\right) \\
& =\sum_{|S| \geqslant k} \int_{0}^{1} \int_{[0, y]^{S}} \int_{[y, 1][n] \backslash S} f(\mathbf{x}) d \mathbf{x} d y
\end{aligned}
$$

Finally, we prove (24) by observing that

$$
\begin{aligned}
& \int_{\mathbb{I}^{n}} f(\mathbf{x}) d \mathbf{x}=\mu_{f}\left(\mathbb{I}^{n}\right)=\mu_{f}\left(\cup_{|S| \geqslant 0}\left\{\mathbf{x} \in \mathbb{I}^{n}: \vee_{i \in S} x_{i} \leqslant y<\wedge_{i \in[n] \backslash S} x_{i}\right\}\right) \\
&=\sum_{|S| \geqslant 0} \int_{0}^{1} \int_{[0, y]^{S}} \int_{[y, 1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y
\end{aligned}
$$

Remark 6. In the special case when $f$ is a probability density function on $\mathbb{I}^{n}$, the integral $\int_{\mathbb{I}^{n}} f(\mathbf{x}) x_{(k)} d \mathbf{x}$ is precisely the expected value $E_{f}\left(X_{k: n}\right)$, which is well investigated in statistics (see [1, 4]).

From Definition 3 and Proposition 17 we derive the following three formulas. The computations are straightforward and thus omitted.

$$
\begin{aligned}
& \left(25 \frac{I(f, k)}{n+1)(n+2)}=\sum_{S \subseteq[n]:|S| \geqslant k-1}(-1)^{|S|+1-k}\binom{|S|+1}{k} \int_{0}^{1} \int_{[0, y]^{S}} \int_{[0,1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y\right. \\
& \left(\frac{26}{} \frac{(f, k)}{n+1)(n+2)}=\sum_{S \subseteq[n]:|S| \geqslant n-k}(-1)^{|S|-n+k-1}\binom{|S|+1}{n-k+1} \int_{0}^{1} \int_{[y, 1]^{S}} \int_{[0,1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y\right. \\
& \left(27 \frac{I(f, k)}{(n+1)(n+2)}=\left(\sum_{S \subseteq[n]:|S|=k-1}-\sum_{S \subseteq[n]:|S|=k}\right) \int_{0}^{1} \int_{[0, y]^{S}} \int_{[y, 1]^{[n] \backslash S}} f(\mathbf{x}) d \mathbf{x} d y\right.
\end{aligned}
$$

## 5. Some examples

We now apply our results to two special classes of functions, namely the multiplicative functions and the Lovász extensions of pseudo-Boolean functions. The latter class includes the so-called discrete Choquet integrals, well-known in aggregation function theory.
5.1. Multiplicative functions. Consider the function $f(\mathbf{x})=\prod_{i=1}^{n} \varphi_{i}\left(x_{i}\right)$, where $\varphi_{i} \in L^{2}(\mathbb{I})$, and set $\Phi_{i}(x)=\int_{0}^{x} \varphi_{i}(t) d t$ for $i=1, \ldots, n$. By using (25), we obtain

$$
\begin{equation*}
\frac{I(f, k)}{(n+1)(n+2)}=\sum_{\substack{S \subseteq[n] \\|S| \geqslant k-1}}(-1)^{|S|+1-k}\binom{|S|+1}{k} \prod_{i \in[n] \backslash S} \Phi_{i}(1) \int_{0}^{1} \prod_{i \in S} \Phi_{i}(y) d y \tag{28}
\end{equation*}
$$

The following result gives a concise expression for $I(f, k)$ when $f$ is symmetric.
Proposition 18. Let $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ be given by $f(\mathbf{x})=\prod_{i=1}^{n} \varphi\left(x_{i}\right)$, where $\varphi \in L^{2}(\mathbb{I})$, and let $\Phi(x)=\int_{0}^{x} \varphi(t) d t$. Then, for every $k \in[n]$, we have

$$
I(f, k)= \begin{cases}\left.\Phi(1)^{n} \int_{0}^{1} D_{z} h(z ; k+1, n-k+2)\right|_{z=\Phi(y) / \Phi(1)} d y, & \text { if } \Phi(1) \neq 0 \\ (-1)^{n-k+1}(n+1) \frac{\Gamma(n+3)}{\Gamma(k+1) \Gamma(n-k+2)} \int_{0}^{1} \Phi(y)^{n} d y, & \text { if } \Phi(1)=0\end{cases}
$$

where $h(z ; a, b)=z^{a-1}(1-z)^{b-1} / B(a, b)$ is the probability density function of the beta distribution with parameters $a$ and $b$.
Proof. Suppose that $\Phi(1) \neq 0$. By using (27), we obtain

$$
\begin{aligned}
& \frac{I(f, k)}{(n+1)(n+2)}=\left(\sum_{\substack{S \subseteq[n] \\
|S|=k-1}}-\sum_{\substack{S \subseteq[n] \\
|S|=k}}\right) \int_{0}^{1} \Phi(y)^{|S|}(\Phi(1)-\Phi(y))^{n-|S|} d y \\
& =\left.\Phi(1)^{n} \int_{0}^{1}\left(\binom{n}{k-1} z^{k-1}(1-z)^{n-k+1}-\binom{n}{k} z^{k}(1-z)^{n-k}\right)\right|_{z=\Phi(y) / \Phi(1)} d y
\end{aligned}
$$

which proves the result. The case $\Phi(1)=0$ follows from (28).

Example 19. Let $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ be given by $f(\mathbf{x})=\left(\prod_{i=1}^{n} x_{i}\right)^{c}$, where $c>-\frac{1}{2}$. For instance, the product function corresponds to $c=1$ and the geometric mean function to $c=1 / n$. We can calculate $I(f, k)$ by using Proposition 18 with $\varphi(x)=$ $x^{c}$. Using the substitution $z=y^{c+1}$ and then integrating by parts, we obtain

$$
I(f, k)=c\left(\frac{1}{c+1}\right)^{n+2} \frac{\Gamma(n+3) \Gamma\left(k-1+\frac{1}{c+1}\right)}{\Gamma(k+1) \Gamma\left(n+1+\frac{1}{c+1}\right)}=\frac{\Gamma\left(k-1+\frac{1}{c+1}\right)}{\Gamma(k+1) \Gamma\left(\frac{1}{c+1}\right)} I(f, 1),
$$

with

$$
I(f, 1)=c\left(\frac{1}{c+1}\right)^{n+2} \frac{\Gamma(n+3) \Gamma\left(\frac{1}{c+1}\right)}{\Gamma\left(n+1+\frac{1}{c+1}\right)}
$$

We observe that $I(f, k) \rightarrow I(f, 1)$ as $c \rightarrow-\frac{1}{2}$. Also, for $c>0$, we have $I(f, k+1)<$ $I(f, k)$ for every $k \in[n-1]$. As expected in this case, the smallest variables are more influent on $f$ than the largest ones.
5.2. Lovász extensions. Recall that an $n$-place (lattice) term function $p: \mathbb{I}^{n} \rightarrow \mathbb{I}$ is a combination of projections $\mathbf{x} \mapsto x_{i}(i \in[n])$ using the fundamental lattice operations $\wedge$ and $\vee$; see [2. For instance,

$$
p\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1} \wedge x_{2}\right) \vee x_{3}
$$

is a 3 -place term function. Note that, since $\mathbb{I}$ is a bounded chain, here the lattice operations $\wedge$ and $\vee$ reduce to the minimum and maximum functions, respectively.

Clearly, any shifted linear combination of $n$-place term functions

$$
f(\mathbf{x})=c_{0}+\sum_{i=1}^{m} c_{i} p_{i}(\mathbf{x})
$$

is a continuous function whose restriction to any standard simplex $\mathbb{I}_{\pi}^{n}\left(\pi \in S_{n}\right)$ is a shifted linear function. According to Singer [11, §2], $f$ is then the Lovász extension of the pseudo-Boolean function $\left.f\right|_{\{0,1\}^{n}}$, that is, the continuous function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ which is defined on each standard simplex $\mathbb{I}_{\pi}^{n}$ as the unique affine function that coincides with $\left.f\right|_{\{0,1\}^{n}}$ at the $n+1$ vertices of $\mathbb{I}_{\pi}^{n}$. Singer showed that a Lovász extension can always be written as

$$
\begin{equation*}
f(\mathbf{x})=f_{n+1}^{\pi}+\sum_{i=1}^{n}\left(f_{i}^{\pi}-f_{i+1}^{\pi}\right) x_{\pi(i)} \quad\left(\mathbf{x} \in \mathbb{I}_{\pi}^{n}\right) \tag{29}
\end{equation*}
$$

with $f_{i}^{\pi}=f\left(\mathbf{1}_{\{\pi(i), \ldots, \pi(n)\}}\right)=v_{f}(\{\pi(i), \ldots, \pi(n)\})$ for $i \in[n+1]$, where the set function $v_{f}: 2^{[n]} \rightarrow \mathbb{R}$ is defined as $v_{f}(S)=f\left(\mathbf{1}_{S}\right)$. In particular, $f_{n+1}^{\pi}=c_{0}=f(\mathbf{0})$. Conversely, any continuous function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ that reduces to an affine function on each standard simplex is a shifted linear combination of term functions:

$$
\begin{equation*}
f(\mathbf{x})=\sum_{S \subseteq[n]} m_{f}(S) x_{1: S} \tag{30}
\end{equation*}
$$

where $m_{f}: 2^{[n]} \rightarrow \mathbb{R}$ is the Möbius transform of $v_{f}$, defined as

$$
m_{f}(S)=\sum_{T \subseteq S}(-1)^{|S|-|T|} v_{f}(T)
$$

Indeed, expression (30) reduces to an affine function on each standard simplex and agrees with $f\left(\mathbf{1}_{S}\right)$ at $\mathbf{1}_{S}$ for every $S \subseteq[n]$. Thus the class of shifted linear combinations of $n$-place term functions is precisely the class of $n$-place Lovász extensions.

Remark 7. A nondecreasing Lovász extension $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ such that $f(\mathbf{0})=0$ is also called a discrete Choquet integral. For general background, see for instance [5].

For every nonempty $S \subseteq[n]$ and every $k \in[|S|]$, the function $o_{k: S}$ is a Lovász extension and, from (19), we have

$$
x_{k: S}=\sum_{\substack{T \subseteq S \\|T| \geqslant k}}(-1)^{|T|-k}\binom{|T|-1}{k-1} x_{|T|: T}
$$

The following proposition gives a concise expression for the index $I\left(\operatorname{os}_{j: S}, k\right)$. We first consider a lemma.

Lemma 20. For every nonempty $S \subseteq[n]$ and every $j \in[|S|]$, we have

$$
\operatorname{Sym}\left(\mathrm{os}_{j: S}\right)=\frac{1}{\binom{n}{|S|}} \sum_{\substack{T \subseteq[n] \\|T|=|S|}} \mathrm{os}_{j: T}
$$

Proof. It is easy to see that $\operatorname{Sym}\left(\mathrm{os}_{j: S}\right)(\mathbf{x})=\frac{1}{n!} \sum_{\pi \in S_{n}} \mathrm{os}_{j: \pi(S)}(\mathbf{x})$. This proves the result for there are exactly $|S|!(n-|S|)$ ! permutations that map $S$ to a given set $T \subseteq[n]$ such that $|T|=|S|$.

Proposition 21. For every nonempty $S \subseteq[n]$, every $j \in[|S|]$, and every $k \in[n]$, we have

$$
\begin{equation*}
I\left(\operatorname{os}_{j: S}, k\right)=\frac{\binom{k-1}{j-1}\binom{n-k}{|S|-j}}{\binom{n}{|S|}} \tag{31}
\end{equation*}
$$

if $0 \leqslant k-j \leqslant n-|S|$, and 0 , otherwise.
Proof. The result follows from Corollary 11, Lemma 15, and Lemma $20{ }^{2}$
The following proposition gives an explicit expression for the index $I(f, k)$ when $f$ is a Lovász extension.

Proposition 22. If $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ is a Lovász extension, then

$$
\begin{equation*}
f(\mathbf{x})=f(\mathbf{0})+\sum_{i=1}^{n} x_{(i)} D_{(i)} f(\mathbf{x}) \tag{32}
\end{equation*}
$$

Moreover, for every $k \in[n]$, we have

$$
\begin{equation*}
I(f, k)=\bar{v}_{f}(n-k+1)-\bar{v}_{f}(n-k)=\sum_{s=1}^{n-k+1}\binom{n-k}{s-1} \bar{m}_{f}(s) \tag{33}
\end{equation*}
$$

where $\bar{v}_{f}(s)=\binom{n}{s}^{-1} \sum_{S \subseteq[n]:|S|=s} v_{f}(S)$ and $\bar{m}_{f}(s)=\binom{n}{s}^{-1} \sum_{S \subseteq[n]:|S|=s} m_{f}(S)$.
Proof. Let $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ be a Lovász extension and let $k \in[n]$. For every $\pi \in S_{n}$ and every $\mathbf{x} \in \mathbb{I}_{\pi}^{n}$, from (29) if follows that $D_{\pi(k)} f(\mathbf{x})=f_{k}^{\pi}-f_{k+1}^{\pi}$. This estalishes (32).

[^1]By Proposition 7, we then obtain

$$
\begin{aligned}
& I(f, k)=\sum_{\pi \in S_{n}} \int_{\mathbb{I}_{\pi}^{n}} h_{k}(\mathbf{x}) D_{\pi(k)} f(\mathbf{x}) d \mathbf{x} \\
& \quad=(n+1)(n+2) \sum_{\pi \in S_{n}}\left(f_{k}^{\pi}-f_{k+1}^{\pi}\right) \int_{\mathbb{I}_{\pi}^{n}}\left(x_{\pi(k+1)}-x_{\pi(k)}\right)\left(x_{\pi(k)}-x_{\pi(k-1)}\right) d \mathbf{x}
\end{aligned}
$$

Since the integral is equal to $1 /(n+2)$ !, we obtain $I(f, k)=\frac{1}{n!} \sum_{\pi \in S_{n}}\left(f_{k}^{\pi}-f_{k+1}^{\pi}\right)$, which, after some algebra, leads to the first equality in (33). Finally, by combining Proposition 50 with (30) and (31) (for $j=1$ ), we completely establish (33).

Remark 8. A function $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ solves equation (32) if and only if, for every $\pi \in S_{n}$, the function $\left.f\right|_{\mathbb{I}_{\pi}^{n}}-f(\mathbf{0})$ is an eigenfunction of the Euler operator with eigenvalue 1. Thus this function reduces to a homogeneous function of degree 1 whenever it is differentiable. Notice however that such a function need not be linear even if $f$ is continuous on $\mathbb{I}^{n}$. For instance, the geometric mean $f(\mathbf{x})=\prod_{i=1}^{n} x_{i}^{1 / n}$ is a continuous function solving (32).

We can readily see that the shifted $L$-statistic functions are precisely the symmetric Lovász extensions. From this observation we derive the following result.

Proposition 23. For any Lovász extension $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$, we have $f_{L}=\operatorname{Sym}(f)$ and

$$
\operatorname{Sym}(f)=f(\mathbf{0})+\sum_{i=1}^{n} I(f, i) \mathrm{os}_{i} .
$$

Proof. Let $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ be a Lovász extension. Then $\operatorname{Sym}(f)$ is a symmetric Lovász extension or, equivalently, a shifted $L$-statistic function. By Propositions 5 and 10, we have $f_{L}=\operatorname{Sym}(f)_{L}=\operatorname{Sym}(f)$. The result then follows since $f_{L}(\mathbf{0})=$ $\operatorname{Sym}(f)(\mathbf{0})=f(\mathbf{0})$.

## 6. Applications

We briefly discuss some applications of the influence index in aggregation theory and statistics. We also introduce a normalized version of the index as well as the coefficient of determination of the approximation problem.
6.1. Influence index in aggregation theory. Several indexes (such as interaction, tolerance, and dispersion indexes) have been proposed and investigated in aggregation theory to better understand the general behavior of aggregation functions with respect to their variables; see [5, Chap. 10]. These indexes enable one to classify the aggregation functions according to their behavioral properties. The index $I(f, k)$ can also be very informative and thus contribute to such a classification. As an example, we have computed this index for the arithmetic mean and geometric mean functions (see Remark 5and Example 19) and we can observe for instance that the smallest variable $x_{(1)}$ has a larger influence on the latter function.

Remark 9. Noteworthy aggregation functions are the so-called conjunctive aggregation functions, that is, nondecreasing functions $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ satisfying $0 \leqslant f(\mathbf{x}) \leqslant$ $x_{(1)}$; see [5, Chap. 3]. Although these functions are bounded from above by $x_{(1)}$,
the index $I(f, k)$ need not be maximum for $k=1$. For instance, for the binary conjunctive aggregation function

$$
f\left(x_{1}, x_{2}\right)= \begin{cases}0, & \text { if } x_{1} \vee x_{2}<\frac{3}{4} \\ x_{1} \wedge x_{2} \wedge \frac{1}{4}, & \text { otherwise }\end{cases}
$$

we have $I(f, 1)=\frac{17}{128}$ and $I(f, 2)=\frac{19}{64}$, and hence $I(f, 1)<I(f, 2)$.
In the framework of aggregation functions, it can be natural to consider and identify the functions $f \in L^{2}\left(\mathbb{I}^{n}\right)$ for which the order statistics are equally influent, that is, such that $I(f, k)=I(f, 1)$ for all $k \in[n]$. As far as the Lovász extensions are concerned, we have the following result, which can be easily derived from Proposition 22 and the immediate identities

$$
\bar{v}_{f}(s)=\sum_{t=0}^{s}\binom{s}{t} \bar{m}_{f}(t) \quad \text { and } \quad \bar{m}_{f}(s)=\sum_{t=0}^{s}(-1)^{s-t}\binom{s}{t} \bar{v}_{f}(t)
$$

Proposition 24. If $f: \mathbb{I}^{n} \rightarrow \mathbb{R}$ is a Lovász extension, then the following are equivalent.
(a) We have $I(f, k)=I(f, 1)$ for all $k \in[n]$.
(b) The sequence $\left(\bar{v}_{f}(s)\right)_{s=0}^{n}$ is in arithmetic progression.
(c) We have $\bar{m}_{f}(s)=0$ for $s=2, \ldots, n$.
6.2. Influence index in statistics. It can be informative to assess the influence of every order statistic on a given statistic to measure, e.g., its behavior with respect to the extreme values. From this information we can also approximate the given statistic by a shifted $L$-statistic. Of course, for $L$-statistics (such as Winsorized means, trimmed means, linearly weighted means, quasi-ranges, Gini's mean difference; see [4, $\S 6.3, \S 8.8, \S 9.4]$ ), the computation of the influence indexes is immediate. However, for some other statistics such as the central moments, the indexes can be computed via (25)-(27).
Example 25. The closest shifted $L$-statistic to the variance $\sigma^{2}=\frac{1}{n} \sum_{i=1}^{n}\left(X_{i}-\bar{X}\right)^{2}$ is given by

$$
\sigma_{L}^{2}=\frac{1-n^{2}}{12 n(n+3)}+\sum_{k=1}^{n} I\left(\sigma^{2}, k\right) X_{(k)}
$$

with $I\left(\sigma^{2}, k\right)=(n+2)(2 k-n-1) /\left(n^{2}(n+3)\right)$, which can be computed from (27) ${ }^{3}$ We then immediately see that the smallest and largest variables are the most influent.
6.3. Normalized index and coefficient of determination. Coming back to the interpretation of the influence index as a covariance (see $\S 3$ ), it is natural to consider the Pearson correlation coefficient instead of that covariance. In this respect, we note that $\sigma^{2}\left(g_{k}\right)=E\left(g_{k}^{2}\right)=I\left(g_{k}, k\right)=2(n+1)(n+2)$, where the latter inequality is immediate since $g_{k} \in V_{L}$.

Definition 26. The normalized influence index is the mapping

$$
r:\left\{f \in L^{2}\left(\mathbb{I}^{n}\right): f \text { is non constant }\right\} \times[n] \rightarrow \mathbb{R}
$$

[^2]defined by
$$
r(f, k)=\frac{I(f, k)}{\sigma(f) \sqrt{2(n+1)(n+2)}}
$$

From this definition it follows that $-1 \leqslant r(f, k) \leqslant 1$, where the bounds are tight. Moreover, this index remains unchanged under interval scale transformations, that is, $r(a f+b, k)=r(f, k)$ for all $a>0$ and $b \in \mathbb{R}$. Finally, we also have $r\left(f^{d}, k\right)=r(f, n-k+1)$.

The coefficient of determination of the best shifted $L$-statistic approximation of a non constant function $f \in L^{2}\left(\mathbb{I}^{n}\right)$ is defined by $R^{2}(f)=\sigma^{2}\left(f_{L}\right) / \sigma^{2}(f)$. We then have

$$
R^{2}(f)=\frac{1}{\sigma^{2}(f)} \sigma^{2}\left(\sum_{j=1}^{n+1} a_{j} x_{(j)}\right)=\frac{1}{\sigma^{2}(f)} \mathbf{a}^{T}\left(M-\mathbf{c c}^{T}\right) \mathbf{a}
$$

where $\mathbf{c}$ is the $(n+1)$ st column of $M$.

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    ${ }^{1}$ An alternative (but equivalent) definition of influence index was previously considered for Boolean functions in [7 and pseudo-Boolean functions in 8.

[^1]:    ${ }^{2}$ The right-hand side of (31) can also be viewed as the multivariate hypergeometric distribution $\binom{1}{1}\binom{k-1}{j-1}\binom{n-k}{|S|-j} /\binom{n}{|S|}$ obtained directly from the proof of Lemma 15

[^2]:    ${ }^{3}$ In terms of Gini's mean difference [4] §9.4], $G=\frac{2}{n(n-1)} \sum_{k=1}^{n}(2 k-n-1) X_{(k)}$, we simply obtain $\sigma_{L}^{2}=\frac{n-1}{12 n(n+3)}(6(n+2) G-(n+1))$.

