Preservation under Substructures modulo Bounded Cores

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Abstract

We investigate a model-theoretic property that generalizes the classical notion of "preservation under substructures". We call this property preservation under substructures modulo bounded cores, and present a syntactic characterization via Σ^0_2 sentences for properties of arbitrary structures definable by FO sentences. As a sharper characterization, we further show that the count of existential quantifiers in the Σ^0_2 sentence equals the size of the smallest bounded core. We also present our results on the sharper characterization for special fragments of FO and also over special classes of structures. We present a (not FO-definable) class of finite structures for which the sharper characterization fails, but for which the classical Łoś-Tarski preservation theorem holds. As a fallout of our studies, we obtain combinatorial proofs of the Łoś-Tarski theorem for some of the aforementioned cases.

Keywords: Model theory, First Order logic, Łoś-Tarski preservation theorem

1 Introduction

Preservation theorems have traditionally been an important area of study in model theory. These theorems provide syntactic characterizations of semantic properties that are preserved under model-theoretic operations. One of the earliest preservation theorems is the Łoś-Tarski theorem, which states that over arbitrary structures, a First Order (FO) sentence is preserved under taking substructures iff it is equivalent to a Π_1^0 sentence [5]. Subsequently many other preservation theorems were studied, e.g. preservation under unions of chains, homomorphisms, direct products, etc. With the advent of finite model theory, the question of whether these theorems hold over finite structures became interesting. It turned out that several preservation theorems fail in the finite [1, 7, 9]. This inspired research on preservation theorems over special classes of finite structures, e.g. those with bounded degree, bounded treewidth etc. These efforts eventually led to some preservation theorems being "recovered" [2, 3]. Among the theorems whose status over the class of all finite structures was open for long was

the homomorphism preservation theorem. This was recently resolved in [10], which showed that the theorem survives in the finite.

In this paper, we look at a generalization of the 'preservation under substructures' property that we call *preservation under substructures modulo bounded cores*. In Section 2, we show that for FO sentences, this property has a syntactic characterization in terms of Σ^0_2 sentences over arbitrary structures. As a sharper characterization, we state our result (but provide the proof later in Section 7) that for core sizes bounded by a number B, there is a syntactic characterization in terms of Σ^0_2 sentences that use atmost B existential quantifiers. In Section 3, we discuss how the notion of *relativization* can be used to prove the sharper characterization in special cases and also discuss its limitations. We present our studies for special classes of FO and over special classes of structures in Sections 4 and 5. As a fallout of our studies, we obtain combinatorial proofs of the classical Łoś-Tarski theorem for some of the aforesaid special cases, and also obtain semantic characterizations of natural subclasses of the Δ^0_2 fragment of FO. In Section 7, we provide the proof of the sharper characterization using tools from classical model theory and some notions that we define. We conclude with questions for future work in Section 8.

We assume that the reader is familiar with standard notation and terminology used in the syntax and semantics of FO (see [8]). A vocabulary τ is a set of predicate, function and constant symbols. In this paper, we will restrict ourselves to finite vocabularies only. A relational vocabulary has only predicate and constant symbols, and a purely relational vocabulary has only predicate symbols. We denote by $FO(\tau)$, the set of all FO formulae over vocabulary τ . A sequence (x_1,\ldots,x_k) of variables is denoted by \bar{x} . We will abbreviate a block of quantifiers of the form $Qx_1 \dots Qx_k$ by $Q\bar{x}$, where $Q \in \{\forall, \exists\}$. By Σ_k^0 (resp. Π_k^0), we mean FO sentences in Prenex Normal Form (PNF) over an arbitrary vocabulary, whose quantifier prefix begins with a \exists (resp. \forall) and consists of k-1 alternations of quantifiers. We use the standard notions of τ structures, substructures and extensions, as in [8]. Given τ -structures M and N, we denote by $M \subseteq N$ that M is a substructure of N (or N is an extension of M). Given M and a subset S (resp. a tuple \bar{a} of elements) of its universe, we denote by M(S)(resp. $M(\bar{a})$) the smallest substructure (under set inclusion ordering of the universe) of M containing S (resp. underlying set of \bar{a}) and call it the substructure of M induced by S (resp. underlying set of \bar{a}). Finally, by size of M, we mean the cardinality of its universe and denote it by |M|. As a final note of convention, whenever we talk of FO definability in the paper, we mean definability via FO sentences (as opposed to theories), unless stated otherwise.

2 Preservation under substructures modulo cores

We denote by \mathbb{PS} the collection of all classes of structures, in any vocabulary, which are closed under taking substructures. This includes classes which are not definable in any logic. We let PS denote the collection of FO definable classes in \mathbb{PS} . We identify classes in PS with their defining FO sentences and will henceforth treat PS as a set of sentences. We now consider a natural generalization of the \mathbb{PS} property. Our discussion will concern arbitrary (finite) vocabularies and arbitrary structures over them.

2.1 The case of finite cores

Definition 1 (*Preservation under substructures modulo finite cores*)

A class of structures S is said to be preserved under substructures modulo a finite core (denoted $S \in \mathbb{PSC}_f$), if for every structure $M \in S$, there exists a finite subset C of elements of M such that if $M_1 \subseteq M$ and M_1 contains C, then $M_1 \in S$. The set C is called a core of M w.r.t. S. If S is clear from context, we will call C as a core of M.

Note that any finite subset of the universe of M containing a core is also a core of M. Also, there can be multiple cores of M having the same size. A *minimal* core of M is a core, no subset of which is a core of M.

We will use \mathbb{PSC}_f to denote the collection of all classes preserved under substructures modulo a finite core. Similarly, we will use PSC_f to denote the collection of FO definable classes in \mathbb{PSC}_f . We identify classes in PSC_f with their defining FO sentences, and will henceforth treat PSC_f as a set of sentences.

Example 1: Let S be the class of all graphs containing cycles. For any graph in S, the vertices of any cycle is a core of the graph. Thus $S \in \mathbb{PSC}_f$.

Note that $\mathbb{PS} \subseteq \mathbb{PSC}_f$ since for any class in \mathbb{PS} and for any structure in the class, any element is a core. However it is easy to check that S in above example is not in \mathbb{PS} ; so \mathbb{PSC}_f strictly generalizes \mathbb{PS} . Further, the FO inexpressibility of S shows that \mathbb{PSC}_f contains classes not definable in FO.

Example 2: Consider $\phi = \exists x \forall y E(x,y)$. In any graph satisfying ϕ , any witness for x is a core of the graph. Thus $\phi \in PSC_f$. In fact, one can put a uniform bound of 1 on the minimal core size for all models of ϕ .

Again it is easy to see that $PS \subsetneq PSC_f$. Specifically, the sentence ϕ in Example 2 is not in PS. This is because a directed graph with exactly two nodes a and b, and having all directed edges except the self loop on a models ϕ but the subgraph induced by a does not model ϕ . Hence $PS \subsetneq PSC_f$. Extending the example above, one can show that for any sentence φ in Σ_2^0 , in any model of φ , any witness for the \exists quantifiers in φ forms a core of the model. Hence $\Sigma_2^0 \subseteq PSC_f$. In fact, for any sentence in Σ_2^0 , the number of \exists quantifiers serves as a uniform bound on the minimal core size for all models. Surprisingly, even for an arbitrary $\phi \in PSC_f$, it is possible to bound the minimal core size for all models!

Towards the result, we use the notions of *chain* and *union of chain* from the literature. The reader is referred to [5] for the definitions. We denote a chain as $M_1 \subseteq M_2 \subseteq \ldots$ and its union as $\bigcup_{i \geq 0} M_i$. We say that a sentence ϕ is *preserved under unions of chains* if for every chain of models of ϕ , the union of the chain is also a model of ϕ . We now recall the following characterization theorem from the '60s [5].

Theorem 1 (Chang-Łoś-Suszko) A sentence ϕ is preserved under unions of chains iff it is equivalent to a Π_2^0 sentence.

Now we have the following theorem.

Theorem 2 A sentence $\phi \in PSC_f$ iff ϕ is equivalent to a Σ_2^0 sentence.

Proof: We infer from Theorem 1 the following equivalences.

 ϕ is equivalent to a Σ_0^2 sentence iff

$$\neg \phi$$
 is equivalent to a Π_0^2 sentence iff $\forall M_1, M_2, \dots ((M_1 \subseteq M_2 \subseteq \dots) \land (M = \bigcup_{i \ge 1} M_i) \land \forall i (M_i \models \neg \phi)) \rightarrow M \models \neg \phi$ iff

$$\forall M_1, M_2, \dots ((M_1 \subseteq M_2 \subseteq \dots) \land (M = \bigcup_{i > 1} M_i) \land (M \models \phi)) \rightarrow \exists i (M_i \models \phi)$$

Assume $\phi \in PSC_f$. Suppose $M_1 \subseteq M_2 \subseteq \ldots$ is a chain, $M = \bigcup_{i>0} M_i$ and $M \models \phi$. Then, there exists a finite core C of M. For any $a \in C$, there exists an ordinal i_a s.t. $a \in M_{i_a}$ (else a would not be in the union M). Since C is finite, let $i = \max(i_a | a \in C)$. Since $i_a \leq i$, we have $M_{i_a} \subseteq M_i$; hence $a \in M_i$ for all $a \in C$. Thus M_i contains C. Since C is a core of M and $M_i \subseteq M$, $M_i \models \phi$ by definition of PSC_f . By the equivalences shown above, ϕ is equivalent to a Σ_2^0 sentence. We have seen earlier that $\Sigma_2^0 \subseteq PSC_f$.

Corollary 1 If $\phi \in PSC_f$, there exists $B \in \mathbb{N}$ such that every model of ϕ has a core of size atmost B.

Proof: Take B to be the number of \exists quantifiers in the equivalent Σ_2^0 sentence.

Given Corollary 1, it is natural to ask if B is computable. In this context, the following recent unpublished result by Rossman [11] is relevant. Let $|\phi|$ denote the size of ϕ .

Theorem 3 (Rossman) There is no recursive function $f: \mathbb{N} \to \mathbb{N}$ such that if $\phi \in PS$, then there is an equivalent Π_1^0 sentence of size at most $f(|\phi|)$. The result holds even for relational vocabularies and further even if PS is replaced with $PS \cap \Sigma_2^0$.

Corollary 2 *There is no recursive function* $f: \mathbb{N} \to \mathbb{N}$ *such that if* $\phi \in PS$ *, then there* is an equivalent Π_1^0 sentence with atmost $f(|\phi|)$ universal variables. The result holds even for relational vocabularies and further even if PS is replaced with $PS \cap \Sigma_2^0$.

Proof: Let $\varphi = \forall^n \bar{z} \psi(\bar{z})$ be a Π_1^0 sentence equivalent to ϕ where $n = f(|\phi|)$. Let k be the number of atomic formulae in ψ . Since ϕ and ψ have the same vocabulary, $k \in O(|\phi| \cdot n^{|\phi|})$. The size of the Disjunctive Normal Form of ψ is therefore bounded above by $O(k \cdot n \cdot 2^k)$. Hence $|\varphi|$ is a recursive function of $|\phi|$ if f is recursive.

Theorem 3 strengthens the non-elementary lower bound given in [6]. Corollary 2 gives us the following.

Lemma 1 There is no recursive function $f: \mathbb{N} \to \mathbb{N}$ s.t. if $\phi \in PSC_f$, then every model of ϕ has a core of size atmost $f(|\phi|)$.

Proof: Consider such a function f. For any sentence ϕ in a relational vocabulary τ s.t. $\phi \in PS$, $\neg \phi$ is equivalent to a Σ_1^0 sentence by Łoś-Tarski theorem. Hence $\neg \phi \in PSC_f$. By assumption about f, the size of minimal models of $\neg \phi$ is bounded above by $n = f(|\phi|) + k$, where k is the number of constants in τ . Therefore, $\neg \phi$ is equivalent to an \exists^n sentence and hence ϕ is equivalent to a \forall^n sentence. Corollary 2 now forbids n, and hence f, from being recursive. It is easy to see that the result extends to vocabularies with functions too (by using functions in a trivial way).

Corollary 1 motivates us to consider sentences with bounded cores since all sentences in PSC_f have bounded cores.

2.2 The case of bounded cores

We first give a more general definition.

Definition 2 (*Preservation under substructures modulo a bounded core*) A class of structures S is said to be preserved under substructures modulo a bounded core (denoted $S \in \mathbb{PSC}$), if $S \in \mathbb{PSC}_f$ and there exists a finite cardinal B dependent only on S such that every structure in S has a core of size atmost B.

The collection of all such classes is denoted by \mathbb{PSC} . Let $\mathbb{PSC}(B)$ be the sub-collection of \mathbb{PSC} in which each class has minimal core sizes bounded by B. Then $\mathbb{PSC} = \bigcup_{B \geq 0} \mathbb{PSC}(B)$. An easy observation is that $\mathbb{PSC}(i) \subseteq \mathbb{PSC}(j)$ for $i \leq j$. As before, \mathbb{PSC} and each $\mathbb{PSC}(B)$ contain non-FO definable classes. As an example, the class of forests is in $\mathbb{PSC}(0)$. Let PSC (resp. PSC(B)) be the FO definable classes in $\mathbb{PSC}(0)$ (resp. $\mathbb{PSC}(B)$). Observe that PSC(0) is exactly PS and $PSC = \bigcup_{B \geq 0} PSC(B)$. Therefore, PSC generalizes PS. Further, the hierarchy in PSC is strict. Consider $\phi \in PSC(k)$ given by $\phi = \exists x_1 \ldots \exists x_k \bigwedge_{1 \leq i < j \leq k} \neg (x_i = x_j)$. Then $\phi \notin PSC(l)$ for l < k. From Corollary 1, we have

Lemma 2 $PSC = PSC_f$.

As noted earlier, a Σ^0_2 sentence ϕ with B existential quantifiers is in PSC_f with minimal core size bounded by B. Hence $\phi \in PSC(B)$. In the converse direction, Theorem 2 and Lemma 2 together imply that for a sentence $\phi \in PSC(B)$, there is an equivalent Σ^0_2 sentence. We can then ask the following sharper question: For $\phi \in PSC(B)$, is there an equivalent Σ^0_2 sentence having B existential quantifiers?

Theorem 4 A sentence $\phi \in PSC(B)$ iff it is equivalent to a Σ_2^0 sentence with B existential quantifiers.

The proof of this theorem uses tools from classical model theory and some notions that we define. We will present it in Section 7. Before that we shall consider Theorem 4 for special fragments of FO and for special classes of structures. Towards this, we first look at the notion of relativization from the literature.

3 Revisiting Relativization

For purposes of our discussion in this and remaining sections of the paper, we will assume relational vocabularies (predicates and constants).

A notion that has proved immensely helpful in proving most of our positive special cases of Theorem 4 is that of *relativization*. Informally speaking, given a sentence ϕ ,

we would like to define a formula (with free variables \bar{x}) which asserts that ϕ is true in the submodel induced by \bar{x} . The following lemma shows the existence of such a formula.

Lemma 3 If τ is a relational vocabulary, for every $FO(\tau)$ sentence ϕ and variables $\bar{x} = (x_1, \ldots, x_k)$, there exists a quantifier-free formula $\phi|_{\bar{x}}$ with free variables \bar{x} such that the following holds: Let M be a model and $\bar{a} = (a_1, \ldots, a_k)$ be a sequence of elements of M. Then

$$(M, a_1, \ldots, a_k) \models \phi|_{\bar{x}} \text{ iff } M(\{a_1, \ldots, a_k\}) \models \phi$$

Proof: Let $X = \{x_1, \dots, x_k\}$ and C be the set of constants in τ . First replace every \forall quantifier in ϕ by $\neg \exists$. Then replace every subformula of ϕ of the form $\exists x \chi(x, y_1, \dots, y_k)$ by $\bigvee_{z \in X \cup C} \chi(z, y_1, \dots, y_k)$.

We refer to $\phi|_{\bar{x}}$ as ' ϕ relativized to \bar{x} '. We shall sometimes denote $\phi|_{\bar{x}}$ as $\phi|_{\{x_1,\ldots,x_k\}}$ (though \bar{x} is a sequence and $\{x_1,\ldots,x_k\}$ is a set).

We refer to $\phi|_{\bar{x}}$ as ' ϕ relativized to \bar{x} '. For clarity of exposition, we will abuse notation and use $\phi|_{\{x_1,\ldots,x_k\}}$ to denote $\phi|_{\bar{x}}$ (although \bar{x} is a sequence and $\{x_1,\ldots,x_k\}$ is a set), whenever convenient.

We begin with the following observation.

Lemma 4 Over any given class C of structures in \mathbb{PS} , if $\phi \leftrightarrow \forall z_1 \dots \forall z_n \varphi$ where φ is quantifier-free, then $\phi \leftrightarrow \psi$ where $\psi = \forall z_1 \dots \forall z_n \phi |_{\{z_1,\dots,z_n\}}$.

Proof: It is easy to see that $\phi \to \psi$. Let $M \in \mathcal{C}$ be s.t. $M \models \psi$. Let \bar{a} be an n-tuple from M. Then, by Lemma 3, $M(\bar{a}) \models \phi$. Since $\mathcal{C} \in \mathbb{PS}$, $M(\bar{a}) \in \mathcal{C}$ so that $M(\bar{a}) \models \forall z_1 \dots \forall z_n \varphi$. Then $M(\bar{a}) \models \varphi(\bar{a})$ and hence $M \models \varphi(\bar{a})$. Then $M \models \forall z_1 \dots \forall z_n \varphi$ and hence $M \models \phi$.

Using Łoś-Tarski theorem and the above lemma, it follows that a sentence ϕ in PS has an equivalent universal sentence whose matrix is ϕ itself relativized to the universal variables. However we give a proof of this latter fact directly using relativization, and hence an alternate proof of the Łoś-Tarski theorem. We emphasize that our proof works only for relational vocabularies (Łoś-Tarski is known to hold for arbitrary vocabularies). This would show that relativization helps us prove Theorem 4 for the case of B=0.

3.1 A proof of Łoś-Tarski theorem using relativization

We first introduce some notation. Given a τ -structure M, we denote by τ_M , the vocabulary obtained by expanding τ with as many constant symbols as the elements of M - one constant per element. We denote by \mathcal{M} the τ_M structure whose τ -reduct is M and in which each constant in τ_M is interpreted as the element of M corresponding to the constant. It is clear that M uniquely determines \mathcal{M} . Finally, $\mathcal{D}(M)$ denotes the diagram of M - the collection of quantifier free τ_M -sentences true in \mathcal{M} .

Theorem 5 (Łoś-Tarski) A FO sentence ϕ is in PS iff there exists an $n \in \mathbb{N}$ such that ϕ is equivalent to $\forall z_1 \dots \forall z_n \phi |_{\{z_1,\dots,z_n\}}$.

Proof:

Consider a set of sentences $\Gamma = \{\xi_k \mid k \in \mathbb{N}, \ \xi_k = \forall z_1 \dots \forall z_k \phi |_{\{z_1, \dots, z_k\}}\}$. Observe that $\xi_{k+1} \to \xi_k$ so that a finite collection of ξ_k s will be equivalent to ξ_{k^*} where k^* is the highest index k appearing in the collection. We will show that $\phi \leftrightarrow \Gamma$. Once we show this, by compactness theorem, $\phi \leftrightarrow \Gamma_1$ for some finite subset Γ_1 of Γ and by the preceding observation, ϕ is equivalent to $\xi_n \in \Gamma_1$ for some n.

If $M \models \phi$, then since $\phi \in PS$, every substructure of it models ϕ - in particular, the substructure induced by any k-elements of M. Then $M \models \xi_k$ for every k and hence $M \models \Gamma$.

Conversely, suppose $M \models \Gamma$. Then every finite substructure of M models ϕ . Let \mathcal{M} be the τ_M structure corresponding to M. Consider any finite subset S of the diagram $\mathcal{D}(M)$ of M. Let C be the finite set of constants referred to in S. Clearly $\mathcal{M}|_{\tau \cup C}$, namely the $(\tau \cup C)$ -reduct of \mathcal{M} models S since $\mathcal{M} \models \mathcal{D}(M)$. Then consider the substructure \mathcal{M}_1 of $\mathcal{M}|_{\tau \cup C}$ induced by the interpretations of the constants of C - this satisfies S. Now since C is finite, so is \mathcal{M}_1 . Then the τ -reduct of \mathcal{M}_1 - a finite substructure of M models ϕ .

Thus $S \cup \{\phi\}$ is satisfiable by \mathcal{M}_1 . Since S was arbitrary, every finite subset of $\mathcal{D}(M) \cup \{\phi\}$ is satisfiable so that by compactness, $\mathcal{D}(M) \cup \{\phi\}$ is satisfiable by some structure say \mathcal{N} . Then the τ -reduct N of \mathcal{N} is s.t. (i) M is embeddable in N and (ii) $N \models \phi$. Since $\phi \in PS$, the embedding of M in N models ϕ and hence $M \models \phi$.

The above proof shows that for $\phi \in PS$, there is an equivalent universal sentence whose matrix is ϕ itself, relativised to the universal variables. In fact, by Lemma 4, there is an optimal (in terms of the number of universal variables) such sentence.

An observation from the proof of Theorem 5 is that, the Łoś-Tarski theorem is true over any class of structures satisfying compactness - hence in particular the class of structures definable by a FO theory (indeed this result is known). But there are classes of structures which are not definable by FO theories but still satisfy compactness: Consider any FO theory having infinite models and consider the class of models of this theory whose cardinality is not equal to a given infinite cardinal. This class satisfies compactness but cannot be definable by any FO theory due to Löwenheim-Skolem theorem. Yet Łoś-Tarski theorem would hold over this class.

Having seen the usefulness of relativization in proving Theorem 4 when B equals 0, it is natural to ask if this technique works for higher values of B too. We answer this negatively.

3.2 Limitations of relativization

We show by a concrete example that relativization cannot be used to prove Theorem 4 in general. This motivates us to derive necessary and sufficient conditions for relativization to work.

Example 3: Consider $\phi = \exists x \forall y E(x,y)$ over $\tau = \{E\}$. Note that ϕ is in PSC(1). Suppose ϕ is equivalent to $\psi = \exists x \forall^n \bar{y} \phi|_{x\bar{y}}$ for some n. Consider the structure

 $M=(\mathbb{Z},\leq)$ namely the integers with usual \leq linear order. Any finite substructure of M satisfies ϕ since it has a minimum element (under the linear order). Then taking x to be any integer, we see that $M\models\psi$. However $M\not\models\phi$ since M has no minimum element - a contradiction. The same argument can be used to show that ϕ cannot be equivalent to any sentence of the form $\exists^n \bar{x} \ \forall^m \bar{y} \ \phi|_{\bar{x}\bar{y}}$.

We now give necessary and sufficient conditions for relativization to work. Towards this, we introduce the following notion. Consider $\phi \in FO(\tau)$ s.t. $\phi \in PSC(B)$. Consider a vocabulary τ_B obtained by expanding τ with B fresh constants. Consider the class S_{ϕ}^{all} of τ_B -structures with the following properties:

- 1. For each $(M,a_1,\ldots,a_B)\in S^{\rm all}_\phi$ where M is a au-structure and $a_1,\ldots,a_B\in M,M\models\phi$ and $\{a_1,\ldots,a_B\}$ forms a core of M w.r.t. ϕ .
- 2. For each model M of ϕ , for each core $C=\{a_1,\ldots,a_l\}$ of M w.r.t. ϕ s.t. $l\leq B$ and for each function $p:\{1,\ldots,B\}\to C$ with range C, it must be that $(M,p(1),\ldots,p(B))\in S^{\mathrm{all}}_{\phi}.$

We now have the following.

Theorem 6 Given $\phi \in PSC(B)$, the following are equivalent.

- 1. S_{ϕ}^{all} is finitely axiomatizable.
- 2. ϕ is equivalent to $\exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$ for some $n \in \mathbb{N}$.
- 3. ϕ is equivalent to a $\exists^B \forall^*$ sentence ψ such that in any model M of ψ and ϕ , the following hold:
 - (a) The underlying set of any witness for ψ is a core of M w.r.t. ϕ .
 - (b) Conversely, if C is a core of M w.r.t. ϕ , x_1, \ldots, x_B are the \exists variables of ψ and $f: \{x_1, \ldots, x_B\} \to C$ is any function with range C, then $(f(x_1), \ldots, f(x_B))$ is witness for ψ in M.

Proof:

- (1) $\xrightarrow{}$ $\xrightarrow{}$ $\xrightarrow{}$ $\xrightarrow{}$ $\xrightarrow{}$ Let S_{ϕ}^{all} be finitely axiomatizable. Check that $S_{\phi}^{\text{all}} \in \mathbb{PS}$ so that by Łoś-Tarski theorem, it is axiomatizable by a Π_1^0 $FO(\tau_B)$ -sentence ψ having say $n \forall$ quantifiers. Further, by Lemma 4, ψ is equivalent to $\gamma = \forall^n \bar{z} \psi|_{\bar{z}}$. Now consider $\varphi = \exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$. Firstly, from Lemma 5, $\phi \to \varphi$. Conversely, suppose $M \models \varphi$. Let a_1, \ldots, a_B be witnesses and consider the τ_B -structure $M_B = (M, a_1, \ldots, a_B)$. Now $M_B \models \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$. We will show that $M_B \models \gamma$. Consider $b_1, \ldots, b_n \in M$ and let $M_1 = M_B(\{b_1, \ldots, b_n\})$. Then $M_1 \models \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$. Check that the τ -reduct of M_1 (i) models ϕ and (ii) contains $\{a_1, \ldots, a_B\}$ as a core. Then $M_1 \in S_{\phi}^{\text{all}}$ and hence $M_1 \models \psi$. Since b_1, \ldots, b_n were arbitrary, $M_B \models \gamma$. Since $\gamma \leftrightarrow \psi$ and ψ axiomatizes S_{ϕ}^{all} , the τ -reduct of M_B , namely M, models ϕ .
- $\underline{(2)} \to \underline{(3)}$: Take ψ to be $\exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$. Consider a model M of ϕ and ψ . The set C of elements of any witness for ψ forms a core of M w.r.t. ψ . Then since $\phi \leftrightarrow \psi$, C is also

a core of M w.r.t. ϕ . Conversely, consider a core C of M w.r.t. ϕ . Then any substructure of M containing C satisfies ϕ . Then check that elements of C form a witness for ψ .

Consider ϕ and M in the Example 3 above. Take any finite substructure M_1 of M - it models ϕ . There is exactly one witness for ϕ in M_1 , namely the least element under \leq . However every element in M_1 serves as a core. The above theorem shows that no $\exists \forall^*$ sentence will be able to capture exactly all the cores through its \exists variable. In the following sections, we shall study Theorem 4 for several special classes of FO and over special structures. Interestingly, in most of the cases in which Theorem 4 turns out true, relativization works! However we also show a case in which relativization does not work, yet Theorem 4 is true.

4 Positive Special Cases for Theorem 4

4.1 Theorem 4 holds for special fragments of FO

Unless otherwise stated, we consider relational vocabularies throughout the section. The following lemma will be repeatedly used in the subsequent results.

Lemma 5 Let $\phi \in PSC(B)$. For every $n \in \mathbb{N}$, ϕ implies $\exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$.

Proof: Suppose $M \models \phi$. Since $\phi \in PSC(B)$, there is a core C of M of size at most B. Interpret \bar{x} to include all the elements of C (in any which way). Since C is a core, for any n-tuple \bar{d} of elements of M, having underlying set D, the substructure of M induced by $C \cup D$ models ϕ . Then $(M, \bar{a}, \bar{d}) \models \phi|_{\bar{x}\bar{y}}$ for all \bar{d} from M.

Lemma 6 Let τ be a monadic vocabulary containing k unary predicates. Let $\phi \in FO(\tau)$ be a sentence of rank r s.t. $\phi \in PSC(B)$. Then ϕ is equivalent to ψ where $\psi = \exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$ where $n = r \times 2^k$. For B = 0, n is optimal i.e. there is an FO sentence in PSC(0) for which any equivalent Π_0^0 sentence has at least n quantifiers.

Proof: That ϕ implies ψ follows from Lemma 5. For the converse, suppose $M \models \psi$ where $n = r \times 2^k$. By an Ehrenfeucht-Fräissé game argument, we can show that M contains a substructure M_S such that (i) $M \equiv_r M_S$, with $|M_S| \leq n$ and (ii) for any extension M' of M_S in M, $M' \equiv_r M_S$. The substructure M_S is obtained by taking up to r elements of each colour $c \in 2^\tau$ present in M. An element a in structure M is said

to have colour c if for every predicate $P \in \Sigma$, $M \models P(a)$ iff $P \in c$. Since $M \models \psi$, there exists witnesses \bar{a} for ψ in M. Choose \bar{b} to be an n-tuple which includes the elements of M_S . This is possible because $|M_S| \leq n$. Then we have, $(M, \bar{a}, \bar{b}) \models \phi|_{\bar{x}\bar{y}}$ so that $M(\bar{a}\bar{b}) \models \phi$. But $M_S \subseteq M(\bar{a}\bar{b}) \subseteq M$ so that $M(\bar{a}\bar{b}) \equiv_r M$. Then $M \models \phi$.

To see the optimality of n for B=0, consider the sentence ϕ which states that there exists at least one colour $c\in 2^{\tau}$ such that there exist at most r-1 elements with colour c. The sentence ϕ can be written as a formula with rank r, as the disjunction over all colours, of sentences of the form, $\exists x_1 \exists x_2 \cdots \exists x_{r-1} \forall x_r (\bigwedge_{i=1}^{r-1} x_r \neq x_i) \rightarrow \neg C(x_r)$. From the preceding paragraph, $\phi \leftrightarrow \forall^n \bar{y} \ \phi|_{\bar{y}}$ where $n=r\times 2^k$. Suppose ϕ is equivalent to a \forall^s sentence for some s< n. Then by Lemma 4, $\phi \leftrightarrow \varphi$ where $\varphi = \forall^s \bar{y} \ \phi|_{\bar{y}}$. Then consider the structure M, which has r elements of each colour. Clearly, $M \not\models \phi$. However check that every s-sized substructure of M models ϕ . Then $M \models \varphi$ and hence $M \models \phi$ - a contradiction.

Lemma 7 Let $S \in \mathbb{PSC}(B)$ be a finite collection of τ -structures so that S is definable by a Σ^0_2 sentence $\phi \in PSC(B)$. Then S is definable by the sentence ψ where $\psi = \exists^B \bar{x} \ \forall^n \bar{y} \ \phi |_{\bar{x}\bar{y}}$ for some $n \in \mathbb{N}$.

Proof: Check that all structures in S must be of finite size so that ϕ exists. Let the size of the largest structure in S be atmost n. Consider ψ . Lemma 5 shows that $\phi \to \psi$. Conversely, suppose $M \models \psi$. Then there exists a witness \bar{a} s.t. any extension of $M(\bar{a})$ within M with atmost n additional elements models ϕ . Since M is of size atmost n, taking the extension M of $M(\bar{a})$, we have $M \models \phi$. Since ϕ defines S so does ψ .

Lemma 8 Consider $\phi \in \Pi_2^0$ given by $\phi = \forall^n \bar{x} \exists^m \bar{y} \beta(\bar{x}, \bar{y})$ where β is quantifier free. If $\phi \in PSC(B)$, then ϕ is equivalent to ψ where $\psi = \exists^B \bar{u} \forall^n \bar{v} \phi|_{\bar{u}\bar{v}}$.

Proof: From Lemma 5, $\phi \to \psi$. For the converse, let $M \models \psi$ and let \bar{a} be a witness. Consider an n-tuple \bar{b} from M. Then $M_1 = M(\bar{a}\bar{b})$ is s.t. $M_1 \models \phi$. Then for $\bar{x} = \bar{b}$, there exists $\bar{y} = \bar{d}$ s.t. \bar{d} is an m-tuple from M_1 and $M_1 \models \beta(\bar{b}, \bar{d})$. Then $M \models \beta(\bar{b}, \bar{d})$ since $M_1 \subseteq M$. Hence $M \models \phi$.

Lemma 9 Suppose $\phi \in PSC(B)$ and $\neg \phi \in PSC(B')$. Then ϕ is equivalent to ψ where $\psi = \exists^B \bar{x} \ \forall^{B'} \bar{y} \ \phi|_{\bar{x}\bar{y}}$.

Proof: From Lemma 5, ϕ implies ψ . For the converse, suppose $M \models \psi$. Then there is a witness \bar{a} for ψ s.t. for any B'-tuple \bar{b} , the substructure induced by $\bar{a}\bar{b}$ i.e. $M(\bar{a}\bar{b})$ models ϕ . Suppose $M \not\models \phi$. Then $M \models \neg \phi$ so that there is a core C of M w.r.t. $\neg \phi$, of size at most B'. Let \bar{d} be a B'-tuple which includes all the elements of C. Then $M(\bar{a}\bar{d}) \models \phi$. But $M(\bar{a}\bar{d}) \subseteq M$ contains C so that $M(\bar{a}\bar{d}) \models \neg \phi$ – a contradiction.

Observe that for the special case of B=0, we get combinatorial proofs of Łoś-Tarski theorem for the fragments mentioned above. Moreover all of these proofs and hence the results hold in the finite. We mention that the result of Lemma 8 holding in the finite was proved by Compton too (see [7]). We were unware of this until recently and have independently arrived at the same result. The reader is referred to Section 6 for our studies on more *positive* cases of Łoś-Tarski in the finite.

Interestingly, Lemma 9 has implications for the Δ^0_2 fragment of FO. Define $\Delta^0_2(k,l)\subseteq \Delta^0_2$ to be the class of sentences which have a $\exists^k\forall^*$ and a $\forall^l\exists^*$ equivalent. Note that $\Delta^0_2=\bigcup_{l,k\geq 0}\Delta^0_2(k,l)$. Lemma 9 gives us the following right away.

Theorem 7 The following are equivalent:

- 1. $\phi \in PSC(k)$ and $\neg \phi \in PSC(l)$.
- 2. ϕ is equivalent to $a \exists^k \forall^l$ and $a \forall^l \exists^k$ sentence.
- 3. $\phi \in \Delta_2^0(k, l)$.

As a corollary, we see that $\Delta_2^0(k,l)$ is a finite class upto equivalence. We are not aware of any other semantic characterization of these natural fragments of Δ_2^0 . This highlights the importance of the notion of cores and the sizes thereof.

4.2 Theorem 4 over special classes of structures

We first look at Theorem 4 over finite words which are finite structures in the vocabulary containing one binary predicate \leq (always interpreted as a linear order) and a finite number of unary predicates (which form a partition of the universe). And we obtain something stronger than Theorem 4. Before that, we mention that the idea of relativization can be naturally extended to MSO. Given ϕ in MSO and a set of variables $Z = \{z_1, \ldots, z_n\}, \phi|_Z$ is obtained by first converting all $\forall X$ to $\neg \exists X$ and then replacing every subformula $\exists X\chi(X,\ldots)$ with $\bigvee_{Y\subseteq Z}((\bigwedge_{z\in Y}X(z) \land \bigwedge_{z\in Z\backslash Y}\neg X(z)) \land \chi(X,\ldots))$. The resulting FO formula is then relativized to Z and simplified to eliminate the (original) SO variables. As before, abusing notation, we use $\phi|_Z$ and $\phi|_{\bar{z}}$ interchangeably.

Note: We at times will refer to the 'structure' connotation of a word and at other times refer to the 'string' connotation of it. This would however be clear from the context (typically language-theoretic notions used for a word would mean we are talking about it as a string whereas model-theoretic notions used for it would mean we are referring to it as a structure).

Theorem 8 Over words, a MSO sentence ϕ is in $\mathbb{PSC}(B)$ iff it is equivalent to ψ where $\psi = \exists^B \bar{x} \forall^k \bar{y} \phi|_{\bar{x}\bar{y}}$ for some $k \in \mathbb{N}$.

Proof sketch: We use the fact that over words, by the Büchi-Elgot-Trakhtenbrot theorem [4], MSO sentences define regular languages. The 'If' direction is easy. For the 'Only if' direction, let the regular language L defined by ϕ be recognized by an n state automaton, say \mathcal{M} . If there is no word of length $> N = (B+1) \times n$ in L, then L is a finite language of finite words and hence from Lemma 7, we are done. Else suppose there is a word of length > N in L. Then consider ψ above for k = N. It is easy to observe that ϕ implies ψ . In the other direction, suppose $w \models \psi$ for some word w. Then there exists a set A of elements i_1, \ldots, i_m s.t. (i) $m \leq B$ and $i_1 < i_2 \cdots < i_m$ and (ii) every substructure of w of size atmost N + m containing A models ϕ . From Lemma 10 below, there exists a substructure w_1 of w containing A such that (i) $|w_1| \leq N$ and (ii) $w_1 \in L$ iff $w \in L$. Then w_1 models ϕ and hence $w \models \phi$. Thus ψ implies ϕ and

Before going into the proof of the lemma, we briefly explain the intuition. Let q_j be the state reached by automaton \mathcal{M} upon reading the subword $w[1\ldots i_j]$. The subword $w[(i_j+1),\ldots i_{j+1}]$ takes \mathcal{M} from q_j to q_{j+1} through a sequence S of states. Since \mathcal{M} has only n states, if $w[(i_j+1),\ldots i_{j+1}]$ is long, then S will contain at least one loop. Then getting rid of the subwords that give rise to loops, we will be able to obtain a subword of $w[(i_j+1),\ldots i_{j+1}]$ that takes \mathcal{M} from q_j to q_{j+1} without causing \mathcal{M} to loop in between. It follows that this subword must be of length at most n. Collecting such subwords of $w[(i_j+1),\ldots i_{j+1}]$ for each j and concatenating them, we get a subword of w of length at most N containing set A that takes \mathcal{M} from the initial state to the same state as w. We now formalize this intuition.

Lemma 10 Let L be a regular language having an n state automaton accepting it. Given a natural number B, consider a word $w \in \Sigma^*$ of length $> N = (B+1) \times n$. Let $A = \{i_1, \ldots, i_m\}$ where $i_1 < i_2 \ldots < i_m$ be a given set of elements from the universe of w. Then there is a substructure w_1 of w containing A such that (i) $|w_1| \leq N$ and (ii) $w_1 \in L$ iff $w \in L$.

Proof:

Let $M=(Q,\Sigma,\delta,q_0,F)$ be a DFA accepting L where $Q=\{q_0,\ldots,q_{n-1}\}$ is the set of states, Σ is the alphabet, δ is the transition function, q_0 is the initial state and F is the set of final states. We use the following notation: If z is a sequence of objects, then we use z(k) to denote the k^{th} element of z and z $[k\ldots l]$ to denote the subsequence of z formed by the $k^{th}, (k+1)^{th}, \ldots l^{th}$ elements of z for k, l s.t. $1 \le k \le l \le$ (length of z).

Let $q(i+1), \ 1 \leq i \leq |w|$ be the state of Q after reading the word $w[1 \dots i]$. We take q(1) to be q_0 . Then let $q = (q(i))_{1 \leq i \leq (|w|+1)}$ be the sequence of these states. We are given $A = \{i_1, \dots, i_m\}$ which is a subset of m elements of the universe of w. Let $i_0 = 1$ and $i_{m+1} = |w| + 1$. For $j \in \{0, \dots, m\}$, consider $q[i_j \dots i_{j+1}]$. Set $p = i_j$ to $s = i_{j+1} - 1$. We collect a set T of indices between p and s using the procedure below:

Initialize i to p.

- 1. If i > s, then stop.
- 2. If i = s, then put i into T and increment i by 1.
- 3. If i < s, then let k s.t. $p \le k \le s$ be the highest index such that q(i) = q(k). Then put k into T and update the value of i to be k + 1.

At the end of this procedure, let the indices in T be k_1,\ldots,k_l where $k_1 < k_2 < \cdots < k_l$ if T is non-empty. Note that T is empty iff $i_j = i_{j+1}$ only if j = 0. Also note that at termination, the value of i must be s+1. Finally note that $q(i_j), q(k_1+1), q(k_2+1), \ldots, q(k_l)$ must all be distinct so that $l \le n$. Then consider the subword w_j of w given by

$$w_j = \begin{cases} \epsilon & \text{if T is empty} \\ w(k_1) \cdot w(k_2) \cdots w(k_l) & \text{if T is non-empty} \end{cases}$$

Observe that $|w_j| \leq n$. Let r_1, \ldots, r_l be the states the automaton M goes through when w_j is applied to state $q(i_j)$.

We consider the following cases:

1. T is non-empty.

Now from the way k_1 was chosen by the above procedure, $q(i_j)=q(k_1)$. Then if M is in state $q(i_j)$, on $w(k_1)$, it moves to state r_1 given by $r_1=q(k_1+1)$. Similarly, the index k_2 is s.t. $q(k_2)=q(k_1+1)$ so that if M is in state r_1 , then on $w(k_2)$, it moves to state r_2 given by $r_2=q(k_2+1)$. Continuing this way we find that on $w(k_l)$, if M is in state r_{l-1} , it moves to state r_l given by $r_l=q(k_l+1)$. Now as observed above, at termination, the value of i must be $s+1=i_{j+1}$. This can happen in only two ways: (a) In the previous iteration of the procedure, step (2) was executed in which case s was put in t - then t - t - t so that t

2. *T* is empty.

Then $w_j = \epsilon$ and $i_j = i_{j+1}$ in which case $w[i_j \dots (i_{j+1} - 1)] = \epsilon$ so that both these words applied to M in state $q(i_j)$, take M to the same state, namely $q(i_{j+1})$.

Then consider the word $w_1 = w_0 \cdot w_1 \cdots w_m$. From the above observations, it follows that w_1 applied to the initial state of M takes M to the same state as w. Then $w_1 \in L$ iff $w \in L$. Further since for each j, $|w_j| \leq n$, we have that $|w_1| \leq (m+1) \times n \leq (B+1) \times n = N$.

Returning to Theorem 8, observe that for the special case of B=0, we obtain Łoś-Tarski theorem for words and also give a bound for the number of $\forall s$ in the equivalent Π_1^0 sentence in terms of the number of states of the automaton for ϕ . We have not encountered this result in our literature survey.

Before proceeding ahead, as a slight diversion, we give a simpler proof of Łoś-Tarski theorem over words. In fact, over words, we have the following stronger result.

Lemma 11 Consider any set S of words which is closed under taking substructures. Then S can be defined by a Π_1^0 sentence.

Proof: Consider $\overline{S} = \Sigma^* \setminus S$ - the complement of S. Since S is closed under taking substructures, \overline{S} is closed under taking extensions. Then consider the set T of minimal words of \overline{S} , i.e. words of \overline{S} for which no subword is contained in \overline{S} . We show that T must be finite. Suppose T were infinite. If we arrange the words of T to form a sequence - which is infinite - then by Higman's lemma, there is some word in the

sequence which is a subword of another in the sequence. That means some word of T is a subword of another word in T. But that contradicts the minimality of the latter word.

Then T is finite. Taking the disjunction of the existential closures of the diagrams of the words of T, we get a Σ_1^0 sentence defining \overline{S} . Then taking the negation of this sentence, we get the desired Π_1^0 sentence defining S.

Thus contrary to the general setting where it is not necessary for a set of structures preserved under substructures to be even FO-expressible, leave alone being definable by a Π_1^0 sentence, over words, Π_1^0 sentences show much greater power.

We return to Theorem 4 now. So far, relativization has worked in all the cases we have seen. We now give an example of a class of structures over which relativization fails, yet Theorem 4 is true.

Consider a subclass $\mathcal C$ of bounded degree graphs in which each graph is a collection (finite or infinite) of *oriented* paths (finite or infinite). For clarity, by oriented path we mean a graph isomorphic to a connected induced subgraph of the graph (V, E) where $V = \mathbb Z$ and $E = \{(i, i+1) \mid i \in \mathbb Z\}$. Observe that $\mathcal C$ can be axiomatized by a theory $\mathcal T$ which asserts that every node has in-degree atmost 1 and out-degree atmost 1 and that there is no directed cycle of length k for each $k \geq 0$. We first show the following.

Lemma 12 For each $B \geq 1$, there is a sentence $\phi \in PSC(B)$ which is not equivalent, over C, to any ψ of the form $\exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$.

Proof: Consider ϕ which asserts that there are at least B elements of total degree atmost 1 where total degree is the sum of in-degree and out-degree. Clearly $\phi \in PSC(B)$ since it is expressible as a $\exists^B \forall^*$ sentence. Suppose ϕ is equivalent to ψ of the form above for some $n \in \mathbb{N}$. Consider $M \in \mathcal{C}$ which is a both-ways infinite path so that every node in M has total degree 2 - then $M \not\models \phi$. Consider B distinct points on this path at a distance of at least 2n from each other and form a B-tuple say \bar{a} with them. Let \bar{b} be any n-tuple from M. Now observe that $M(\bar{a}\bar{b})$ is a finite structure which has at least B distinct paths (0-sized paths included). Then $M(\bar{a}\bar{b}) \models \phi$ so that $(M, \bar{a}, \bar{b}) \models \phi|_{\bar{x}\bar{y}}$. Since \bar{b} was arbitrary, $M \models \psi$ so that $M \models \phi$. Contradiction.

However Theorem 4 holds over C!

Theorem 9 Over the class C of graphs defined above, $\phi \in PSC(B)$ iff ϕ is equivalent to a $\exists^B \forall^*$ sentence.

Proof: If $\tau = \{E\}$ is the vocabulary of ϕ , let τ_B be a vocabulary obtained by addding B fresh constants c_1, \ldots, c_B to τ . Given a class $\mathcal S$ of τ -structures, define $\mathcal S_B$ to be the class of $all\ \tau_B$ -structures s.t. the τ -reduct of each structure in $\mathcal S_B$ is in $\mathcal S$. Then the proof can be divided into two main steps. Below \equiv denotes elementary equivalence. **Step 1**: Given ϕ , define class $\mathcal C' \subseteq \mathcal C$ such that for every structure $A \in \mathcal C_B$, there exists structure $D \in \mathcal C'_B$ such that $A \equiv D$ (Property I). Since compactness theorem holds over $\mathcal C_B$ (as $\mathcal C_B$ is defined by the same theory $\mathcal T$ as $\mathcal C$), it also holds over $\mathcal C'_B$.

Step 2: Show that ϕ is equivalent to an $\exists^B \forall^*$ sentence over \mathcal{C}' , hence showing the same over \mathcal{C} as well.

Note: The conditions in **Step 1** imply that for every $A \in \mathcal{C}$, there exists $D \in \mathcal{C}'$ such that $A \equiv D$. Then since compactness theorem holds over \mathcal{C} , it also holds over \mathcal{C}' . Suppose the rank of ϕ is m. We define \mathcal{C}' to be the set of graphs $G \in \mathcal{C}$ such that either (a) there exists a bound n_G (dependent on G) such that all paths in G have length less than n_G (this does not mean that G is finite - there could be infinite paths of the same length in G) or (b) there are at least (B+m+2) paths in G which are infinite in both directions. It can be shown that \mathcal{C}' satisfies Property I (See A below). We proceed assuming this to be true.

Now, to show **Step 2**, we use the following approach.

Let $P \in \mathcal{C}'$ be s.t. $P \models \phi$. Choose a core Z in P (recall that $\phi \in PSC(B)$). Let $M_P \in \mathcal{C}'_B$ be a τ_B -structure whose τ -reduct is P and in which each element of Z is assigned to some constant. Let Γ^{M_P} be the set of all \forall^* sentences true in M_P . We can show that if $M' \in \mathcal{C}'_B$ is such that $M' \models \Gamma^{M_P}$, then $M' \models \phi$ (See \mathbf{B} below. We proceed assuming this to be true). That is, if every finite substructure of M' is embeddable in M_P , then $M' \models \phi$. Then over \mathcal{C}'_B , $\Gamma^{M_P} \to \phi$. Now, since \mathcal{C}'_B satisfies compactness theorem, there exists a finite subset $\Gamma^{M_P}_0$ of Γ^{M_P} such that $\Gamma^{M_P}_0 \to \phi$ over \mathcal{C}'_B . Note that, since $\Gamma^{M_P}_0$ is a conjunction of \forall^* sentences, we can assume that $\Gamma^{M_P}_0$ is a single \forall^* sentence.

Let ϕ_P be the τ -sentence of the form $\exists^B \forall^*$ obtained by replacing the B constants in $\Gamma_0^{M_P}$ by B fresh variables and existentially quantifying these variables. Then check that $\phi_P \to \phi$. It is easy to see that $\phi \to \bigvee_{P \in \mathcal{C}', P \models \phi} \phi_P$ (If $P \models \phi$, then interpret the \exists quantifiers in ϕ_P as the chosen core Z mentioned above). By compactness theorem over \mathcal{C}' , there exists a finite set of structures, say $\{P_1,\cdots,P_m\}$ such that $P_i \in \mathcal{C}'$, $P_i \models \phi$ and $\phi \to \bigvee_{i=1}^{i=m} \phi_{P_i}$. Then, we have $\phi \leftrightarrow \bigvee_{i=0}^{i=m} \phi_{P_i}$ over \mathcal{C}' . Since each ϕ_{P_i} is of the form $\exists^B \forall^*, \bigvee_{i=0}^{i=m} \phi_{P_i}$ is also of the same form. That completes **Step 2** and completes the proof.

Below we shall be referring to the notions of 'ball type of radius r' (or simply r-ball type), 'disjoint unions' (denoted by \sqcup) and 'm-equivalence' (denoted by \equiv_m). We shall also use Hanf's theorem. The reader is referred to [8] for these concepts.

A. C' satisfies Property I

Suppose $A \in \mathcal{C}_B$. If there exists a bound n_A , such that all paths in A have length less than n_A , then $A \in \mathcal{C}_B'$ and hence we are done. Contrarily, suppose that there is no such bound n_A . This means that either there are paths of arbitrarily large lengths in A or there is at least one infinite path in A (Let us mark this inference as [*]). Now, construct structure $D \in \mathcal{C}_B'$, where $D = A \sqcup \bigsqcup_{i=1}^{k+m+2} P$, where P is a path which is infinite in both directions and \sqcup denotes disjoint union. We show that $A \equiv D$, by showing that for every $n \in \mathbb{N}$, $A \equiv_n D$. By Hanf's theorem [8], given n, there exist numbers r and q, dependent only on n, such that $A \equiv_n D$ if for each ball type ξ of radius r, the number of instances of ξ in A and D are either equal or are both are greater than q. By adding (B + m + 2) paths, we are introducing infinite copies of just one r-ball type ξ in D, namely the 2r + 1 length path with the ball center as the midpoint. However,

this type ξ was already present infinitely many times in A (due to [*]). Hence Hanf's condition holds for every type ξ , and thus, $A \equiv D$.

B. If $M_1 \in \mathcal{C}_B'$ is such that $M_1 \models \Gamma^{M_P}$, then $M_1 \models \phi$

Before, we proceed, we state and prove the following lemma. Below, an 'outwardly' (resp. 'inwardly') infinite path is an oriented infinite path with an end point which has an outgoing (resp. incoming) edge and no incoming (resp. outgoing) edge.

Lemma 13 For every $m \in \mathbb{N}$ and structure $G \in \mathcal{C}$, there exists a substructure $G^m \subseteq \mathcal{C}$ G, such that $G^m \equiv_m G$ and G^m has

- atmost finitely many finite paths
- atmost m paths which are outwardly-infinite
- atmost m paths which are inwardly-infinite
- atmost 1 path which is bidirectionally-infinite

Proof: By Hanf's Theorem, there exists $t_m \in \mathbb{N}$, such that any two paths of length greater than t_m are m-equivalent. For any graph $G \in \mathcal{C}$, define the following,

- for $i \in \mathbb{N}$, let a_i^G be the number of i length paths
- $-a_{\uparrow}^{G}$ be the number of outwardly-infinite paths
- $-a_{\perp}^{G}$ be the number of inwardly-infinite paths
- $-\,a_{\updownarrow}^{\dot{G}}$ be the number of bidirectionally-infinite paths

Given G, consider $G^m \subseteq G$ given as,

- for $i \in \{0, \dots, t_m\}$, $a_i^{G^m} = \min(a_i^G, m)$
- $-a_{t_{m+1}}^{G^{m}} = \min(\sum_{i=t_{m+1}}^{\infty} a_{i}^{G}, m)$ $\text{ for } i > (t_{m} + 1), a_{i}^{G^{m}} = 0$ $a_{\uparrow}^{G^{m}} = \min(a_{\uparrow}^{G}, m)$ $a_{\downarrow}^{G^{m}} = \min(a_{\downarrow}^{G}, m)$

- $-a_{\uparrow}^{\check{G}^m} = \min(a_{\uparrow}^{\check{G}}, 1)$

By Hanf's theorem, it is easy to see that $G^m \subseteq G$ and $G^m \equiv_m G$.

Suppose $M_1 \in \mathcal{C}'_B$ is such that $M_1 \models \Gamma^{M_P}$. To show that $M_1 \models \phi$, we show that there exists a substructure M_2 of M_P such that $M_1 \equiv_m M_2$ (recall that P is a model of ϕ and M_P is the expansion of P with the elements of a chosen core Z as interpretations of the B constants). Since $\phi \in PSC(B)$, $P \models \phi$, and any substructure of M_P would contain the core Z of P, we have that $M_2 \models \phi$. And since $M_1 \equiv_m M_2$, we would have $M_1 \models \phi$.

Consider the partition of M_P into two parts $M_{P,1}$ and $M_{P,2}$, where $M_{P,1}$ is substructure containing all those paths in M_P which contain the interpretation of atleast one of the constants c_1, \dots, c_B and $M_{P,2}$ contains all the paths in M_P which are not in $M_{P,1}$. Similarly, consider the partition of M_1 into $M_{1,1}$ and $M_{1,2}$. There are two cases to consider.

Case 1: There exists a bound n_P such that all paths in P (and M_P) have length less than n_P

Note that since $M_1 \models \Gamma^{M_P}$, for every finite substructure of M_1 , there exists an isomorphic substructure of M_P . And since all paths in M_P have length less than n_P , we have that all paths in M_1 have length less than n_P as well. Consider the substructure $M_1^S = M_{1,1} \sqcup M_{1,2}^m \subseteq M_1$ (where $M_{1,2}^m$ is as defined in Lemma 13). Clearly, $M_1^S \equiv_m M_1$. Moreover, since both $M_{1,1}$ and $M_{1,2}^m$ are finite, M_1^S is finite, hence there exists a substructure $M_2 \subseteq M_P$, such that M_1^S and M_2 are isomorphic. And since $M_2 \models \phi$ (see above for the reasoning), we have $M_1^S \models \phi$ and hence $M_1 \models \phi$ (since $M_1^S \equiv_m M_1$).

Case 2: There are at least (B+m+2) paths in M_P which are infinite in both directions. Consider a path L in M_1 containing the interpretation a_i of a constant c_i . Since $M_1 \models \Gamma^{M_P}$, one can see that L must be a subpath of some path in M_P - infact subpath of some path in $M_{P,1}$. Thus, arguing similarly for each path $L \subseteq M_{1,1}$, we have $M_{1,1} \subseteq M_{P,1}$. Also, since there are (B+m+2) bidirectional-infinite paths in M_P , at least (m+2) of these would be present in $M_{P,2}$. Now, since $M_{1,2}^m \subseteq M_{1,2}$ (as defined in Lemma 13) contains,

- finitely many finite paths all of these can be embedded in a single bidirectional infinite path
- at most m outwardly-infinite and at most m inwardly-infinite paths all of these can be embedded in m bidirectional-infinite paths
- atmost 1 bidirectional-infinite path: can be embedded in a single bidirectional-infinite path.

it follows that $M_{1,2}^m$ can be embedded into $M_{P,2}$. Thus, $M_1^S = M_{1,1} \sqcup M_{1,2}^m \subseteq M_{P,1} \sqcup M_{P,2} = M_P$. Hence $M_1^S \models \phi$. And since $M_{1,2}^m \equiv_m M_{1,2}$, we have $M_1^S \equiv_m M_1$, and hence $M_1 \models \phi$.

Thus, we have shown that if $M_1 \in \mathcal{C}_B'$ and $M_1 \models \Gamma^{M_P}$, then $M_1 \models \phi$.

We now look at some classes of structures over which Theorem 4 fails.

5 Theorem 4 fails over special classes of structures

We first look at the class \mathcal{F} of all finite structures. Łoś-Tarski theorem fails over this class and hence so does Theorem 4 (for B=0). However, we have the following stronger result. We prove it for relational vocabularies (constants permitted).

Lemma 14 For relational vocabularies, Theorem 4 fails, over \mathcal{F} , for each B > 0.

Proof: We refer to [1] for the counterexample χ for Łoś-Tarski in the finite. Let τ be the vocabulary of χ (i.e. $\{\leq, S, a, b\}$) along with a unary predicate U. Let us call an element x as having colour 0 in a structure if U(x) is true in the structure and having colour 1 otherwise. Let φ be a sentence asserting that there are exactly B elements having colour 0 and these are different from a and b. Then consider $\phi = \neg \chi \wedge \varphi$. Check

that since $\neg \chi$ is preserved under substructures in the finite, in any model of ϕ , the B elements of colour 0 form a core of the model w.r.t. ϕ . Then $\phi \in PSC(B)$. Suppose ϕ is equivalent to ψ given by $\exists^B \bar{x} \forall^n \bar{y} \beta$ where β is quantifier-free. Observe that in any model of ϕ and ψ , any witness for ψ must include all the B elements of colour 0 (else the substructure formed by the witness would not model φ and hence ϕ , though it would model ψ). Consider the structure $M = (\{0, 1, \dots, B + 2n + 3\}, \leq, S, a, b, U)$ where \leq is the usual linear order on numbers, S is the (full) successor relation of \leq , a = 0, b = B + 2n + 3 and $U = \{1, \dots, B\}$. Now $M \not\models \phi$ since $M \not\models \neg \chi$. Consider M_1 which is identical to M except that S(B+n+1,y) is false in M_1 for all y. Then $M_1 \models \phi$ so that $M_1 \models \psi$. Any witness \bar{a} for ψ must include all the B colour 0 elements of M_1 . Then choose exactly the same value, namely \bar{a} , from M to assign to \bar{x} . Choose any \bar{b} as \bar{y} from M. Check that it is possible to choose \bar{d} as \bar{y} from M_1 s.t. $M(\bar{a}\bar{b})$ is isomorphic to $M_1(\bar{a}\bar{d})$ under the isomorphism f given by f(0) = 0, f(B + 2n + 3) = B + 2n + 3, $f(a_i) = a_i$ and $f(b_i) = d_i$ where $\bar{a}=(a_1,\ldots,a_B), \bar{b}=(b_1,\ldots,b_n)$ and $\bar{d}=(d_1,\ldots,d_n)$. Then since $M_1\models\beta(\bar{a},\bar{d}),$ $M \models \beta(\bar{a}, \bar{b})$. Then M models ψ , and hence ϕ . But that is a contradiction.

The example expressed by χ can also be written as a sentence in a purely relational vocabulary. The sentence ϕ below is over the vocabulary $\tau = \{ \leq, S, U \}$. We leave it to the reader to reason out (in the same manner as in [1]) that ϕ is preserved under substructures in the finite but is not equivalent to any universal sentence.

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\begin{array}{rcl} \phi &=& \chi_1 \wedge \chi_2 \wedge \chi_3 \text{ where} \\ \chi_1 &=& \forall x \forall y \forall z \quad \left( (x \leq x) \, \wedge \, \left( (x \leq y) \vee (y \leq x) \right) \wedge \\ & & \left( (x \leq y) \wedge (y \leq z) \right) \rightarrow (x \leq z) \right) \\ \chi_2 &=& \forall x \forall y \, S(x,y) \rightarrow \forall z (((x \leq z) \wedge (x \neq z)) \rightarrow (y \leq z)) \\ \chi_3 &=& \exists z \forall x_1 \forall x_2 \, \left( \bigwedge_{i=1}^{i=2} \neg U(x_i) \wedge (x_1 \neq x_2) \right) \rightarrow \left( \chi_4(x_1,x_2,z) \vee \chi_4(x_2,x_1,z) \right) \\ \chi_4(x_1,x_2,z) &=& \forall y \quad \left( (x_1 \leq y) \wedge (y \leq x_2) \wedge \\ & & \left( (y \neq x_1) \wedge (y \neq x_2) \right) \rightarrow U(y) \wedge \\ & & \left( (z \neq x_2) \wedge \neg S(z,y) \right) \end{array}
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Then one can do a similar proof as above to show that for purely relational vocabularies too, for each $B \ge 0$, Theorem 4 fails over \mathcal{F} .

So far, in all the cases we have seen, it has always been the case that Theorem 4 and Łoś-Tarski theorem either are both true or are both false. We then finally have the following result which is our first instance of a class of structures over which Łoś-Tarski theorem holds but Theorem 4 fails.

Theorem 10 Over the class C of graphs in which each graph is a finite collection of finite undirected paths, for each $B \geq 2$, there is a sentence $\phi \in PSC(B)$ which is not equivalent to any $\exists^B \forall^*$ sentence. However, Łoś-Tarski theorem holds over C.

Proof: Łoś-Tarski theorem holds from the results of Dawar et al. over bounded degree structures [2]. As a counterexample to Theorem 4 for $B \ge 2$, consider condition D_1 , parametrized by B, which asserts that there are at least B paths (0 length included) in the graph. We show that this is FO definable because the following equivalent condition

 D_2 , parametrized by B, is FO definable: (The number of nodes of degree 0) + $\frac{1}{2}\times$ (the number of nodes of degree 1) $\geq B$. We briefly explain this equivalence between D_1 and D_2 . Consider a graph satisfying D_1 . Let k be the number of 0-length paths so that there are atleast B-k paths of length ≥ 1 . Each of the latter paths has exactly 2 nodes of degree 1. Then it is easy to check that condition D_2 holds. Conversely, suppose a graph satisfies D_2 , but it has less than B paths. Let k be the number of 0-length paths so that there are atmost B-1-k paths of length ≥ 1 . Each of the latter paths has exactly 2 nodes of degree 1. Then, (the number of nodes of degree 0) + $\frac{1}{2}\times$ (the number of nodes of degree 1) $\leq (k+\frac{1}{2}\times 2\times (B-1-k)) < B$ – contradicting D_1 . Then D_2 implies D_1 .

Then, given B, D_1 is expressible by a FO sentence ϕ since D_2 is FO expressible (the latter is easy to see).

To see that ϕ is in PSC(B), in any model, observe that the set of nodes formed by picking up one end point each of B distinct paths is a core.

Now suppose that ϕ is equivalent over \mathcal{C} to $\psi = \exists^B \bar{x} \ \forall^n \bar{y} \ \beta(\bar{x}, \bar{y})$ for some $n \in \mathbb{N}$ where β is quantifier-free. Consider a graph M which has exactly $\left\lceil \frac{B}{2} \right\rceil$ paths, each of length $\geq 5n$ (There is nothing sacrosanct about the number 5 - it is just sufficiently large for our purposes). By definition, $M \not\models \phi$ and hence $M \not\models \psi$. Label the end points of these paths as $p_1, p_2, p_3, \ldots, p_{2 \cdot k}$ where $k = \lceil \frac{B}{2} \rceil$. Now consider a graph N having exactly B paths, each of length $\geq 5n$. By definition, $N \models \phi$ and hence $N \models \psi$. Then there exists a witness $\bar{a}=(a_1,\ldots,a_B)$ in N for ψ . Observe that no two of the a_i s can be in the same path else taking the substructure of N formed by just the paths containing \bar{a} , one would get a model of ψ and hence ϕ - but the number of paths in this model would be $\leq B-1$, giving a contradiction. We now choose points b_1,\ldots,b_B in M as follows. For $i \in \{1, \dots, B\}$, if a_i is at a distance of atmost n from any end point in N, then choose b_i to be at the same distance from p_i in M. Else choose b_i to be at a distance of n from p_i in M. Assigning $\bar{b} = (b_1, \dots, b_B)$ as \bar{x} , choose any \bar{d} as \bar{y} from M. Check that it is possible to choose \bar{e} as \bar{y} from N s.t. $M(\bar{b}\bar{d})$ is isomorphic to $N(\bar{a}\bar{e})$ under the isomorphism f given by $f(b_i)=a_i, f(d_j)=e_j$ where $\bar{d}=(d_1,\ldots,d_n)$ and $\bar{e}=(e_1,\ldots,e_n)$. Since $N\models\beta(\bar{a},\bar{e}),\,M\models\beta(\bar{b},\bar{d})$. Then Mmodels ψ – a contradiction.

Important Note: For B=2, the sentence ϕ above is equivalent to asserting that either (i) there are atleast 2 nodes of degree exactly 0 or (ii) there are atleast 3 nodes of degree atmost 1. Consider the following condition for $B\geq 2$ whose special case for B=2 is the condition just mentioned: Either (i) there are atleast B nodes of degree exactly 0 or (ii) there are atleast B+1 nodes of degree atmost 1. This condition, for a given B, is easily seen to be expressible as a FO sentence ξ (in fact, ξ is of the form $\exists^{B+1}\forall^*$). But for B>2, $\xi\notin PSC(B)$. To see this, consider a graph M containing exactly 2 paths $\overline{P_1}$ and $\overline{P_2}$ of length ≥ 1 and B-3 paths of length 0 (the total number of paths is then <B). We will show that M has no core (w.r.t. ξ) of size atmost B. Firstly, $M\models\xi$ since M has B+1 nodes of degree atmost 1. If $\xi\in PSC(B)$, then M has a core C of size atmost B. There are 2 cases: (a) One of P_1 or P_2 has atmost 1 core element. (b) Both P_1 and P_2 have atleast 2 core elements. In case of (b), note that atleast one of the 0-length paths will not contain any core element. Then consider the substructure M_1 of M without this path - this contains all core elements and hence

must satisfy ξ . However, there are exactly B elements of degree atmost 1 in M_1 and hence M_1 violates ξ . In case of (a), there are two subcases: (i) One of P_1 or P_2 , say P_1 w.l.o.g., contains no core element. Then the substructure M_1 of M which is all of M, but excluding P_1 , contains all core elements and must hence model ξ . But M_1 contains exactly B-1 nodes of degree atmost 1; so it violates ξ . (ii) One of P_1 or P_2 , say P_1 w.l.o.g., contains exactly 1 core element say a. Let M_1 be the substructure of M without P_1 . Consider the disjoint union M_3 of M_1 and the substructure M_2 of M induced by a. Then $M_3 \subseteq M$ contains all core elements and must hence model ξ . But M_3 contains exactly B nodes of degree atmost 1; so it violates ξ .

In all cases, we have a contradiction. Hence M has no core of size $\leq B$. Hence $\xi \notin PSC(B)$.

Interestingly however, Theorem 4 holds over \mathcal{C} for B=1 as we shall see in the next Lemma. We also give a simpler proof for the case of B=0 i.e. Łoś-Tarski over \mathcal{C} .

Lemma 15 Over C, for $B \leq 1$, $\phi \in PSC(B)$ iff ϕ is equivalent to ψ where $\psi = \exists^B \bar{x} \ \forall^n \bar{y} \ \phi|_{\bar{x}\bar{y}}$ for some $n \in \mathbb{N}$.

Proof: Let the quantifier rank of ϕ be m. By Hanf's theorem, we have the following:

- A There exists a number $t_m \in \mathbb{N}$ such that any two undirected paths of length greater than t_m are m-equivalent.
- B There exists a number $s_m \in \mathbb{N}$ such that given a structure G = (P, a) where $P \in \mathcal{C}$ is (finite) path of length greater than s_m and a is a designated element of P, there is a substructure $G_1 = (P_1, a)$ of G s.t. (i) P_1 is a subpath of P containing the designated element a, (ii) $|P_1| \leq s_m$ and (iii) $G \equiv_m G_1$.
- C For any graph $G \in \mathcal{C}$, let a_i^G be the number of undirected paths of length i in G. Now, given graph $G \in \mathcal{C}$, we consider a graph $G^m \subseteq G$ as follows (similar to the method in the proof of Theorem 9):

$$- \text{ for } i \in \{0, \cdots, t_m\}, a_i^{G^m} = \min(a_i^G, m)$$

$$- a_{t_m+1}^{G^m} = \min(\sum_{i=t_m+1}^{\infty} a_i^G, m)$$

$$- \text{ for } i > (t_m+1), a_i^{G^m} = 0$$

By Hanf's theorem, it is easy to verify that $G^m \equiv_m G$.

Now consider the statement of the (current) lemma for B=1. Let $n=s_m+\sum_{i=t_m+1}^{i=t_m+1} (m\cdot (i+1))$ and consider ψ given by $\psi=\exists x\ \forall^n \bar{y}\ \phi|_{x\bar{y}}$. That $\phi\to\psi$ follows from Lemma 5. For the converse, suppose $G\models\psi$. Let a be a witness and let P be the path in G on which a appears. Consider the vocabulary $\tau_1=\{E\}\cup\{c_1\}$ where c_1 is a fresh constant and consider $\mathcal{G}=(G,a)$ - the τ_1 -structure obtained by expanding G with a as the interpretation for c_1 . Let $\mathcal{G}=\mathcal{G}_1\sqcup G_2$ where $\mathcal{G}_1=(P,a)$ and $G_2\in\mathcal{C}$ is the collection of all paths in G other than P. Note that we have abused the \sqcup notation slightly but the idea of separating P and a from the rest of G is clear. Now,

– Let $\mathcal{G}_1'\subseteq\mathcal{G}_1$ be the structure ensured by [B] above. Then (i) $|\mathcal{G}_1'|\leq s_m$ and (ii) $\mathcal{G}_1'\equiv_m\mathcal{G}_1$.

- Let
$$G_2^m$$
 be as given by [C] above. Then (i) $G_2^m \subseteq G_2$, (ii) $|G_2^m| \le \sum_{i=0}^{i=t_m+1} m \cdot (i+1)$ and (iii) $G_2^m \equiv_m G_2$.

Then
$$\mathcal{G}'=(\mathcal{G}_1'\sqcup G_2^m)\equiv_m(\mathcal{G}_1\sqcup G_2)=\mathcal{G}.$$
 Also $\mathcal{G}'\subseteq\mathcal{G}.$ Note that $|\mathcal{G}'|\leq s_m+\sum_{i=0}^{i=t_m+1}m\cdot(i+1)=n.$ Now since $G\models\psi,$ choose $x=a$ and $\bar{y}=\bar{d}$ where \bar{d} is any tuple containing exactly the elements of \mathcal{G}' - this is possible since $|\mathcal{G}'|\leq n$ as we just saw. Then $(G,a,\bar{d})\models\phi|_{\bar{x}\bar{y}}$ so that $\mathcal{G}'\models\phi.$ Then $\mathcal{G}\models\phi$ and hence $G\models\phi.$ For $B=0$, there is no \mathcal{G}_1 and hence no \mathcal{G}_1' . It is easy to see that the same proof goes through.

6 Additional observations on Łoś-Tarski theorem over the class of all finite structures

We will refer to truth or failure of Łoś-Tarski over the class of all finite structures simply as the truth or failure of Łoś-Tarski 'in the finite'.

Now as observed earlier in Sections 4 and 5, while Łoś-Tarski fails in the finite, there are special fragments of FO for which Łoś-Tarski is *true* in the finite. We present below two additional fragments of FO for which Łoś-Tarski is true in the finite. This would follow from their combinatorial proofs and hence we state the results below for arbitrary structures.

Lemma 16 Consider ϕ of the form $\exists x \forall y \psi(x,y)$ in a purely relational vocabulary τ . If $\phi \in PS$, then ϕ is equivalent to $\varphi = \forall z_1 \dots \forall z_n \phi|_{\{z_1,\dots,z_n\}}$ where $n = 2^{|\tau|}$. Further, this bound is tight i.e. there is a $\exists \forall$ sentence in PS which is not equivalent to a universal sentence with less than n quantifiers.

Proof:

From Lemma 5, it follows that if $M \models \phi$ then $M \models \varphi$. Therefore to prove the lemma, it suffices to show that if $M \models \varphi$, that is, every substructure of M with size atmost n is a model of ϕ , then infact $M \models \phi$. We prove it by contradiction, so assume that $M \models \varphi \land \neg \phi$. The main idea is to use M to come up with a structure which models ϕ , but which has a substructure which is a non-model of ϕ . This contradicts that $\phi \in PS$. [Note that |M| > n for such an M, since if $|M| \le n$ and $M \models \varphi$ then $M \models \phi$ as well.]

Since every substructure of M with size atmost n models ϕ , every 1 sized substructure of M is a model of ϕ , and hence $\psi(x,x)$ is true for every $x \in M$ (recall that $\phi = \exists x \forall y \psi(x,y)$). Now note that $n=2^{|\tau|}$ is the number of all 1-types possible over the vocabulary τ upto equivalence (An i-type of τ is a quantifier-free formula over τ which uses just i variables. The number of i types is finite upto equivalence. See [8] where our i-type is called t-type). Denote the 1-types as $\{\sigma_0, \cdots, \sigma_{n-1}\}$, and $\sigma_i(x)$ denotes that t is of 1-type t-type t-type of t-type t-t

M. Since $M \models \forall x \exists y \neg \psi(x,y)$, there exists a y_0 such that $\psi(x_0,y_0)$ is false in M. However, since every substructure of size atmost n is a model of ϕ , the substructure $M(\{x_0,y_0\}) \models \phi$ and hence $\psi(y_0,x_0)$ must be true in M (since either x_0 or y_0 must act as a witness for x in ϕ . But $\psi(x_0,y_0)$ is false. Hence x_0 cannot be the witness). Let y_0 be of 1-type σ_k .

Suppose that it is possible to have a structure A with just two elements $\{a_0,a_1\}$ such that $\sigma_i(a_0)$, $\sigma_i(a_1)$ and $\neg \psi(a_0,a_1)$ hold. Then consider the structure X with universe $\{a_0,a_1,a_2,b\}$ such that (i) $\sigma_i(a_j)$ holds for $j\in\{0,1,2\}$ (ii) $\sigma_k(b)$ holds

(iii) $\neg \psi(a_j, a_{(j+1) \bmod 3})$ holds for $j \in \{0, 1, 2\}$ (iv) $\psi(b, a_j)$ holds for $j \in \{0, 1, 2\}$ and (v) $\psi(b, b)$. Such a structure exists because all the 1-types and 2-types have been *copied* from other structures, namely, (i), (iii) are copied from A and (ii), (iv), (v) copied from M. Clearly, $X \models \phi$, since $b \in X$ acts as a witness for x in ϕ . However, the substructure of X induced by $\{a_0, a_1, a_2\} \not\models \phi$. This contradicts the given assumption of $\phi \in PS$. Hence, it is not possible to have a structure A as assumed, and hence taking a structure A' with two elements a_0, a_1 such that $\sigma_i(a_0), \sigma_i(a_1)$ hold, necessitates that $\psi(a_0, a_1)$ must hold (Note that for every 1-type σ_i in M, one can construct such an A'_i).

Consider M' to be a substructure of M which contains exactly one element of each 1-type present in M. Clearly $|M'| \leq n$ and hence $M' \models \phi$. Thus, there exists $x_1 \in M'$ such that for every $y_1 \in M'$, $\psi(x_1, y_1)$ holds. Suppose that $\sigma_l(x_1)$ holds. Construct an extension \bar{M} of M with an additional element z_0 such that (i) $\sigma_l(z_0)$ holds (ii) $\forall y \in M \ \psi(z_0, y)$ holds (iii) $\psi(z_0, z_0)$ holds. Such a structure \bar{M} exists because all the 1-types and 2-types have been copied from other structures, namely (i), (iii) are copied from M, (ii) is copied from M for y satisfying $\neg \sigma_i(y)$, and for y satisfying $\sigma_i(y)$, the 2-type is copied from A'_l . Clearly, $\bar{M} \models \phi$ as $z_0 \in \bar{M}$ acts a witness for x in ϕ . However, $M \subseteq \bar{M}$ and $M \not\models \phi$. This again contradicts that $\phi \in PS$. Hence, our original assumption that there exists M such that $M \models \varphi \land \neg \phi$ is incorrect. Then $\varphi \to \phi$.

To prove the optimality of the bound, consider the following example over a vocabulary of k unary predicates. We construct a formula ϕ such that the smallest n for which $\phi \leftrightarrow \forall z_1 \cdots \forall z_n \phi|_{\{z_1,\ldots,z_n\}}$ is infact $n=2^k$. Suppose for contradiction that $\phi \leftrightarrow \forall z_1 \ldots \forall z_n \psi$. Then by Lemma 4, $\phi \leftrightarrow \forall z_1 \cdots \forall z_{n-1} \phi|_{\{z_1,\ldots,z_{n-1}\}}$. Let $\{\sigma_0,\cdots,\sigma_{n-1}\}$ be the set of all 1-types.

Define $\phi = \exists x \forall y \bigwedge_{i=0}^{n-1} (\sigma_i(x) \to \neg \sigma_{(i+1) \bmod n}(y))$. It is easy to check that the semantic interpretation of ϕ implies that $M \models \phi$ if and only if there exists at least one 1-type σ_j which is *not* present in M. Now consider the structure M which has exactly one copy of each 1-type σ_i . Clearly, in every substructure of M which has size less than or equal to n-1, there exists at least one 1-type which is missing. Hence $M \models \forall z_1 \cdots \forall z_{n-1} \phi |_{\{z_1, \dots, z_{n-1}\}}$. However, $M \not\models \phi$ as all 1-types are present in M. This is a contradiction. Hence $\phi \not\leftrightarrow \forall z_1 \cdots \forall z_{n-1} \phi |_{\{z_1, \dots, z_{n-1}\}}$, and thus, the bound $n=2^{|\tau|}$ is optimal.

Lemma 17 Let τ be a purely relational vocabulary and ϕ be a sentence in $FO(\tau)$ s.t. (i) $\phi = \exists x_1 \ldots \exists x_k \forall y \psi(x_1, \ldots, x_k, y)$ where ψ is quantifier free and no \exists variable is compared with a \forall variable using equality (ii) $\phi \in PS$. Then ϕ is equivalent to $\varphi = \forall z_1 \ldots \forall z_n \phi|_{\{z_1, \ldots, z_n\}}$ where n is $2^{|\tau|}$.

Proof:

From Lemma 5, we have $\phi \to \varphi$. Therefore to prove the lemma, it suffices to show that if $M \models \varphi$, that is, every substructure of M with size atmost n is a model of ϕ , then infact $M \models \phi$. We prove it by contradiction, so assume that $M \models \varphi \land \neg \phi$. The main idea is to use M to come up with a structure which models ϕ , but which has a substructure which is a non-model of ϕ . This contradicts that $\phi \in PS$. [Note that |M| > n, since if $|M| \le n$ and $M \models \varphi$ then $M \models \phi$ as well.]

Consider M' to be a substructure of M which contains exactly one element of each 1-type present in M. Clearly $|M'| \leq n$ and hence $M' \models \phi$. Thus, there exists $a_1, \cdots, a_k \in M'$ such that for every $b \in M'$, $\psi(a_1, \cdots, a_k, b)$ holds. Construct an extension \bar{M} of M with k additional elements $\{z_1, \cdots, z_k\}$ such that (i) $\bar{M}(\{z_1, \cdots, z_k\})$ is isomorphic to $M'(\{a_1, \cdots, a_k\})$ via the isomorphism $\underline{f}(z_i) = a_i$

(ii) $\forall y \in M \ \psi(z_1, \cdots, z_k, y)$ holds. Such a structure \bar{M} exists because all r-types $(r \leq k+1)$ have been obtained by copying predicate values from other structures as now explained. The types in (i) are copied from M'. The types in (ii) are copied as they are from M' as follows: suppose $y_0 \in M'$ has the same 1-type as $y \in M$, then r-type $\{z_1, \cdots, z_k, y\}$ in \bar{M} (where r is the number of distinct elements present in $\{z_1, \cdots, z_k, y\}$) is obtained by having all propositional statements, $\alpha(z_1, \cdots, z_k, y)$ to have the same value in \bar{M} as $\alpha(a_1, \cdots, a_k, y_0)$ in M', where there is no equality between y and z_j in α . Then, $\psi(z_1, \cdots, z_k, y_0)$ is true in \bar{M} , as $\psi(a_1, \cdots, a_k, y_0)$ is true in M'. Also, since there are no equality comparisons between z_j and y in ψ , $\psi(z_1, \cdots, z_k, y)$ has the same value as $\psi(a_1, \cdots, a_k, y_0)$, even if y_0 was infact one of the a_j s itself. Thus, we have $\bar{M} \models \phi$ as $z_1, \cdots, z_k \in \bar{M}$ act as witnesses for x_1, \cdots, x_k in ϕ . However, $M \subseteq \bar{M}$ and $M \not\models \phi$. This contradicts that $\phi \in PS$. Hence, our original assumption that there exists M such that $M \models \varphi \land \neg \phi$ is incorrect. Then $\varphi \to \phi$.

We now make the following important observation given our results. Over the class of all finite structures and for purely relational vocabularies, the following hold:

- 1. Łoś-Tarski holds trivially for the Σ_1^0 and Π_1^0 fragments of FO. A Σ_1^0 sentence in PS is actually *valid*. There is nothing to do in the Π_1^0 case.
- 2. By Lemma 8, Łoś-Tarski holds for Π_2^0 .
- 3. The counterexample to Łoś-Tarski in the finite, given as a purely relational sentence ϕ after Lemma 14 in Section 5, is an $\exists \forall^4$ sentence. Then Łoś-Tarski fails in the finite for Σ^0_k and Π^0_k for all $k \geq 3$.
- 4. By Lemmas 16 and 17, for the $\exists \forall$ fragment (with equality) of Σ_2^0 and the $\exists^* \forall$ fragment (with restricted equality) of Σ_2^0 , Łoś-Tarski holds.

This then leaves open only the following cases to investigate for Łoś-Tarski in the finite for purely relational vocabularies.

- 1. Full $\exists^* \forall$ fragment (in particular, the 'with-equality' case)
- $2. \exists^* \forall^2$

- 3. $\exists^* \forall^3$
- 4. $\exists \forall^4$ without equality

Any resolution of all these cases would give a <u>complete characterization</u> of the dividing line in the class of prefix fragments of FO, over purely relational vocabularies, between those prefix fragments for which Łoś-Tarski holds in the finite and those for which it does not!

We are currently trying to see if Lemmas 16 and 17 go through for relational vocabularies too (constants permitted). If so, then observing that the counterexample χ mentioned in the proof of Lemma 14 is a $\exists \forall^3$ sentence, the only cases left to investigate would be the above cases of (1) and (2) and finally the $\exists \forall^3$ fragment without equality. With any resolution of these cases, we would get a complete characterization of the dividing line in the class of prefix fragments of FO, over relational vocabularies, between those prefix fragments for which Łoś-Tarski holds in the finite and those for which it does not.

7 Proof of Theorem 4

We first introduce some notations. Given a vocabulary τ , we denote by τ_k , the vocabulary obtained by expanding τ with k-fresh constants, say c_1,\ldots,c_k . Given a τ -structure M and k elements b_1,\ldots,b_k from M, we denote by (M,b_1,\ldots,b_k) , the τ_k -structure whose τ -reduct is M and in which the constant c_i is interpreted as b_i for $1 \leq i \leq k$. Finally, for a τ -structure M, we denote by |M|, the power of M, i.e. the cardinality of the universe of M.

We begin with the following definition.

Definition 3 (k-cover) Given a τ -structure M, we call a set K of τ -structures as a k-cover of M if (i) $N \subseteq M$ for each $N \in K$ (ii) the union of the universes of the elements of K is the universe of M and (iii) for every atmost k-sized subset S of the universe of M, there exists an element of K containing S. We call M as the union of K and denote M as $\bigcup K$.

Note that given M, there always exists a k-cover of it - choose the set K above as $\{M\}$.

Definition 4 (Preservation under k-covers) A $FO(\tau)$ -sentence ϕ is said to be preserved under k-covers, if for all τ -structures M and all k-covers K of M, if every structure in K satisfies ϕ , then M satisfies ϕ .

We will assume familiarity with the notion of saturations described in [5] and recall now the following theorems from [5] which we will use subsequently.

Proposition 1 (A special case of Proposition 5.1.1(iii) in [5]) Given an infinite cardinal λ and a λ -saturated structure M, for every k-tuple (a_1, \ldots, a_k) of elements from M where $k \in \mathbb{N}$, (M, a_1, \ldots, a_k) is also λ -saturated.

Proposition 2 (Proposition 5.1.2(ii) in [5]) M is finite iff M is λ -saturated for all cardinals λ .

Theorem 11 (A special case of Lemma 5.1.4 in [5]) Let τ be a finite vocabulary, λ be an infinite cardinal and M be a τ -structure such that $\omega \leq |M| \leq 2^{\lambda}$. Then there is a β -saturated elementary extension of M for $\beta \geq \lambda$.

Theorem 12 (Lemma 5.2.1 in [5]) Given τ -structures M and N and a cardinal λ , suppose that (i) M is λ -saturated (ii) $\lambda \geq |N|$ and (iii) every existential sentence true in N is also true in M. Then N is embeddable in M.

Putting Theorem 11 and Proposition 2 together we get the following.

Corollary 3 For every τ -structure M, there exists a β -saturated elementary extension of M for some cardinal $\beta \geq \omega$.

Towards our syntactic characterization, we first prove the following.

Lemma 18 Given a finite vocabulary τ , consider a $FO(\tau)$ -sentence ϕ which is preserved under k-covers and let Γ be the set of all $\forall^k \exists^*$ consequences of ϕ . Then for all infinite cardinals λ , for every λ -saturated structure M, if $M \models \Gamma$, then $M \models \phi$.

Proof:

If ϕ is either unsatisfiable or valid, then the result is immediate.

Else, consider M satisfying the assumptions above. To show that $M \models \phi$, it suffices to show that for every atmost k-sized subset S of the universe of M, there is a substructure M_s of M containing S such that M_s models ϕ . Then the set $K = \{M_s | S \text{ is an atmost } k\text{-sized subset of the universe of } M\}$ forms a k-cover of M. Further since ϕ is preserved under k-covers, $M \models \phi$.

Let a_1,\ldots,a_k be the elements of a subset S of the universe of M. To show the existence of M_s , it suffices to show that there exists a τ_k -structure N s.t. (i) N is of power atmost λ (ii) every \exists^* sentence true in N is also true in (M,a_1,\ldots,a_k) (ii) $N\models\phi$. Since M is λ -saturated, by Proposition 1, (M,a_1,\ldots,a_k) is also λ -saturated. Then from Theorem 12, N is embeddable into (M,a_1,\ldots,a_k) . Then the τ -reduct of the copy of N in (M,a_1,\ldots,a_k) can be taken to be M_s referred to above.

We now show the existence of N to complete the proof.

Let P be the set of all \forall^* sentences of $FO(\tau_k)$ which are true in (M, a_1, \ldots, a_k) . Consider the set $T = \{\phi\} \cup P$. Suppose T is unsatisfiable. Then by Compactness theorem, there is a finite subset of T which is unsatisfiable. Since P is closed under taking finite conjunctions and since each of P and ϕ is satisfiable, there exists a sentence ψ in P s.t. $\{\phi,\psi\}$ is unsatisfiable. Then $\phi \to \neg \psi$. Now ϕ is a $FO(\tau)$ sentence while ψ is a $FO(\tau_k)$ sentence. Then by \forall -introduction, $\phi \to \varphi$ where $\varphi = \forall x_1 \ldots \forall x_k \neg \psi[c_1 \mapsto x_1; \ldots; c_k \mapsto x_k]$ where x_1, \ldots, x_k are k fresh variables and $c_i \mapsto x_i$ denotes replacement of c_i by x_i . Now note that since ψ is a \forall^* sentence, $\neg \psi$ is a \exists^* sentence (in $FO(\tau_k)$) and hence φ is a $\forall^k \exists^*$ sentence (in $FO(\tau)$). Then $\varphi \in \Gamma$ so that $M \models \varphi$. Then $(M, a_1, \ldots, a_k) \models \neg \psi$. This contradicts the fact that $\psi \in P$.

Then T is satisfiable. By Löwenheim-Skolem theorem, there is a model N of T of power atmost λ . Since N models every \forall^* sentence true in (M, a_1, \ldots, a_k) , it follows that every \exists^* sentence true in N is true in (M, a_1, \ldots, a_k) . Finally, since $N \models \phi, N$ is indeed as desired.

Theorem 13 Given a finite vocabulary τ , a $FO(\tau)$ -sentence ϕ is preserved under k-covers iff it is equivalent to a $\forall^k \exists^*$ sentence.

Proof:

Let Γ be the set of all $\forall^k\exists^*$ consequences of ϕ . It is easy to see that $\phi \to \Gamma$. For the converse direction, suppose $M \models \Gamma$. By Corollary 3, there is a β -saturated elementary extension M^+ of M for some $\beta \geq \omega$. Then $M^+ \models \Gamma$. Then from Lemma 18, it follows that $M^+ \models \phi$. Since M^+ is elementarily equivalent to M, we have that $M \models \phi$. Then $\Gamma \to \phi$ and hence $\phi \leftrightarrow \Gamma$. By Compactness theorem, ϕ is equivalent to a finite conjunction of sentences of Γ . Since Γ is closed under finite conjunctions, ϕ is equivalent to a $\forall^k\exists^*$ sentence.

We now prove Theorem 4.

Theorem 4 Given a finite vocabulary τ , a $FO(\tau)$ sentence ϕ is in PSC(B) iff it is equivalent to a $\exists^B \forall^*$ sentence.

Proof: We infer from Theorem 13 the following equivalences.

 ϕ is equivalent to a $\exists^B \forall^*$ sentence iff

 $\neg \phi$ is equivalent to a $\forall^B \exists^*$ sentence iff

For all τ -structures M and all B-covers K of M, if $\forall N \in K, \ N \models \neg \phi$, then $M \models \neg \phi$ iff

For all τ -structures M and all B-covers K of M, if $M \models \phi$ then $\exists N \in K, N \models \phi$

Assume $\phi \in PSC(B)$. Suppose K is a B-cover of M and that $M \models \phi$. Since $\phi \in PSC(B)$, there exists a core C of M of size atmost B. Then by definition of B-cover, there exists $N \in K$ s.t. (i) N contains C and (ii) $N \subseteq M$. Then since C is a core of M, $N \models \phi$ by definition of PSC(B). Then by the equivalences shown above, ϕ is equivalent to a $\exists^B \forall^*$ sentence. It is easy to see that an $\exists^B \forall^*$ sentence is in PSC(B).

8 Conclusion and Future Work

For future work, we would like to investigate cases for which combinatorial proofs of Theorem 4 can be obtained. This would potentially improve our understanding of the conditions under which combinatorial proofs can be obtained for the Łoś-Tarski theorem as well. An important direction of future work is to investigate whether Theorem 4 holds for important classes of finite structures for which the Łoś-Tarski theorem holds. Examples of such classes include those considered by Atserias et al. in [2]. We have also partially investigated how preservation theorems can be used to show FO inexpressibility for many typical examples (see [12]). We would like to pursue this line of

work as well in future.

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References

- [1] N. Alechina and Y. Gurevich. Syntax vs. semantics on finite structures. In *Structures in Logic and Computer Science*. A Selection of Essays in Honor of A. Ehrenfeucht, pages 14–33. Springer, 1997.
- [2] A. Atserias, A. Dawar, and M. Grohe. Preservation under extensions on well-behaved finite structures. *SIAM J. Comput.*, 38(4):1364–1381, 2008.
- [3] A. Atserias, A. Dawar, and P. G. Kolaitis. On preservation under homomorphisms and unions of conjunctive queries. *J. ACM*, 53(2):208–237, 2006.
- [4] J. R. Büchi. Weak second-order arithmetic and finite automata. Z. Math. Logik Grundlagen Math. 6, pages 66–92, 1960.
- [5] C. C. Chang and H. J. Keisler. *Model Theory*. Elsevier Science Publishers, 3rd edition, 1990.
- [6] A. Dawar, M. Grohe, S. Kreutzer, and N. Schweikardt. Model theory makes formulas large. In *ICALP*, pages 913–924, 2007.
- [7] Y. Gurevich. Toward logic tailored for computational complexity. In *COMPU-TATION AND PROOF THEORY*, pages 175–216. Springer, 1984.
- [8] L. Libkin. Elements of Finite Model Theory. Springer, 2004.
- [9] E. Rosen. Some aspects of model theory and finite structures. *Bulletin of Symbolic Logic*, 8(3):380–403, 2002.
- [10] B. Rossman. Homomorphism preservation theorems. J. ACM, 55(3), 2008.
- [11] B. Rossman. Personal Communication. 2012.
- [12] A. Sankaran, N. Limaye, A. Sundararaman, and S. Chakraborty. *Using Preservation Theorems for Inexpressibility Results in First Order Logic*. Technical report, 2012. URL: http://www.cfdvs.iitb.ac.in/reports/index.php.