Non-Termination Sets of Simple Linear Loops

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Abstract. A simple linear loop is a simple while loop with linear assignments and linear loop guards. If a simple linear loop has only two program variables, we give a complete algorithm for computing the set of all the inputs on which the loop does not terminate. For the case of more program variables, we show that the non-termination set cannot be described by Tarski formulae in general.

Keywords: Simple linear loop, termination, non-termination set, eigenvalue, Tarski formula

1 Introduction

Termination of programs is an important property of programs and one of the main research topics in the field of program verification. It is well known that the following so-called "uniform halting problem" is undecidable in general.

Using only a finite amount of time, determine whether a given program will always finish running or could execute forever.

However, there are some well known techniques for deciding termination of some special kinds of programs. A popular technique is to use ranking functions. A ranking function for a loop maps the values of the loop variables to a well-founded domain; further, the values of the map decrease on each iteration. A linear ranking function is a ranking function that is a linear combination of the loop variables and constants. Some methods for the synthesis of ranking functions and some heuristics concerning how to automatically generate linear ranking functions for linear programs have been proposed, for example, in Colón and Sipma [3], Dams et al. [4] and

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Podelski and Rybalchenko [6]. Podelski and Rybalchenko [6] provided an efficient and complete synthesis method based on linear programming to construct linear ranking functions. Chen et al. [2] proposed a method to generate nonlinear ranking functions based on semi-algebraic system solving. The existence of ranking function is only a sufficient condition on the termination of a program. There are programs, which terminate, but do not have ranking functions. Another popular technique based on well-orders, presented in Lee et al. [5], is size-change principle. The well-founded data can ensure that there are no infinitely descents, which guarantees termination of programs.

For linear loops, some other methods based on calculating eigenvectors of matrices have been proposed. Tiwari [7] proved that the termination problem of a class of linear programs (simple loops with linear loop conditions and updates) over the reals is decidable through Jordan form and eigenvector computation. Braverman [1] proved that it is also decidable over the integers. Xia et al. [8] considered the termination problems of simple loops with linear updates and polynomial loop conditions, and proved that the termination problem of such loops over the integers is undecidable. In [9], Xia et al. provided a novel symbolic decision procedure for termination of simple linear loops, which is as efficient as the numerical one given in [7].

A counter-example to termination is an infinite program execution. In program verification, the search for counter-examples to termination is as important as the search for proofs of termination. In fact, these are the two folds of termination analysis of programs. Gupta et al. [10] proposed a method for searching counter-examples to termination, which first enumerates lasso-shaped candidate paths for counter-examples and proves the feasibility of a given lasso by solving the existence of a recurrent set as a template-based constraint satisfaction problem. Gulwani et al. [11] proposed a constraint-based approach to a wide class of program analyses and weakest precondition and strongest postcondition inference. The approach can be applied to generating most-general counter-examples to termination.

In this paper, we consider the set of all inputs on which a given program does not terminate. The set is called NT throughout the paper. For simple linear loops, we are interested in whether the NT is decidable and how to compute it if it is decidable. Similar problems was also considered in [12]. Our contributions in this paper are as follows. First, for homogeneous linear loops (see Section 2 for the definition) with only two program variables, we give a complete algorithm for computing the NT. For the case of more program variables, we show that the NT cannot be described by Tarski formulae in general.

The rest of this paper is organized as follows. Section 2 introduces some notations and basic results on simple linear loops. Section 3 presents an algorithm for computing the NT of homogeneous linear loops with only two program variables. The correctness of the algorithm is proved by a series of lemmas. For linear loops with more than two program variables, it is proved in Section 4 that the NT is not a semi-algebraic set in general, i.e., it cannot be described by Tarski formulae in general. The paper is concluded in Section 5.

2 Preliminaries

In this paper, the domain of inputs of programs is $\mathbb{R}$, the field of real numbers. A *simple linear loop* in general form over $\mathbb{R}$ can be formulated as

$$\texttt{P1}: \quad \text{while } \ (B\boldsymbol{x} > \boldsymbol{b}) \ \{\boldsymbol{x} := A\boldsymbol{x} + \boldsymbol{c}\}$$

where $\boldsymbol{b}, \boldsymbol{c}$ are real vectors, $A_{n\times n}, B_{m\times n}$ are real matrices. $B\boldsymbol{x} > \boldsymbol{b}$ is a conjunction of m linear inequalities in $\boldsymbol{x}$ and $\boldsymbol{x} := A\boldsymbol{x} + \boldsymbol{c}$ is a linear assignment on the program variables $\boldsymbol{x}$.

Definition 1. [7] *The* non-termination set *of a program is the set of all inputs on which the program does not terminate. It is denoted by* NT *in this paper.*

In particular,

$$\mathrm{NT}(\texttt{P1}) = \{\boldsymbol{x} \in \mathbb{R}^n | \texttt{P1} \text{ does not terminate on } \boldsymbol{x}\} \ .$$

We list some related results in [7].

Proposition 1. [7] *For a simple linear loop* `P1`*, the following is true.*

- *The termination of* `P1` *is decidable.*

– *If A has no positive eigenvalues, the* NT *is empty.*
– *The* NT *is convex.*

In this paper, only the following *homogeneous case* is considered.

$$\texttt{P2}: \quad \text{while } (B\boldsymbol{x} > 0)\ \{\boldsymbol{x} := A\boldsymbol{x}\}\ .$$

Let $B_1, \ldots, B_m$ be the rows of B. Consider the following loops

$$L_i: \quad \text{while } (B_i\boldsymbol{x} > 0)\ \{\boldsymbol{x} := A\boldsymbol{x}\}\ .$$

Obviously, NT(P2)=$\bigcap_{i=1}^{m}$ NT(L_i). Therefore, without loss of generality, we assume throughout this paper that $m = 1$, *i.e.*, there is only one inequality as the loop guard. The following is a simple example of such loops.

$$\text{while } (4x_1 + x_2 > 0) \quad \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}\ .$$

That is $B = (4, 1), A = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}$.

3 Two-variable case

To make things clear, we restate the problem for this two-variable case as follows.

For a given homogeneous linear loop P2 *with exactly two program variables and only one inequality as the loop guard, compute* NT(P2).

For simplicity, we denote the program variables by x_1, x_2 and use NT instead of NT(P2) in this section. If $\boldsymbol{\alpha}$ is a non-zero point in the plane, we denote by $\overrightarrow{\boldsymbol{\alpha}}$ a ray starting from the origin of plane and going through the point $\boldsymbol{\alpha}$.

Proposition 2. NT *must be one of the following:*
(1) an empty set;
(2) a ray starting from the origin;
(3) a sector between two rays starting from the origin.

Proof. We view an input (x_1, x_2) as a point in the real plane with origin O. If there exists a point $M(x_1, x_2) \in$ NT, any point $\boldsymbol{P}$ on the ray $\overrightarrow{\boldsymbol{OM}}$ can be written as $\boldsymbol{P} = kM = (kx_1, kx_2)$ for a positive number k. So $BA^n(kx_1, kx_2)^T = k^n BA^n(x_1, x_2)^T > 0$ for any $n \in \mathbb{N}$. That means $\boldsymbol{P} \in$ NT. Therefore, it is clear from the item 3 of Proposition 1 that the conclusion is true.

By the above proposition, the key point for computing the NT is to compute the ray(s) which is (are) the boundary of NT. We give the following algorithm to compute the ray(s) (and thus the NT) for `P2` if the NT is not empty. The algorithm, as can be expected, is mainly based on the computation of eigenvalues and eigenvectors of A. The correctness of our algorithm will be proved by a series of lemmas following the algorithm.

Algorithm 1. `NonTermination`

Input: Matrices $A_{2\times 2}$ and $B_{1\times 2}$.
Output: The NT of `P2` with A and B.

if $A = \mathbf{0}$ *or* $B = \mathbf{0}$ **then**
    return $\emptyset$;
Compute the eigenvalues of A and denote them by λ_1, λ_2;
if $\lambda_1 \not> 0 \wedge \lambda_2 \not> 0$ **then**
    return $\emptyset$; // Proposition 1
Take $\boldsymbol{\alpha_0} \in \mathbb{R}^2 \setminus \{0\}$ such that $B\boldsymbol{\alpha_0} = 0$ and $BA\boldsymbol{\alpha_0} \geq 0$;
if $BA\boldsymbol{\alpha_0} = 0$ **then**
    choose $\boldsymbol{\xi}$ such that $B\boldsymbol{\xi} > 0$
    if $B(A\boldsymbol{\xi}) > 0$ **then**
        return $\{\boldsymbol{x} | \boldsymbol{x} \in \mathbb{R}^2, B\boldsymbol{x} > 0\}$ // Lemma 4
    else
        return $\emptyset$ // Lemma 5
if $\lambda_1 = 0 \vee \lambda_2 = 0$ **then**
    return $\{\boldsymbol{x} | \boldsymbol{x} \in \mathbb{R}^2, B\boldsymbol{x} > 0, BA\boldsymbol{x} > 0\}$; // Lemma 6
Suppose $\lambda_1 \geq \lambda_2$
if $\lambda_1 \geq \lambda_2 > 0$ **then**
    choose an eigenvector $\boldsymbol{\beta_2}$ related to λ_2 such that $B\boldsymbol{\beta_2} \geq 0$;
    return $\{\boldsymbol{x} | \boldsymbol{x} = k_1\boldsymbol{\alpha_0} + k_2\boldsymbol{\beta_2}, k_1 \geq 0, k_2 > 0\}$; // Lemmas 7 and 8
if $\lambda_1 > 0 \wedge \lambda_2 < 0$ **then**
    if $\lambda_1 \geq |\lambda_2|$ **then**
        let $\boldsymbol{\alpha_{-1}} = A^{-1}\boldsymbol{\alpha_0}$ and return $\{\boldsymbol{x} | \boldsymbol{x} = k_1\boldsymbol{\alpha_0} + k_2\boldsymbol{\alpha_{-1}}, k_1 > 0, k_2 > 0\}$;
    if $\lambda_1 < |\lambda_2|$ **then**
        choose an eigenvector $\boldsymbol{\beta}$ related to λ_1 such that $B\boldsymbol{\beta} > 0$ and
        return $\{\boldsymbol{x} | \boldsymbol{x} = k\boldsymbol{\beta}, k > 0\}$ // Lemma 10

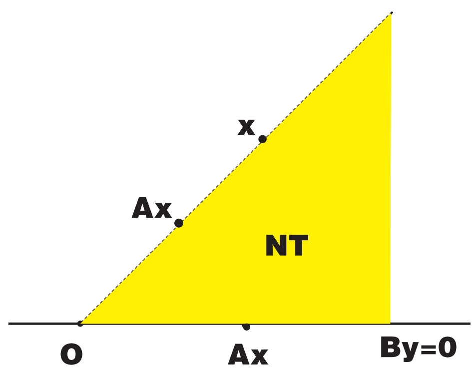

Fig. 1. Lemma 1

Lemma 1. *Suppose* NT *is not empty and* ∂NT *is the boundary of* NT*. If* $\boldsymbol{x} \in \partial\mathrm{NT}$ *and* $B\boldsymbol{x} \neq 0$*, then* $A\boldsymbol{x} \in \partial\mathrm{NT}$*.*

Proof. Obviously, B is a linear map from $\mathbb{R}^2$ to $\mathbb{R}$. Because $B\boldsymbol{y} > 0$ for all $\boldsymbol{y} \in \mathrm{NT}$, we have $B\boldsymbol{x} \geq 0$. And thus $B\boldsymbol{x} > 0$ by the assumption that $B\boldsymbol{x} \neq 0$. Hence, there exists an open ball $o_1(\boldsymbol{x}, r_1)$ such that $B\boldsymbol{y} > 0$ for all $\boldsymbol{y} \in o_1(\boldsymbol{x}, r_1)$.

Let F be the linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that $F(\boldsymbol{y}) = A\boldsymbol{y}$ for any $\boldsymbol{y} \in \mathbb{R}^2$ and hence F is continuous. So for any neighborhood $o(A\boldsymbol{x}, r)$ of $A\boldsymbol{x}$, there exists a positive real number r_2 such that $o_2(\boldsymbol{x}, r_2) \subseteq o_1(\boldsymbol{x}, r_1)$ and $F(o_2(\boldsymbol{x}, r_2)) \subseteq o(A\boldsymbol{x}, r)$. Because $\boldsymbol{x} \in \partial\mathrm{NT}$, there exist $\boldsymbol{y}, \boldsymbol{z} \in o_2(\boldsymbol{x}, r_2)$ such that $\boldsymbol{y} \in \mathrm{NT}$ and $\boldsymbol{z} \notin \mathrm{NT}$. Then $A(\boldsymbol{y})$, $A(\boldsymbol{z}) \in o(A\boldsymbol{x}, r)$, $A(\boldsymbol{y}) \in \mathrm{NT}$ and $A(\boldsymbol{z}) \notin \mathrm{NT}$. It is followed that there are both terminating and non-terminating inputs in any neighborhood of $A\boldsymbol{x}$. Therefore, $A\boldsymbol{x} \in \partial\mathrm{NT}$.

Lemma 2. *Suppose* NT *is neither empty nor a ray and* $\partial\mathrm{NT} \cap \{\boldsymbol{x} | B\boldsymbol{x} = 0\} = \{(0,0)\}$*. If* $B\boldsymbol{y} = 0$ *and* $BA\boldsymbol{y} > 0$*, then* $A\boldsymbol{y} \in \mathrm{NT}$*.*

Proof. By Proposition 2, ∂NT consists of two rays. Let l_1, l_2 be the two rays. Since neither l_1 nor l_2 is on $Bx = 0$, l_1 and l_2 are not collinear. So we can choose two points $\boldsymbol{z} \in l_1$ and $\boldsymbol{v} \in l_2$ such that $B\boldsymbol{z} > 0$, $B\boldsymbol{v} > 0$ and $\boldsymbol{y} = t_1\boldsymbol{z} + t_2\boldsymbol{v}$ for some $t_1 \in \mathbb{R}, t_2 \in \mathbb{R}$. By Lemma 1, $A\boldsymbol{z}$ and $A\boldsymbol{v}$ must be on the boundary of NT, i.e., l_1 or l_2. Thus, we have at most four possible cases as follows.

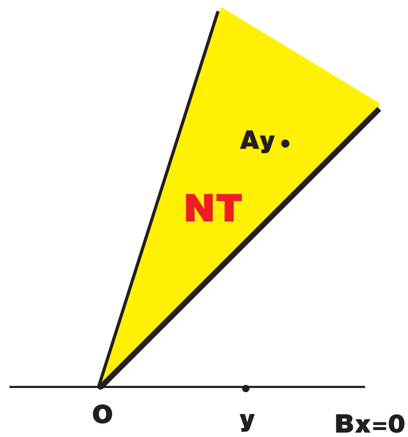

Fig. 2. Lemma 2

(1) $A\boldsymbol{z} = k_1\boldsymbol{z}, A\boldsymbol{v} = k_2\boldsymbol{v}$, (i.e., $A\boldsymbol{z} \in l_1, A\boldsymbol{v} \in l_2$)
(2) $A\boldsymbol{z} = k_1\boldsymbol{z}, A\boldsymbol{v} = k_2\boldsymbol{z}$, (i.e., $A\boldsymbol{z} \in l_1, A\boldsymbol{v} \in l_1$)
(3) $A\boldsymbol{z} = k_1\boldsymbol{v}, A\boldsymbol{v} = k_2\boldsymbol{v}$, (i.e., $A\boldsymbol{z} \in l_2, A\boldsymbol{v} \in l_2$)
(4) $A\boldsymbol{z} = k_1\boldsymbol{v}, A\boldsymbol{v} = k_2\boldsymbol{z}$, (i.e., $A\boldsymbol{z} \in l_2, A\boldsymbol{v} \in l_1$)

where $k_1 > 0, k_2 > 0$.

Case (1). Because $B\boldsymbol{y} = t_1B\boldsymbol{z} + t_2B\boldsymbol{v} = 0$ and

$$BA\boldsymbol{y} = BA(t_1\boldsymbol{z} + t_2\boldsymbol{v}) = t_1k_1B\boldsymbol{z} + t_2k_2B\boldsymbol{v} > 0,$$

we have $t_1t_2 < 0$. Without loss of generality, assume that $t_1 > 0$ and $t_2 < 0$. We denote $t_1B\boldsymbol{z}$ by P. Note that $P > 0$ and $t_2B\boldsymbol{v} = -P$. Since $BA\boldsymbol{y} = (k_1 - k_2)P > 0$, we have $k_1 > k_2 > 0$ and

$$BA^n(A\boldsymbol{y}) = k_1^{n+1}t_1B\boldsymbol{z} + k_2^{n+1}t_2B\boldsymbol{v} = k_1^{n+1}P - k_2^{n+1}P > 0$$

for any $n \in \mathbb{N}$. By the definition of NT, $A\boldsymbol{y} \in$ NT.

Case (2). Because $BA\boldsymbol{y} = (t_1k_1 + t_2k_2)B\boldsymbol{z} > 0$, we have

$$BA^n(A\boldsymbol{y}) = k_1^n(t_1k_1 + t_2k_2)B\boldsymbol{z} > 0$$

for any $n \in \mathbb{N}$. By the definition of NT, we have $A\boldsymbol{y} \in$ NT.

Case (3). Similarly as Case (2), we can prove $A\boldsymbol{y} \in$ NT.

Case (4). We shall show that this case cannot happen. Let

$$S = \{\boldsymbol{x} | \boldsymbol{x} = r_1\boldsymbol{y} + r_2A\boldsymbol{y}, r_1 > 0, r_2 > 0\}$$

be the sector between the two rays $\overrightarrow{\boldsymbol{y}}$ and $\overrightarrow{\boldsymbol{A}\boldsymbol{y}}$. For any $\boldsymbol{w} \in S$, we have $B\boldsymbol{w} = r_1 B\boldsymbol{y} + r_2 BA\boldsymbol{y} = r_2 BA\boldsymbol{y} > 0$.

Because

$$A^2\boldsymbol{y} = A(t_1k_1\boldsymbol{v} + t_2k_2\boldsymbol{z}) = t_1k_1k_2\boldsymbol{z} + t_2k_1k_2\boldsymbol{v} = k_1k_2\boldsymbol{y},$$

we have $A\boldsymbol{w} = r_1A\boldsymbol{y} + r_2A^2\boldsymbol{y} = r_1A\boldsymbol{y} + r_2k_1k_2\boldsymbol{y} \in S$. Therefore, $\boldsymbol{w} \in \mathrm{NT}$ and $S \subseteq \mathrm{NT}$. As $\overrightarrow{\boldsymbol{y}}$ is a boundary of S and $B\boldsymbol{y} = 0$, $\overrightarrow{\boldsymbol{y}}$ is contained in $\partial\mathrm{NT}$, which contradicts with the assumption of the lemma. So (4) cannot happen.

In summary, $A\boldsymbol{y} \in \mathrm{NT}$.

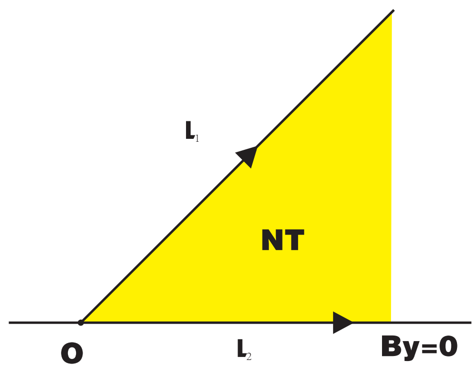

Fig. 3. Lemma 3

Lemma 3. *If $\partial\mathrm{NT}$ is composed of two rays l_1 and l_2, then either l_1 or l_2 is on $B\boldsymbol{x} = 0$.*

Proof. Assume neither l_1 nor l_2 is on $B\boldsymbol{x} = 0$. Choose a point $\boldsymbol{y}$ such that $\boldsymbol{y} \neq \boldsymbol{0}$, $B\boldsymbol{y} = 0$ and $BA\boldsymbol{y} \geq 0$.

Suppose $BA\boldsymbol{y} = 0$. As NT is not empty, there exists $\boldsymbol{z} \in \mathrm{NT}$. Hence $A\boldsymbol{y}$ can be rewritten as $A\boldsymbol{y} = h_1\boldsymbol{z} + h_2\boldsymbol{y}$ for some $h_1 \in \mathbb{R}, h_2 \in \mathbb{R}$. As a result of $BA\boldsymbol{y} = h_1B\boldsymbol{z} + h_2B\boldsymbol{y} = h_1B\boldsymbol{z} = 0$, $h_1 = 0$. Note that

$$A^n\boldsymbol{y} = h_2^n\boldsymbol{y}, BA^n\boldsymbol{y} = h_2^nB\boldsymbol{y} = 0 \ . \tag{1}$$

According to Eq.(1) and $\boldsymbol{z} \in \mathrm{NT}$, we have $BA^n(k_1\boldsymbol{z} + k_2\boldsymbol{y}) = k_1BA^n\boldsymbol{z} + k_2BA^n\boldsymbol{y} = k_1BA^n\boldsymbol{z} > 0$ for any $k_1 > 0, n \in \mathbb{N}$. Hence

$\{\boldsymbol{x}|\boldsymbol{x} = k_1\boldsymbol{z} + k_2\boldsymbol{y}, k_1 > 0\} \subseteq \mathrm{NT}$. Therefore, $\{\boldsymbol{x}|B\boldsymbol{x} = 0\} = \partial\mathrm{NT}$, which contradicts with the assumption.

If $BA\boldsymbol{y} > 0$, $A\boldsymbol{y} \in \mathrm{NT}$ follows from Lemma 2. Let $S = \{\boldsymbol{x}|k_1\boldsymbol{y} + k_2A\boldsymbol{y}, k_1 > 0, k_2 > 0\}$. And we have $BA^n\boldsymbol{z} = k_1BA^ny + k_2BA^{n+1}\boldsymbol{y} > 0$ for any $n \in \mathbb{N}$, $\boldsymbol{z} \in S$. Thus $\boldsymbol{z} \in \mathrm{NT}$ and $S \subseteq \mathrm{NT}$. By the method of choosing $\boldsymbol{y}$, $\overrightarrow{\boldsymbol{y}} \subseteq \partial\mathrm{NT}$. That means $\overrightarrow{\boldsymbol{y}}$ is l_1 or l_2, which contradicts with the assumption.

Lemma 4. *Suppose A has positive eigenvalues and has an eigenvector $\boldsymbol{\alpha}$ satisfying $B\boldsymbol{\alpha} = 0$. If $\boldsymbol{\xi}$ is a vector such that $B\boldsymbol{\xi} > 0$ and $BA\boldsymbol{\xi} > 0$, then $\mathrm{NT} = \{\boldsymbol{x}|B\boldsymbol{x} > 0\}$.*

Proof. For any $\boldsymbol{y} \in \{\boldsymbol{x}|B\boldsymbol{x} > 0\}$, it can be written as $\boldsymbol{y} = k_1\boldsymbol{\xi} + k_2\boldsymbol{\alpha}$ for some $k_1 \in \mathbb{R}, k_2 \in \mathbb{R}$. As $B\boldsymbol{y} = k_1B\boldsymbol{\xi} + k_2B\boldsymbol{\alpha} = k_1B\boldsymbol{\xi} > 0$, we have $k_1 > 0$. Thus $BA\boldsymbol{y} = k_1BA\boldsymbol{\xi} + k_2BA\boldsymbol{\alpha} = k_1BA\boldsymbol{\xi} > 0$ and $A\boldsymbol{y} \in \{\boldsymbol{x}|B\boldsymbol{x} > 0\}$. By the definition of NT, we have $\{\boldsymbol{x}|B\boldsymbol{x} > 0\} \subseteq \mathrm{NT}$ and hence $\mathrm{NT} = \{\boldsymbol{x}|B\boldsymbol{x} > 0\}$.

Lemma 5. *Suppose A has positive eigenvalues and has an eigenvector $\boldsymbol{\alpha}$ satisfying $B\boldsymbol{\alpha} = 0$. If there is a vector $\boldsymbol{\xi}$ such that $B\boldsymbol{\xi} > 0$ and $BA\boldsymbol{\xi} \le 0$, then $\mathrm{NT} = \emptyset$.*

Proof. For any $\boldsymbol{y} \in \{\boldsymbol{x}|B\boldsymbol{x} > 0\}$, it can be written as $\boldsymbol{y} = k_1\boldsymbol{\alpha} + k_2\boldsymbol{\xi}$ for some $k_1 \in \mathbb{R}, k_2 \in \mathbb{R}$. Since $B\boldsymbol{y} = k_2B\boldsymbol{\xi} > 0$, we have $k_2 > 0$. And because $BA\boldsymbol{y} = k_2BA\boldsymbol{\xi} \le 0$, $\mathrm{NT} = \emptyset$.

Lemma 6. *Suppose A has a positive eigenvalue and a zero eigenvalue. If $\boldsymbol{\gamma}$ is an eigenvector related to the positive eigenvalue such that $B\boldsymbol{\gamma} > 0$, then $\mathrm{NT} = \{\boldsymbol{x}|B\boldsymbol{x} > 0, BA\boldsymbol{x} > 0\}$.*

Proof. Let $\boldsymbol{\beta}$ be an eigenvector with respect to eigenvalue 0 and λ be the positive eigenvalue. Let S be the set $\{\boldsymbol{x}|B\boldsymbol{x} > 0, BA\boldsymbol{x} > 0\}$. For any $\boldsymbol{y} \in S$, it can be written as $k_1\boldsymbol{\beta} + k_2\boldsymbol{\gamma}$ for some $k_1 \in \mathbb{R}, k_2 \in \mathbb{R}$. We have $BA\boldsymbol{y} = k_2\lambda B\boldsymbol{\gamma} > 0$, thus $k_2 > 0$. Note that $BA^n\boldsymbol{y} = k_2\lambda^n\boldsymbol{\gamma} > 0$ for any $n \in \mathbb{N}$, hence $S \subseteq \mathrm{NT}$. Because $\{\boldsymbol{x}|B\boldsymbol{x} \le 0 \vee BA\boldsymbol{x} \le 0\} \cap \mathrm{NT} = \emptyset$, $\mathrm{NT} = \{\boldsymbol{x}|B\boldsymbol{x} > 0, BA\boldsymbol{x} > 0\}$.

Lemma 7. *Suppose A has two positive eigenvalues $\lambda_1 > \lambda_2 > 0$ and two eigenvectors $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ related to λ_1 and λ_2, respectively, such that $B\boldsymbol{\beta_1} > 0, B\boldsymbol{\beta_2} > 0$. If $\boldsymbol{\alpha}$ is a vector such that $B\boldsymbol{\alpha} = 0$ and $BA\boldsymbol{\alpha} > 0$, then $\mathrm{NT} = \{\boldsymbol{x}|\boldsymbol{x} = k_1\boldsymbol{\alpha} + k_2\boldsymbol{\beta_2}, k_1 \ge 0, k_2 > 0\}$.*

Proof. It is easy to know $\boldsymbol{\beta_1}, \boldsymbol{\beta_2} \in \mathrm{NT}$, thus NT is neither empty nor a ray. By Lemma 3 there is a $\overrightarrow{\boldsymbol{y}} \subseteq \partial\mathrm{NT}$ and $\boldsymbol{y}$ satisfies $B\boldsymbol{y} = 0$. Since for any $\boldsymbol{z} \in \partial\mathrm{NT}$, we have $BA\boldsymbol{z} \geq 0$. So $BA\boldsymbol{y} \geq 0$ and hence $\overrightarrow{\boldsymbol{\alpha}} = \overrightarrow{\boldsymbol{y}}$. In other word, $\overrightarrow{\boldsymbol{\alpha}}$ is one ray of $\partial\mathrm{NT}$. Let the other ray of $\partial\mathrm{NT}$ be l. As $-BA\boldsymbol{\alpha} < 0$, $\overrightarrow{-\boldsymbol{\alpha}}$ is not l. By Lemma 1, we have $Al \in \partial\mathrm{NT}$. So l is one of $\overrightarrow{\boldsymbol{\beta_1}}, \overrightarrow{\boldsymbol{\beta_2}}$ and $\overrightarrow{\boldsymbol{A^{-1}}\alpha}$. By directly checking, we know $\overrightarrow{\boldsymbol{\beta_2}}$ is l and so $\mathrm{NT} = \{\boldsymbol{x} | \boldsymbol{x} = k_1\boldsymbol{\alpha} + k_2\boldsymbol{\beta_2}, k_1 \geq 0, k_2 > 0\}$.

Lemma 8. *Assume that A has one positive eigenvalue λ with multiplicity 2 and only one eigenvector $\boldsymbol{\beta}$ satisfying $B\boldsymbol{\beta} > 0$. If $\boldsymbol{\alpha}$ is a vector such that $B\boldsymbol{\alpha} = 0$ and $BA\boldsymbol{\alpha} > 0$, then $\mathrm{NT} = \{\boldsymbol{x} | \boldsymbol{x} = h_1\boldsymbol{\alpha} + h_2\boldsymbol{\beta}, k_1 \geq 0, k_2 > 0\}$.*

Proof. By the theory of Jordan normal form in linear algebra, there exists a vector $\boldsymbol{\beta_1}$ such that $A\boldsymbol{\beta_1} = \boldsymbol{\beta} + \lambda\boldsymbol{\beta_1}$ and $\boldsymbol{\beta}$ and $\boldsymbol{\beta_1}$ are linearly independent.

Let $\boldsymbol{\alpha_1} = A\boldsymbol{\alpha}$. We claim that

$$\forall n \in \mathbb{N}.(BA^n\boldsymbol{\alpha_1} > 0 \wedge \exists h_2 > 0.(A^n\boldsymbol{\alpha_1} = h_1\boldsymbol{\beta} + h_2\boldsymbol{\beta_1})). \qquad (2)$$

To prove this claim we use induction on the value of n.

Suppose $\boldsymbol{\alpha} = h_1\boldsymbol{\beta} + h_2\boldsymbol{\beta_1}$. If $n = 0$, then $\boldsymbol{\alpha_1} = A\boldsymbol{\alpha} = (h_1\lambda + h_2)\boldsymbol{\beta} + h_2\lambda\boldsymbol{\beta_1}$. Because $B\boldsymbol{\alpha_1} = \lambda B\boldsymbol{\alpha} + h_2B\boldsymbol{\beta} = h_2B\boldsymbol{\beta} > 0$, we have $h_2 > 0$.

Now assume that the claim is true for $n-1$. Let $A^{n-1}\boldsymbol{\alpha_1} = h_1\boldsymbol{\beta} + h_2\boldsymbol{\beta_1}$ where $h_2 > 0$. Because $A^n\boldsymbol{\alpha_1} = A(A^{n-1}\boldsymbol{\alpha_1}) = (\lambda h_1 + h_2)\boldsymbol{\beta} + \lambda h_2\boldsymbol{\beta_1}$, we have $\lambda h_2 > 0$ and $BA^n\boldsymbol{\alpha_1} = \lambda BA^{n-1}\boldsymbol{\alpha_1} + h_2B\boldsymbol{\beta} > 0$. So the claim is true for any $n \in \mathbb{N}$ and we have $\boldsymbol{\alpha_1} \in \mathrm{NT}$.

Obviously, $\boldsymbol{\beta} \in \mathrm{NT}$ and $\boldsymbol{\beta}$ and $\boldsymbol{\alpha_1}$ are linearly independent, so NT is not a ray. By Lemma 3, $\overrightarrow{\boldsymbol{\alpha}} \subseteq \partial\mathrm{NT}$.

Let the other ray of $\partial\mathrm{NT}$ be l. As $-BA\boldsymbol{\alpha} < 0$, $\overrightarrow{-\boldsymbol{\alpha}}$ is not l. By Lemma 1, $Al = l$ or $Al = \overrightarrow{\boldsymbol{\alpha}}$. So l must be $\overrightarrow{\boldsymbol{\beta}}$ or $\overrightarrow{\boldsymbol{A^{-1}\alpha}}$. By directly checking, we know l is $\overrightarrow{\boldsymbol{\beta}}$ and thus $\mathrm{NT} = \{\boldsymbol{x} | \boldsymbol{x} = k_1\boldsymbol{\alpha} + k_2\boldsymbol{\beta}, k_1 \geq 0, k_2 > 0\}$.

Lemma 9. *Suppose A has a positive eigenvalue λ_1 and a negative eigenvalue λ_2 with $\lambda_1 \geq |\lambda_2|$ and two eigenvectors $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ related to λ_1 and λ_2, respectively, such that $B\boldsymbol{\beta_1} > 0, B\boldsymbol{\beta_2} > 0$. Suppose $\boldsymbol{\alpha}$ is a vector such that $B\boldsymbol{\alpha} = 0$ and $BA\boldsymbol{\alpha} > 0$. Let $\boldsymbol{\alpha_{-1}} = A^{-1}\boldsymbol{\alpha}$, $\boldsymbol{\alpha_1} = A\boldsymbol{\alpha}$. Then $\mathrm{NT} = \{k_1\boldsymbol{\alpha} + k_2\boldsymbol{\alpha_{-1}}, k_1 > 0, k_2 > 0\}$.*

Proof. Let $\boldsymbol{\alpha_{-1}} = h_1\boldsymbol{\beta_1} + h_2\boldsymbol{\beta_2}$. So $\boldsymbol{\alpha} = A\boldsymbol{\alpha_{-1}} = h_1\lambda_1\boldsymbol{\beta_1} + h_2\lambda_2\boldsymbol{\beta_2}$ and $\boldsymbol{\alpha_1} = A\boldsymbol{\alpha} = h_1\lambda_1^2\boldsymbol{\beta_1} + h_2\lambda_2^2\boldsymbol{\beta_2}$. Because $B\boldsymbol{\alpha} = 0$ and $B\boldsymbol{\alpha_1} > 0$, h_1, h_2 and $A\boldsymbol{\alpha_{-1}}$ are all positive.

Note that $\boldsymbol{\alpha_1} = (-\lambda_1\lambda_2)\boldsymbol{\alpha_{-1}} + (\lambda_1 + \lambda_2)\boldsymbol{\alpha}$ where $-\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 \geq 0$. Let $S = \{\boldsymbol{x}|\boldsymbol{x} = k_1\boldsymbol{\alpha} + k_2\boldsymbol{\alpha_{-1}},\ k_1 > 0,\ k_2 > 0\}$. Since $B\boldsymbol{y} = k_2B\boldsymbol{\alpha_{-1}} > 0$ and $A\boldsymbol{y} = (k_2 + k_1(\lambda_1 + \lambda_2))\boldsymbol{\alpha} - k_1\lambda_1\lambda_2\boldsymbol{\alpha_{-1}} \in S$ for any $\boldsymbol{y} \in S$, we have $\mathrm{NT} \supseteq S$.

Let $\boldsymbol{y} = k_1\boldsymbol{\alpha}+k_2\boldsymbol{\alpha_{-1}}$. Because $B\boldsymbol{y} = k_2B\boldsymbol{\alpha_{-1}} \leq 0$ for any $k_2 \leq 0$ and $BA\boldsymbol{y} = k_1B\boldsymbol{\alpha_1} \leq 0$ for any $k_1 \leq 0$, we have $\mathrm{NT} = S$.

Lemma 10. *Suppose A has a positive eigenvalue λ_1 and a negative eigenvalue λ_2 such that $\lambda_1 < |\lambda_2|$. If there are two eigenvectors $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ related to λ_1 and λ_2, respectively, such that $B\boldsymbol{\beta_1} > 0$ and $B\boldsymbol{\beta_2} > 0$, then $\mathrm{NT} = \{\boldsymbol{x}|\boldsymbol{x} = k\boldsymbol{\beta_1}, k > 0\}$.*

Proof. Consider any $\boldsymbol{\beta} = k_1\boldsymbol{\beta_1} + k_2\boldsymbol{\beta_2} \in \mathbb{R}^2$.

If $k_2 \neq 0$, because $A^n(k_1\boldsymbol{\beta_1} + k_2\boldsymbol{\beta_2}) = k_1\lambda_1^n\boldsymbol{\beta_1} + k_2\lambda_2^n\boldsymbol{\beta_2}$ and

$$BA^n(k_1\boldsymbol{\beta_1} + k_2\boldsymbol{\beta_2})BA^{n+1}(k_1\boldsymbol{\beta_1} + k_2\boldsymbol{\beta_2}) < 0$$

when n is large enough, $k_1\boldsymbol{\beta_1} + k_2\boldsymbol{\beta_2} \notin \mathrm{NT}$.

If $k_2 = 0$, obviously, $\mathrm{NT} \supseteq \{\boldsymbol{x}|\boldsymbol{x} = k\boldsymbol{\beta_1}, k > 0\}$ and $Bk\boldsymbol{\beta_1} \notin \mathrm{NT}$ for any $k \leq 0$.

So $\mathrm{NT} = \{\boldsymbol{x}|\boldsymbol{x} = k\boldsymbol{\beta_1}, k > 0\}$.

Now, the correctness of our algorithm `NonTermination` can be easily obtained as follows.

Theorem 1. *The algorithm* `NonTermination` *is correct.*

Proof. First, the termination of `NonTermination` is obvious because there are no loops and no iterations in it. Second, it is also clear that the algorithm discusses all the cases of eigenvalues of A, respectively. According to Lemmas 4-10 (each of them corresponds to a certain case in the algorithm as commented in the algorithm), the output of the algorithm in each case is correct.

Example 1. Compute the NT of the following loop.

$$\text{while } (4x_1 + x_2 > 0) \quad \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

Herein, $B = (4, 1), A = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}$.

The computation of `NonTermination` on the loop is:

Line 1. $B \neq 0$ and $A \neq 0$.

Line 4. A has a positive eigenvalue $-1 + \sqrt{17}$.

Line 6. Let $\boldsymbol{\alpha_0} = (-1, 4)^T, \boldsymbol{\alpha_1} = A\boldsymbol{\alpha_0} = (18, -4)^T$.

Line 7. $B\boldsymbol{\alpha_1} = 68 \neq 0$.

Line 13. The two eigenvalues of A are $-1 + \sqrt{17}, -1 - \sqrt{17}$, respectively. Neither of them is 0.

Line 19. A has two eigenvalues, of which one is positive and the other negative.

Line 20. The absolute value of the negative eigenvalue is greater than the positive eigenvalue.

Line 22. The eigenvector with respect to the positive eigenvalue is $\boldsymbol{\beta} = (1, \frac{\sqrt{17}+1}{4})^T$ and $B\boldsymbol{\beta} > 0$. Return $\{\boldsymbol{x} | \boldsymbol{x} = k\boldsymbol{\beta}, k > 0\}$.

4 More variables

Theorem 2. *In general,* NT *is not a semi-algebraic set.*

Remark 1. All Tarski formulae are in the form of conjunctions or/and disjunctions of polynomial equalities and/or inequalities, so, in other words, semi-algebraic sets are exactly the sets defined by Tarski formulae. By Theorem 2, we can conclude that the non-termination sets of linear loops with more than two variables cannot be defined by Tarski formulae in general.

Remark 2. It should be noticed that all polynomial invariants are semi-algebraic sets.

In order to prove the above theorem, we give an example to demonstrate its NT is not a semi-algebraic set.

Proposition 3. *Let a linear loop with three program variables be as follows.*

$$\texttt{P3} : \text{while } (x_1 + 2x_2 + x_3 \geq 0) \quad \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\}.$$

Then NT(`P3`) *is not a semi-algebraic set.*

The conclusion can be proved by using the following lemmas. For simplicity, NT(`P3`) is denoted by NT in this section.

Lemma 11. *Denote by τ the following set*

$$\{9(x_1^2 + x_2^2) - x_3^2 < 0, x_3 > 0\},$$

then $\tau \subseteq \mathrm{NT}$.

Proof. For any $(x_1, x_2, x_3) \in \tau$, we have $x_3 > 3|x_1|, x_3 > 3|x_2|$ and thus $x_1 + 2x_2 + x_3 > 0$. Because $A(x_1, x_2, x_3)^T = (2x_1, 3x_2, 5x_3)^T$ and $9(4x_1^2 + 9x_2^2) - 25x_3^2 < 0$, $A(x_1, x_2, x_3)^T \in \tau$. Therefore $\tau \subseteq \mathrm{NT}$.

Lemma 12. $\partial\mathrm{NT} \subseteq \mathrm{NT}$.

Proof. Because the loop guard is of the form $B(x_1, x_2, x_3)^T \geq 0$, NT is a closed set. So the conclusion is correct. Furthermore, for any $(x_1, x_2, x_3) \in \partial\mathrm{NT}, x_1 + 2x_2 + x_3 \geq 0$.

Lemma 13. *If* $(x_1, x_2, x_3) \in \mathrm{NT}$ *and* $A(x_1, x_2, x_3)^T \in \partial\mathrm{NT}$, *then* $(x_1, x_2, x_3) \in \partial\mathrm{NT}$.

Proof. Let $\boldsymbol{x} = (x_1, x_2, x_3)$. If the conclusion is not true, there exists a ball $o(\boldsymbol{x}, r) \subseteq \mathrm{NT}$. Because $A\boldsymbol{x}^T \in \partial\mathrm{NT}$, there exists $\boldsymbol{x'}$ such that $|A\boldsymbol{x} - \boldsymbol{x'}| < r$ and $\boldsymbol{x'}$ is not in NT.

Since $|A^{-1}\boldsymbol{x'} - \boldsymbol{x}| < |\boldsymbol{x'} - A\boldsymbol{x}| < r$, $A^{-1}\boldsymbol{x'} \in o(\boldsymbol{x}, r)$. So $A^{-1}\boldsymbol{x'} \in \mathrm{NT}$ and thus $\boldsymbol{x'} \in \mathrm{NT}$, which is a contradiction.

Lemma 14. $\{(\frac{1}{2^n}, -\frac{1}{3^n}, \frac{1}{5^n})\}_{n=0}^{\infty} \subseteq \partial\mathrm{NT}$.

Proof. Let $\boldsymbol{p}_n = (\frac{1}{2^n}, -\frac{1}{3^n}, \frac{1}{5^n}), n \geq 0$. We use induction on the value of n.

When $n = 0$, because $B\boldsymbol{p}_0 = B(1, -1, 1)^T = 0$ and

$$BA^k\boldsymbol{p}_0 = 2^k - 2 \times 3^k + 5^k > 0 \quad \text{for any } k \in \mathbb{N}^+,$$

we have $\boldsymbol{p}_0 \in \partial\mathrm{NT}$.

Now assume that the conclusion holds for $n-1$. So, $A\boldsymbol{p}_n = \boldsymbol{p}_{n-1} \in \partial\mathrm{NT} \subseteq \mathrm{NT}$. By Lemma 13, $\boldsymbol{p}_n \in \partial\mathrm{NT}$.

Lemma 15. *For any non-zero polynomial* $f(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$, *there exists an* N *such that* $f(\frac{1}{2^n}, -\frac{1}{3^n}, \frac{1}{5^n}) \neq 0$ *for all* $n > N$.

Proof. Assume that the conclusion does not hold. Then there exists a subsequence $\{((\frac{1}{2})^{n_k}, -(\frac{1}{3})^{n_k}, (\frac{1}{5})^{n_k})\}_{k=1}^{\infty}$ such that f vanishes on each point of it.

Let $f = b_1 x_1^{\alpha_1} x_2^{\beta_1} x_3^{\gamma_1} + ... + b_s x_1^{\alpha_s} x_2^{\beta_s} x_3^{\gamma_s}$ where $b_i \in \mathbb{R}, b_i \neq 0, \alpha_i \in \mathbb{N}, \beta_i \in \mathbb{N}, \gamma_i \in \mathbb{N}$, and $(\alpha_i, \beta_i, \gamma_i) \neq (\alpha_j, \beta_j, \gamma_j)$ for $i \neq j$.

Obviously $s \geq 1$ because $f \not\equiv 0$. Let $t_i = (\frac{1}{2})^{\alpha_i}(\frac{1}{3})^{\beta_i}(\frac{1}{5})^{\gamma_i}$.

It is an obvious fact that $2^{\alpha_j}3^{\beta_j}5^{\gamma_j} \neq 2^{\alpha_i}3^{\beta_i}5^{\gamma_i}$ for $i \neq j$. Hence $t_1, t_2, ..., t_s$ are pairwise distinct. Without loss of generality, let $t_1 > t_2 > ... > t_s$.

For every $j > 1$, we have $\lim_{k\to\infty} (\frac{t_j}{t_1})^{n_k} = 0$. Thus

$$\lim_{k\to\infty} \left|\frac{f((\frac{1}{2})^{n_k}, -(\frac{1}{3})^{n_k}, (\frac{1}{5})^{n_k})}{((\frac{1}{2})^{\alpha_1}(\frac{1}{3})^{\beta_1}(\frac{1}{5})^{\gamma_1})^{n_k}}\right| = |b_1| \neq 0 \ .$$

This contradicts with $f((\frac{1}{2})^{n_k}, -(\frac{1}{3})^{n_k}, (\frac{1}{5})^{n_k}) = 0$. Therefore the conclusion follows.

Using the above lemmas, we can now prove Theorem 2.

Proof. Denote by S the sequence $\{(\frac{1}{2})^n, -(\frac{1}{3})^n, (\frac{1}{5})^n)\}$. By Lemma 14, $S \subseteq \partial\mathrm{NT}$.

Assume NT is a semi-algebraic set. Then there exist finite many polynomials $f_{i,j} \in \mathbb{R}[x_1, x_2, x_3]$ and $\triangleleft_{i,j} \in \{<, =\}$ for $i = 1, ..., s$ and $j = 1, ..., r_i$ such that

$$\mathrm{NT} = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{(x_1, x_2, x_3) \in \mathbb{R}^3 | f_{i,j} \triangleleft_{i,j} 0\}. \tag{3}$$

Because $S \subseteq \partial\mathrm{NT} \subseteq \{f_{i,j} = 0\}_{i,j}$, for any $x \in S$, there exists a polynomial $f_{i,j}$ such that $f_{i,j}(x) = 0$. By pigeonhole principle there exists an $f_{i,j}$ and a subsequence S_1 of S such that $f_{i,j}$ vanishes on S_1, which contradicts with Lemma 15.

5 Conclusion

In this paper, we consider whether the NT of a simple linear loop is decidable and how to compute it if it is decidable. For homogeneous linear loops with only two program variables, we give a complete algorithm for computing the NT. For the case of more program variables, we show that the NT cannot be described by Tarski formulae in general.

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