

Equivalence of Games with Probabilistic Uncertainty and Partial-observation Games

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Abstract. We introduce games with probabilistic uncertainty, a natural model for controller synthesis in which the controller observes the state of the system through imprecise sensors that provide correct information about the current state with a fixed probability. That is, in each step, the sensors return an observed state, and given the observed state, there is a probability distribution (due to the estimation error) over the actual current state. The controller must base its decision on the observed state (rather than the actual current state, which it does not know). On the other hand, we assume that the environment can perfectly observe the current state. We show that our model can be reduced in polynomial time to standard partial-observation stochastic games, and vice-versa. As a consequence we establish the precise decidability frontier for the new class of games, and for most of the decidable problems establish optimal complexity results.

1 Introduction

In a control system, a controller interacts with its environment through sensors and actuators. The controller observes the state of the environment through a set of sensors, computes a control signal that depends on the history of observed sensor readings, and feeds the control signal to the environment through actuators. The state of the environment is then updated as a function of the control signal as well as a disturbance signal that models external inputs to the environment. In a *reactive* setting, the sense-compute-actuate cycle repeats forever, resulting in an infinite trace of environment states. The objective of the controller is to ensure that the trace belongs to a given specification of “good” traces. The controller synthesis problem asks, given the dynamical law that specifies how the environment state changes according to the controller inputs and external disturbances, and a specification of good traces, to synthesize a control law that ensures that the environment traces are good, no matter how external disturbances behave.

Controller synthesis has been studied extensively for deterministic games with ω -regular specifications [5,14,13]. In this setting, the problem is modeled as a game on a graph. The vertices of the graph represent system states, and are divided into “controller states” and “disturbance states.” At a controller state, the controller chooses an outgoing edge and moves to a neighboring vertex along this edge. At a disturbance state, the disturbance chooses an outgoing edge and moves along this edge. This continues ad infinitum, defining a sequence of states. If this sequence satisfies the specification, the controller wins; otherwise, the disturbance wins. The games are called *perfect observation*, since both players have exact knowledge of the current state and the history of the game.

The study of perfect-observation deterministic games have been extended to systems with *partial observation*, in which the controller can only observe part of the environment’s state [15,7], and to *stochastic dynamics* [12,8,10,11], in which the state updates happen according to a probabilistic law.

The “standard model” of partial-observation stochastic games [7,3,2] is described as an extension to the above graph model, by fixing an equivalence relation on the vertices (the “observation function”), and stipulating that the controller only sees the equivalence class of the current vertex, not the particular vertex the state is in. In addition, the transitions of the graph are stochastic: the controller and the disturbance each choose some move, and the next vertex is chosen according to a probability distribution based on the current vertex and the chosen move.

In this paper, we introduce a different, albeit natural, model of probabilistic uncertainty in controller synthesis. Consider a state given by n bits. The sensors used to measure the state are typically not perfect, and observing the state through the sensor results in some bits being flipped with some known probability (probabilistic noise). In applications where the controller observes the state bits through a network, then the probabilistic noise in the communication channels results in bits being flipped with some known probability (according to the classical Shannon’s communication channel model). Thus, the controller observes n bits through the sensor, and this estimate defines a probability

distribution over the state space for the current state. In contrast, we allow the disturbance to precisely observe the state, corresponding to a worst case assumption on the disturbance. The objective of the controller is to find a strategy that ensures that the system satisfies the specification under this probabilistic uncertainty on the current state. We distinguish between two models of the disturbance. In the first model, the disturbance observes the correct sequence of states as well as both the observation of the controller and the sequence of controller moves. In the second model, the disturbance observes the correct sequence of states as well as the sequence of controller moves (but not the observation of the controller). It turns out that the two models give rise to subtle differences in defining the probability measures on the games, as well as different complexities in the solution algorithms.

Our model (which we refer to as games with probabilistic uncertainty) is inspired by analogous models of state estimation under probabilistic noise in continuous control systems. We believe this model of games with probabilistic uncertainty naturally captures the behavior of many sensor-based control systems. Intuitively, the standard model of partial-observation games represent “partial but correct information” where the controller can observe correctly only the first $k < n$ bits of the state (i.e., the observation is partial as the controller observes only a part of the state bits, but the information about the observed state bits is always correct). In contrast, our model of games with probabilistic uncertainty represent “complete but uncertain information” where the controller can observe all the n bits of the state but with uncertainty of observation (i.e., the controller can observe all the bits, but each bit is correct with some probability). Since the type of uncertain information in our model is very different from the standard models of partial-observation games studied in the literature, the relationship between them is not immediate.

Our main contribution, along with the introduction of the natural model of games with probabilistic uncertainty, is establishing the equivalence of the new class of games and partial-observation games. Our main technical result is a polynomial-time reduction from this new model of games with probabilistic uncertainty to standard partial-observation games, and a converse reduction from partially-observable Markov decision processes (POMDPs) to games with probabilistic uncertainty. The results to establish the equivalence of the two classes of games which represent two different notions of information (partial but correct vs complete but uncertain) are quite intricate. For example, for the new class of games the inductive definition of probability measure is subtle and different from the classical definition of probability measure for probabilistic systems [17,9]. This is because the controller observes a history that can be completely different from the actual history, whereas the environment (or disturbance) observes the actual history. We first inductively define a probability measure of observed history, given the actual history, and use it to define the probability measure inductively. We show how our polynomial constructions for reduction capture the subtleties in the probability measure, and by establishing precise mapping of strategies (which is at the heart of the proof of correctness of the reduction) we obtain the desired equivalence result.

In the positive direction, our reduction allows us to solve controller synthesis problems for games with probabilistic uncertainty against ω -regular specifications, using algorithms of [7,2]. In the negative direction, we get lower bounds on the hardness of problems by using known lower bounds for POMDPs using the hardness results of [1,6]. In particular, with our reductions we establish precisely the decidability frontier of games with probabilistic uncertainty for various classes of parity objectives (a canonical form to express ω -regular specifications); and for most of the decidable problems we establish EXPTIME-complete bounds, and in some cases 2EXPTIME upper bounds and EXPTIME lower bounds (see Table 1). Moreover, our reduction allows the rich body of algorithms (such as symbolic and anti-chain based algorithms [7,2]) for partial-observation games, along with any future algorithmic developments for partial-observation games, to be applicable to solve games with probabilistic uncertainty. In summary, our results provide precise decidability frontier, optimal complexity (in most cases), and algorithmic solutions for games with probabilistic uncertainty, that is a natural model for control problems with state estimation under probabilistic noise.

2 Games with Probabilistic Uncertainty

In this section we introduce a class of games with probabilistic imperfect information, and call them games with probabilistic uncertainty.

Probability distribution. A probability distribution on a finite set A is a function $\kappa : A \rightarrow [0, 1]$ such that $\sum_{a \in A} \kappa(a) = 1$. We denote by $\mathcal{D}(A)$ the set of probability distributions on A .

Game structures with probabilistic uncertainty. A game structure with probabilistic uncertainty consists of a tuple $\mathcal{G} = (L, \Sigma_I, \Sigma_O, \Delta, \text{un})$, where (a) L is a set of locations; (b) Σ_I and Σ_O are two sets of input and output alphabets,

respectively; (c) $\Delta : L \times \Sigma_I \times \Sigma_O \rightarrow \mathcal{D}(L)$ is a probabilistic transition function that given a location, an input and an output letter gives the probability distribution over the next locations; and (d) $\text{un} : L \rightarrow \mathcal{D}(L)$ is the *probabilistic uncertainty function* that given the true current location describes the probability distribution of the observed location. If un is the identity function we obtain perfect-observation games.

Intuitively, a game proceeds as follows. The game starts at some location $\ell \in L$. Player 1 observes a state drawn from the distribution $\text{un}(\ell)$, which represents a potentially faulty observation process. Intuitively, at every step the player can observe the value of all variables that corresponds to the state of the game, but there is a probability that the observed value of some variables is incorrect. Player 2 observes the “correct” state ℓ . Given the observation of the history of the game so far, Player 1 picks an input alphabet $\sigma^i \in \Sigma_i$. Player 2 then picks an output letter $\sigma^o \in \Sigma_o$: we consider two variants, (1) Player 2 only observes the history of correct locations and the moves of the players; and (2) Player 2 observes the history of correct locations, the moves of the players, and also observes the history of observed locations of Player 1. The state of the game is updated to ℓ' with probability $\Delta(\ell, \sigma^i, \sigma^o)(\ell')$. This process is repeated ad infinitum.

Plays. A play of \mathcal{G} is a sequence $\rho = \ell_0 \sigma_0^i \sigma_0^o \ell_1 \sigma_1^i \sigma_1^o \dots$ of locations, input letter, and output letter, such that for all $j \geq 0$ we have $\Delta(\ell_j, \sigma_j^i, \sigma_j^o)(\ell_{j+1}) > 0$. The *prefix up to* ℓ_n of the play ρ is denoted by $\rho(n)$, its *length* is $|\rho(n)| = n+1$ and its *last element* is $\text{Last}(\rho(n)) = \ell_n$. The set of plays in \mathcal{G} is denoted by $\text{Plays}(\mathcal{G})$, and the set of corresponding finite prefixes is denoted $\text{Pref}(\mathcal{G})$.

Strategies. A strategy for Player 1 observes the finite prefix of a play and then selects an input letter (pure strategies) or a probability distribution over input letters in Σ_i . Formally, a pure strategy for Player 1 is a function $\alpha : \text{Pref}(\mathcal{G}) \rightarrow \Sigma_i$, and a randomized strategy for Player 1 is a function $\alpha : \text{Pref}(\mathcal{G}) \rightarrow \mathcal{D}(\Sigma_i)$. Similarly, pure and randomized strategies for Player 2 are defined as functions $\beta : \text{Pref}(\mathcal{G}) \times \Sigma_i \rightarrow \Sigma_o$ and $\beta : \text{Pref}(\mathcal{G}) \times \Sigma_i \rightarrow \mathcal{D}(\Sigma_o)$, respectively. Note that Player 2 sees Player 1’s choice of input action at each step. In the case where Player 2 observes also the history of observed locations, the pure and randomized strategies are defined as functions $\beta : \text{Pref}(\mathcal{G}) \times \text{Pref}(\mathcal{G}) \times \Sigma_i \rightarrow \Sigma_o$ and $\beta : \text{Pref}(\mathcal{G}) \times \text{Pref}(\mathcal{G}) \times \Sigma_i \rightarrow \mathcal{D}(\Sigma_o)$, respectively, where the output letter is chosen based on the original history and observed history. We refer to strategies that observes both histories as “all-powerful” strategies for Player 2.

Outcomes. The *outcome* of two randomized strategies α for Player 1 and β for Player 2 from a location $\ell \in L$ is the set of plays $\rho = \ell_0 \sigma_0^i \sigma_0^o \dots$ such that (1) $\ell = \ell_0$, (2) there exists a sequence $\ell'_0 \ell'_1 \dots$ such that $\text{un}(\ell_j)(\ell'_j) > 0$ for each $j \geq 0$, (3) for each $j \geq 0$, we have $\alpha(\ell_0 \sigma_0^i \sigma_0^o \dots \ell'_j)(\sigma_j^i) > 0$ and $\beta(\rho(j), \sigma_j^i)(\sigma_j^o) > 0$ (if β is an all-powerful strategy, then $\beta(\rho(j), \ell'_0 \sigma_0^i \sigma_0^o \ell'_1 \dots \ell'_j, \sigma_j^i)(\sigma_j^o) > 0$), and $\Delta(\ell_j, \sigma_j^i, \sigma_j^o)(\ell_{j+1}) > 0$. The primed sequence $\ell'_0 \ell'_1 \dots$ gives the sequence of observations made by Player 1 using the probabilistic uncertainty function. Note that this sequence may be incorrect with some probability due to probabilistic uncertainty in the observation. We denote this set of plays as $\text{Outcome}(\mathcal{G}, \ell, \alpha, \beta)$. The outcome of two pure strategies is defined analogously, considering pure strategies as degenerate randomized strategies which pick a letter with probability one. The *outcome set* of the pure (resp. randomized) strategy α for Player 1 in \mathcal{G} is the set $\text{Outcome}_1(\mathcal{G}, \ell, \alpha)$ of plays ρ such that there exists a pure (resp. randomized) strategy β for Player 2 with $\rho \in \text{Outcome}(\mathcal{G}, \ell, \alpha, \beta)$. The outcome set $\text{Outcome}_2(\mathcal{G}, \ell, \beta)$ for Player 2 is defined symmetrically.

Probability measure. Given strategies α and β , we define the probability measure $\text{Pr}_{\ell_0}^{\alpha, \beta}(\cdot)$. The definition of the probability measure is subtle and non-standard as the prefix that Player 1 observes can be completely different from the original history. For a finite prefix $\rho \in \text{Pref}(\mathcal{G})$, let $\text{Cone}(\rho)$ denote the set of plays with ρ as prefix. We will define $\text{Pr}_{\ell_0}^{\alpha, \beta}(\cdot)$ for cones, and then by Caratheodory extension theorem [4] there is a unique extension to all measurable sets of paths. To define the probability measure we also need to define a function $\text{ObsSeq}(\rho)$, that given a finite prefix ρ , gives the probability distribution over finite prefixes ρ' , such that $\text{ObsSeq}(\rho)(\rho')$ denotes the probability of observing ρ' given the correct prefix is ρ . The base case is as follows:

$$\text{Pr}_{\ell_0}^{\alpha, \beta}(\text{Cone}(\ell_0)) = 1; \quad \text{ObsSeq}(\ell_0)(\ell') = \text{un}(\ell_0)(\ell').$$

The inductive definition of ObsSeq is as follows: for a prefix ρ of length $n+1$

$$\text{ObsSeq}(\rho \sigma_n^i \sigma_n^o \ell_{n+1})(\rho' \sigma_n^i \sigma_n^o \ell'_{n+1}) = \text{ObsSeq}(\rho)(\rho') \cdot \text{un}(\ell_{n+1})(\ell'_{n+1})$$

Given a sequence $\rho = \ell_0 \sigma_0^i \sigma_0^o \ell_1 \sigma_1^i \sigma_1^o \dots \ell_n$, we define $\text{ActMt}(\rho) = \{\tilde{\rho} = \tilde{\ell}_0 \tilde{\sigma}_0^i \tilde{\sigma}_0^o \ell_1 \tilde{\sigma}_1^i \tilde{\sigma}_1^o \dots \tilde{\ell}_n \mid \forall 1 \leq j \leq n-1. \tilde{\sigma}_j^i = \sigma_j^i \text{ and } \tilde{\sigma}_j^o = \sigma_j^o\}$ the sequences of same length as ρ such that the sequence of input and output letter matches

(i.e., the set of action-matching prefixes). Note that for non action-matching prefixes the observation sequence function always assigns probability zero. The inductive case for the probability measure is as follows: for a prefix ρ of length $n + 1$ with last state ℓ_n , we have

$$\Pr_{\ell_0}^{\alpha, \beta}(\text{Cone}(\rho \sigma_n^i \sigma_n^o \ell_{n+1})) = \Pr_{\ell_0}^{\alpha, \beta}(\text{Cone}(\rho)) \cdot \left(\sum_{\rho' \in \text{ActMt}(\rho)} \text{ObsSeq}(\rho)(\rho') \cdot \alpha(\rho')(\sigma_n^i) \cdot \beta(\rho \sigma_n^i)(\sigma_n^o) \cdot \Delta(\ell_n, \sigma_n^i, \sigma_n^o)(\ell_{n+1}) \right);$$

i.e., $\text{ObsSeq}(\rho)(\rho')$ gives the probability to observe ρ' , then $\alpha(\rho')(\sigma_n^i)$ denotes the probability to play σ_n^i given the strategy and observed sequence ρ' , and since Player 2 observes the correct sequence the probability to play σ_n^o is given by $\beta(\rho \sigma_n^i)(\sigma_n^o)$ (Player 2 observes ρ), and the final term $\Delta(\ell_n, \sigma_n^i, \sigma_n^o)(\ell_{n+1})$ gives the transition probability. If β is an all-powerful strategy, then β observes both the correct history ρ and the observed history ρ' , and then the definition is as follows:

$$\Pr_{\ell_0}^{\alpha, \beta}(\text{Cone}(\rho \sigma_n^i \sigma_n^o \ell_{n+1})) = \Pr_{\ell_0}^{\alpha, \beta}(\text{Cone}(\rho)) \cdot \left(\sum_{\rho' \in \text{ActMt}(\rho)} \text{ObsSeq}(\rho)(\rho') \cdot \alpha(\rho')(\sigma_n^i) \cdot \beta(\rho, \rho', \sigma_n^i)(\sigma_n^o) \cdot \Delta(\ell_n, \sigma_n^i, \sigma_n^o)(\ell_{n+1}) \right).$$

Winning objectives. An *objective* for Player 1 in \mathcal{G} is a set $\phi \subseteq \text{Plays}(\mathcal{G})$ of plays. A play $\rho \in \text{Plays}(\mathcal{G})$ satisfies the objective ϕ , denoted $\rho \models \phi$, if $\rho \in \phi$. We consider ω -regular objectives specified as parity objectives (a canonical form to express all ω -regular objectives [16]). For a play $\rho = \ell_0 \sigma_0^i \sigma_0^o \dots$, we denote by ρ_k the k -th location ℓ_k of the play and denote by $\text{Inf}(\rho)$ the set of locations that occur infinitely often in ρ , that is, $\text{Inf}(\rho) = \{\ell \mid \forall i \exists j : j > i \text{ and } \ell_j = \ell\}$. We consider the following classes of objectives.

1. *Reachability and safety objectives.* Given a set $\mathcal{T} \subseteq L$ of target locations, the *reachability* objective $\text{Reach}(\mathcal{T})$ requires that a location in \mathcal{T} be visited at least once, that is, $\text{Reach}(\mathcal{T}) = \{\rho \mid \exists k \geq 0 \cdot \rho_k \in \mathcal{T}\}$. Dually, the *safety* objective $\text{Safe}(\mathcal{T})$ requires that only states in \mathcal{T} be visited. Formally, $\text{Safe}(\mathcal{T}) = \{\rho \mid \forall k \geq 0 \cdot \rho_k \in \mathcal{T}\}$.
2. *Büchi and coBüchi objectives.* Let $\mathcal{T} \subseteq L$ be a set of target locations. The *Büchi* objective $\text{Buchi}(\mathcal{T})$ requires that a state in \mathcal{T} be visited infinitely often, that is, $\text{Buchi}(\mathcal{T}) = \{\rho \mid \text{Inf}(\rho) \cap \mathcal{T} \neq \emptyset\}$. Dually, the *coBüchi* objective $\text{coBuchi}(\mathcal{T})$ requires that only states in \mathcal{T} be visited infinitely often. Formally, $\text{coBuchi}(\mathcal{T}) = \{\rho \mid \text{Inf}(\rho) \subseteq \mathcal{T}\}$.
3. *Parity objectives.* For $d \in \mathbb{N}$, let $p : L \rightarrow \{0, 1, \dots, d\}$ be a *priority function*, which maps each state to a nonnegative integer priority. The *parity* objective $\text{Parity}(p)$ requires that the minimum priority that occurs infinitely often be even. Formally, $\text{Parity}(p) = \{\rho \mid \min\{p(\ell) \mid \ell \in \text{Inf}(\rho)\} \text{ is even}\}$. The Büchi and coBüchi objectives are the special cases of parity objectives with two priorities, $p : L \rightarrow \{0, 1\}$ and $p : L \rightarrow \{1, 2\}$, respectively.

Sure, almost-sure and positive winning. An *event* is a measurable set of plays, and given strategies α and β for the two players, the probabilities of events are uniquely defined. For an objective ϕ , assumed to be Borel, we denote by $\Pr_{\ell}^{\alpha, \beta}(\phi)$ the probability that ϕ is satisfied by the play obtained from the starting location ℓ when the strategies α and β are used. Given a game \mathcal{G} , an objective ϕ , and a location ℓ , we consider the following winning modes: (1) a strategy α for Player 1 is *sure winning* for the objective ϕ from $\ell \in L$ if $\text{Outcome}(\mathcal{G}, \ell, \alpha, \beta) \subseteq \phi$ for all strategies β for Player 2; (2) a strategy α for Player 1 is *almost-sure winning* for the objective ϕ from $\ell \in L$ if $\Pr_{\ell}^{\alpha, \beta}(\phi) = 1$ for all strategies β for Player 2; and (3) a strategy α for Player 1 is *positive winning* for the objective ϕ from $\ell \in L$ if $\Pr_{\ell}^{\alpha, \beta}(\phi) > 0$ for all strategies β for Player 2.

Qualitative analysis of a game consists of the computation of the sure, almost-sure and positive winning sets. The sure (resp. almost-sure and positive) winning decision problem for an objective consists of a game and a starting location ℓ , and asks whether there is a sure (resp. almost-sure and positive) winning strategy from ℓ .

3 Partial-observation Stochastic Games

We now recall the usual definition of partial-observation games and their subclasses. We focus on partial-observation turn-based probabilistic games, where at each round one of the players is in charge of choosing the next action and the

transition function is probabilistic. We will present a polynomial time reduction of games with probabilistic uncertainty to these games.

Partial-observation games. A *partial-observation stochastic game* (for short partial-observation game or simply a *game*) is a tuple $G = \langle S_1 \cup S_2, A_1, A_2, \delta_1 \cup \delta_2, \mathcal{O}_1, \mathcal{O}_2 \rangle$ with the following components:

1. (*State space*). $S = S_1 \cup S_2$ is a finite set of states, where $S_1 \cap S_2 = \emptyset$ (i.e., S_1 and S_2 are disjoint), states in S_1 are Player 1 states, and states in S_2 are Player 2 states.
2. (*Actions*). A_i ($i = 1, 2$) is a finite set of actions for Player i .
3. (*Transition function*). For $i \in \{1, 2\}$, the probabilistic transition function for Player i is the function $\delta_i : S_i \times A_i \rightarrow \mathcal{D}(S_{3-i})$ that maps a state $s_i \in S_i$ and an action $a_i \in A_i$ to the probability distribution $\delta_i(s_i, a_i)$ over the successor states in S_{3-i} (i.e., games are alternating).
4. (*Observations*). $\mathcal{O}_1 \subseteq 2^S$ is a finite set of observations for Player 1 that partitions the state space S , and similarly \mathcal{O}_2 is the observations for Player 2. These partitions uniquely define functions $\text{obs}_i : S \rightarrow \mathcal{O}_i$, for $i \in \{1, 2\}$, that map each state to its observation such that $s \in \text{obs}_i(s)$ for all $s \in S$. We will also consider the special case of one-sided games, where Player 2 is perfectly informed (has complete observation), i.e., $\mathcal{O}_2 = S$, and $\text{obs}_2(s) = s$ for all $s \in S$ (i.e., the partition consists of singleton states).

Special Class: POMDPs. We will consider one special class of partial-observation games called *partial-observable Markov decision processes* (POMDPs), where the action set for Player 2 is a singleton (i.e., there is effectively only Player 1 and stochastic transitions). Hence we will omit the action set and observation for Player 2 and represent a POMDP as the following tuple $G = \langle S, A, \delta, \mathcal{O} \rangle$, where $\delta : S \times A \rightarrow \mathcal{D}(S)$.

Plays. In a game, in each turn, for $i \in \{1, 2\}$, if the current state s is in S_i , then Player i chooses an action $a \in A_i$, and the successor state is chosen by sampling the probability distribution $\delta_i(s, a)$. A *play* in G is an infinite sequence of states and actions $\rho = s_0 a_0 s_1 a_1 \dots$ such that for all $j \geq 0$, if $s_j \in S_i$, for $i \in \{1, 2\}$, then $a_j \in A_i$ such that $\delta_i(s_j, a_j)(s_{j+1}) > 0$. The definitions of prefix and length are analogous to the definitions in Section 2. For $i \in \{1, 2\}$, we denote by $\text{Pref}_i(G)$ the set of finite prefixes in G that end in a state in S_i . The *observation sequence* of $\rho = s_0 a_0 s_1 a_1 \dots$ for Player i ($i = 1, 2$) is the unique infinite sequence of observations and actions, i.e., $\text{obs}(\rho) = o_0 a_0 o_1 a_1 o_2 \dots$ such that $s_j \in o_j$ for all $j \geq 0$. The observation sequence for finite sequences (prefix of plays) is defined analogously.

Strategies. A *pure strategy* in G for Player 1 is a function $\alpha : \text{Pref}_1(G) \rightarrow A_1$. A *randomized strategy* in G for Player 1 is a function $\alpha : \text{Pref}_1(G) \rightarrow \mathcal{D}(A_1)$. A (pure or randomized) strategy α for Player 1 is *observation-based* if for all prefixes $\rho, \rho' \in \text{Pref}_1(G)$, if $\text{obs}(\rho) = \text{obs}(\rho')$, then $\alpha(\rho) = \alpha(\rho')$. We omit analogous definitions of strategies for Player 2. We denote by $\mathcal{A}_G, \mathcal{A}_G^O, \mathcal{A}_G^P, \mathcal{B}_G, \mathcal{B}_G^O, \mathcal{B}_G^P$ the set of all Player-1 strategies in G , the set of all observation-based Player-1 strategies, the set of all pure Player-1 strategies, the set of all Player-2 strategies in G , the set of all observation-based Player-2 strategies, and the set of all pure Player-2 strategies, respectively. In the setting where Player 1 has partial-observation and Player 2 has complete observation, the set \mathcal{B}_G of all strategies coincides with the set \mathcal{B}_G^O of all observation-based strategies. We will require the players to play observation-based strategies.

Outcomes. The *outcome* of two randomized strategies α (for Player 1) and β (for Player 2) from a state s in G is the set of plays $\rho = s_0 a_0 s_1 a_1 \dots \in \text{Plays}(G)$, with $s_0 = s$, where for all $j \geq 0$, if $s_j \in S_1$ (resp. $s_j \in S_2$), then $\alpha(\rho(j))(a_j) > 0$ (resp. $\beta(\rho(j))(a_j) > 0$) and $\delta_1(s_j, a_j)(s_{j+1}) > 0$ (resp. $\delta_2(s_j, a_j)(s_{j+1}) > 0$). This set is denoted $\text{Outcome}(G, s, \alpha, \beta)$. The outcome of two pure strategies is defined analogously by viewing pure strategies as randomized strategies that play their chosen action with probability one. The *outcome set* of the pure (resp. randomized) strategy α for Player 1 in G is the set $\text{Outcome}_1(G, s, \alpha)$ of plays ρ such that there exists a pure (resp. randomized) strategy β for Player 2 with $\rho \in \text{Outcome}(G, s, \alpha, \beta)$. The outcome set $\text{Outcome}_2(G, s, \beta)$ for Player 2 is defined symmetrically.

Probability measure. We define the probability measure $\text{Pr}_s^{\alpha, \beta}(\cdot)$ as follows: for a finite prefix ρ , let $\text{Cone}(\rho)$ denote the set of plays with ρ as prefix. Then we have $\text{Pr}_s^{\alpha, \beta}(\text{Cone}(s)) = 1$, and for a prefix of length n ending in a Player 1 state s_n we have

$$\text{Pr}_s^{\alpha, \beta}(\text{Cone}(\rho a_n s_{n+1})) = \text{Pr}_s^{\alpha, \beta}(\text{Cone}(\rho)) \cdot \alpha(\rho)(a_n) \cdot \delta_1(s_n, a_n)(s_{n+1});$$

and the definition when s_n is a Player 2 state is similar. For a set Q of finite prefixes, we write $\Pr_s^{\alpha,\beta}(\text{Cone}(Q))$ for $\Pr_s^{\alpha,\beta}(\bigcup_{\rho \in Q} \text{Cone}(\rho))$.

The winning modes sure, almost-sure, and positive are defined analogously to Section 2, where we restrict the players to play an observation-based strategy. From the results of [7,2,1,3,6] we obtain the following theorem summarizing the results for partial-observation games and POMDPs.

Theorem 1 ([7,2,1,3,6]). *The following assertions hold:*

1. (One-sided games and POMDPs). *The sure, almost-sure and positive winning for safety objectives; the sure and almost-sure winning for reachability objectives and Büchi objectives; the sure and positive winning for coBüchi objectives; and the sure winning for parity objectives are EXPTIME-complete for one-sided partial-observation games (Player 2 perfectly informed) and POMDPs. The positive winning problem for reachability objectives is PTIME-complete both for one-sided partial-observation games and POMDPs.*
2. (General partial-observation games). *The sure, almost-sure winning for safety objectives, the sure winning for parity objectives are EXPTIME-complete for partial-observation games; the almost-sure winning for reachability objectives and Büchi objectives, and the positive winning for safety and coBüchi objectives are 2EXPTIME-complete for partial-observation games. The positive winning problem for reachability objectives is EXPTIME-complete.*
3. (Undecidability results). *The positive winning problem for Büchi objectives, the almost-sure winning problem for coBüchi objectives, and the positive and almost-sure winning problems for parity objectives are undecidable for POMDPs.*

4 Reduction: Games with Probabilistic Uncertainty to Partial-observation Games

We now present a reduction of games with probabilistic uncertainty to classical partial-observation games. Let $G = (L, \Sigma_I, \Sigma_O, \Delta, \text{un})$ be a game with probabilistic uncertainty and we construct a partial-observation game $H = (L \times L \cup L \times L \times \Sigma_I, A_1 = \Sigma_I, A_2 = \Sigma_O, \delta = \delta_1 \cup \delta_2, \mathcal{O}_1, \mathcal{O}_2)$ as follows (below as δ_1 and δ_2 would be clear from context, we simply use δ for simplicity):

1. The transition function δ_1 is deterministic and for $(\ell_1, \ell_2) \in L \times L$ and $\sigma_I \in \Sigma_I$ we have

$$\delta((\ell_1, \ell_2), \sigma_I) = (\ell_1, \ell_2, \sigma_I)$$

2. The transition function δ_2 captures both Δ and un and is defined as follows: for $(\ell_1, \ell_2, \sigma_I) \in L \times L \times \Sigma_I$ and $\sigma_O \in \Sigma_O$ we have

$$\delta((\ell_1, \ell_2, \sigma_I), \sigma_O)(\ell'_1, \ell'_2) = \Delta(\ell_1, \sigma_I, \sigma_O)(\ell'_1) \cdot \text{un}(\ell'_1)(\ell'_2).$$

Intuitively, the first component of the game H keeps track of the real state of the game G , and the second component keeps track of the information available from probabilistic uncertainty. Hence Player 1 is only allowed to observe the second component which is the probability distribution over the observable state given the current state.

3. The observation mapping is as follows: we have $\mathcal{O}_1 = L$; and $\text{obs}_1(\ell_1, \ell_2) = \text{obs}_1(\ell_1, \ell_2, \sigma_I) = \ell_2$, i.e., only the second component is observable. We will consider two cases for \mathcal{O}_2 : for the reduction of all-powerful strategies we will consider Player 2 has complete-observation, and in the other case we have $\mathcal{O}_2 = L$ and Player 2 observes the first component that represents the correct history: i.e., $\text{obs}_2(\ell_1, \ell_2) = \text{obs}_2(\ell_1, \ell_2, \sigma_I) = \ell_1$.
4. For a parity objective in G given by priority function $p_G : L \rightarrow \{0, 1, \dots, d\}$, we consider the priority function p_H in H as follows: $p_H((\ell, \ell')) = p_H((\ell, \ell', \sigma_I)) = p_G(\ell)$, for all $\ell, \ell' \in L$ and $\sigma_I \in \Sigma_I$.

Correspondence of strategies. We will now establish the correspondence of probabilistic uncertain strategies in G and the observation based strategies in H . We present a few notations. For simplicity of presentation, we will use a slight abuse of notation: given a history (or finite prefix) $\rho_H = s_0 a_0 s_1 a_1 s_2 a_2 \dots s_{2n}$ in H we will represent the history as $s_0 a_0 a_1 s_2 a_2 a_3 s_3 \dots s_{2n}$ as the intermediate state is always uniquely defined by the state and the action. Intuitively this is removing the stuttering and does not affect parity objectives.

Mapping of strategies from G to H. Given a history $\rho_H = s_0 a_0 a_1 s_2 a_2 a_3 s_3 \dots s_{2n}$ in H , such that $s_{2i} = (\ell_{2i}^1, \ell_{2i}^2)$, we consider two histories in G as follows:

$$g_1(\rho_H) = \ell_0^1 a_0 a_1 \ell_2^1 a_2 a_3 \dots \ell_{2n}^1; \quad g_2(\rho_H) = \ell_0^2 a_0 a_1 \ell_2^2 a_2 a_3 \dots \ell_{2n}^2.$$

Intuitively, g_1 gives the first component (which is the correct history) and g_2 gives the second component (which is the observed history). We now define the mapping of strategies from G to H : given strategy α_G for Player 1, a strategy β_G for Player 2, and an all-powerful strategy β_G^A for Player 2, in the game G , we define the corresponding strategies in H as follows: for a history ρ_H and an action a_i for Player 1 we have

$$\begin{aligned} \alpha_H(\rho_H) &= \alpha_G(g_2(\rho_H)); \\ \beta_H(\rho_H a_i) &= \beta_G(g_1(\rho_H) a_i); \\ \beta_H^C(\rho_H a_i) &= \beta_G^A(g_1(\rho_H), g_2(\rho_H), a_i). \end{aligned}$$

Note that α_H and β_H are observation-based strategies, and β_H^C is a strategy with complete-observation, i.e., all-powerful strategies are mapped to complete-observation strategies. Hence for all-powerful strategies the reduction is to one-sided games. We will use \hat{g} to denote the mapping of strategies, i.e., $\alpha_H = \hat{g}(\alpha_G)$, $\beta_H = \hat{g}(\beta_G)$, and $\beta_H^C = \hat{g}(\beta_G^A)$.

Mapping of strategies from H to G. We now present the mapping in the other direction. Let $\rho_G^1 = \ell_0^1 \sigma_0^i \sigma_0^o \ell_1^1 \sigma_1^i \sigma_1^o \dots \ell_n^1$, and $\rho_G^2 = \ell_0^2 \sigma_0^i \sigma_0^o \ell_1^2 \sigma_1^i \sigma_1^o \dots \ell_n^2$ be two prefixes in G . Intuitively, the first represent the correct history and the second the observed history. Then we consider the following set of histories in H :

$$h_1(\rho_G^1) = \{\rho_H \mid g_1(\rho_H) = \rho_G^1\}; \quad h_2(\rho_G^2) = \{\rho_H \mid g_2(\rho_H) = \rho_G^2\};$$

and

$$h_{12}(\rho_G^1, \rho_G^2) = (\ell_0^1, \ell_0^2) \sigma_0^i \sigma_0^o (\ell_1^1, \ell_1^2) \sigma_1^i \sigma_1^o \dots (\ell_n^1, \ell_n^2).$$

We now define the mapping of strategies. Given an observation-based strategy $\alpha_H \in \mathcal{A}_H^O$ for Player 1, observation-based strategy $\beta_H \in \mathcal{B}_H^O$ for Player 2, and complete observation-based strategy $\beta_H^C \in \mathcal{B}_H$, we define the following strategies in G : for a correct history ρ_G^1 , observed history ρ_G^2 , and input σ^i we have

$$\begin{aligned} \beta_G(\rho_G^1 \sigma^i) &= \beta_H(\rho_H \sigma^i); \quad \rho_H \in h_1(\rho_G^1); \\ \alpha_G(\rho_G^2) &= \alpha_H(\rho_H); \quad \rho_H \in h_2(\rho_G^2); \\ \beta_G^A(\rho_G^1, \rho_G^2, \sigma^i) &= \beta_H^C(h_{12}(\rho_G^1, \rho_G^2), \sigma^i). \end{aligned}$$

Note that since β_H is observation-based it plays the same for all $\rho_H \in h_1(\rho_G^1)$, and similarly, since α_H is observation-based it plays the same for all $\rho_H \in h_2(\rho_G^2)$. Also observe that the strategy β_G^A is an all-powerful strategy. We will use \hat{h} to denote the mapping of strategies, i.e., $\alpha_G = \hat{h}(\alpha_H)$, $\beta_G = \hat{h}(\beta_H)$, and $\beta_G^A = \hat{h}(\beta_H^C)$.

Given a starting state $\ell_0 \in G$, consider the following probability distribution μ in H : $\mu(\ell_0, \ell) = \text{un}(\ell_0)(\ell)$. Given the mapping of strategies, our goal is to establish the equivalences of the probability measure. We introduce some notations required to establish the equivalence. For $j \geq 0$, we denote by (τ_j^1, τ_j^2) the pair of random variables to denote the j -th Player 1 state of the game H , and by θ_j^i and θ_j^o the random variables for the actions following the j -th state. Our first lemma establishes a connection of the probability of observing the second component in H given the first component along with function `ObsSeq`. We introduce notations to define two events: given two prefixes $\rho_G^1 = \ell_0^1 \sigma_0^i \sigma_0^o \ell_1^1 \sigma_1^i \sigma_1^o \dots \ell_n^1$, and $\rho_G^2 = \ell_0^2 \sigma_0^i \sigma_0^o \ell_1^2 \sigma_1^i \sigma_1^o \dots \ell_n^2$ in G , let $\mathcal{E}_{1,2}(\rho_G^1, \rho_G^2)$ denote the event that for all $0 \leq j \leq n$ we have $\tau_j^1 = \ell_j^1, \tau_j^2 = \ell_j^2$ and for all $0 \leq j \leq n-1$ we have $\theta_j^i = \sigma_j^i, \theta_j^o = \sigma_j^o$; and $\mathcal{E}_1(\rho_G^1)$ denote the event that for all $0 \leq j \leq n$ we have $\tau_j^1 = \ell_j^1$ and for all $0 \leq j \leq n-1$ we have $\theta_j^i = \sigma_j^i, \theta_j^o = \sigma_j^o$.

Lemma 1. *Let $\rho_G^1 = \ell_0^1 \sigma_0^i \sigma_0^o \ell_1^1 \sigma_1^i \sigma_1^o \dots \ell_n^1$, and $\rho_G^2 = \ell_0^2 \sigma_0^i \sigma_0^o \ell_1^2 \sigma_1^i \sigma_1^o \dots \ell_n^2$ be two prefixes in G . Then for all strategies α_H and β_H , the probability that the second component sequence in H is ρ_G^2 , given the first component sequence is ρ_G^1 is `ObsSeq`(ρ_G^1)(ρ_G^2), i.e., formally*

$$\text{Pr}_\mu^{\alpha_H, \beta_H}(\mathcal{E}_{1,2}(\rho_G^1, \rho_G^2) \mid \mathcal{E}_1(\rho_G^1)) = \text{ObsSeq}(\rho_G^1)(\rho_G^2).$$

Proof. The proof is by induction on the length of the prefixes. The base case is as follows: let the length of prefixes ρ_G^1 and ρ_G^2 be 1, with $\rho_G^1 = \ell_0$ and $\rho_G^2 = \ell$. Then we have

$$\text{ObsSeq}(\ell_0)(\ell) = \mu(\ell_0, \ell);$$

as required. We now consider the inductive case: we consider prefixes $\rho_G^1 \sigma_n^i \sigma_n^o \ell_{n+1}^1$ and $\rho_G^2 \sigma_n^i \sigma_n^o \ell_{n+1}^2$. Let us consider the events $\mathcal{E}_{n+1}^1 = \mathcal{E}_{1,2}(\rho_G^1 \sigma_n^i \sigma_n^o \ell_{n+1}^1, \rho_G^2 \sigma_n^i \sigma_n^o \ell_{n+1}^2)$ and $\mathcal{E}_{n+1}^2 = \mathcal{E}_1(\rho_G^1 \sigma_n^i \sigma_n^o \ell_{n+1}^1)$. Let $\bar{\mathcal{E}}_{n+1}^1$ denote the event that $\tau_n^1 = \ell_n^1, \tau_n^2 = \ell_n^2, \tau_{n+1}^1 = \ell_{n+1}^1, \tau_{n+1}^2 = \ell_{n+1}^2, \theta_n^1 = \sigma_n^i$, and $\theta_n^2 = \sigma_n^o$; and $\bar{\mathcal{E}}_{n+1}^2$ denote the event that $\tau_n^1 = \ell_n^1, \tau_{n+1}^1 = \ell_{n+1}^1, \theta_n^1 = \sigma_n^i$, and $\theta_n^2 = \sigma_n^o$. Then by definition we have

$$\begin{aligned} \Pr_{\mu}^{\alpha_H, \beta_H}(\bar{\mathcal{E}}_{n+1}^1 \mid \bar{\mathcal{E}}_{n+1}^2) &= \frac{\delta((\ell_n^1, \ell_n^2, \sigma_n^i, \sigma_n^o)(\ell_{n+1}^1, \ell_{n+1}^2))}{\sum_{\tilde{\ell}_{n+1}^2, \tilde{\ell}_{n+1}^1} \delta((\ell_n^1, \ell_n^2, \sigma_n^i, \sigma_n^o)(\ell_{n+1}^1, \tilde{\ell}_{n+1}^2))} \\ &\quad \text{(In the numerator all choices are fixed, and} \\ &\quad \text{in denominator are all possible choices of the second component)} \\ &= \frac{\Delta(\ell_n^1, \sigma_n^i, \sigma_n^o)(\ell_{n+1}^1) \cdot \text{un}(\ell_{n+1}^1)(\ell_{n+1}^2)}{\Delta(\ell_n^1, \sigma_n^i, \sigma_n^o)(\ell_{n+1}^1) \cdot \sum_{\tilde{\ell}_{n+1}^2} \text{un}(\ell_{n+1}^1)(\tilde{\ell}_{n+1}^2)} \\ &= \text{un}(\ell_{n+1}^1)(\ell_{n+1}^2) \quad \text{(Since } \sum_{\tilde{\ell}_{n+1}^2} \text{un}(\ell_{n+1}^1)(\tilde{\ell}_{n+1}^2) = 1) \end{aligned}$$

Note that the crucial fact used in the above proof is in the second equality and the fact is that for all $\tilde{\ell}_n^2$ we have $\delta((\ell_n^1, \tilde{\ell}_n^2, \sigma_n^i, \sigma_n^o)(\ell_{n+1}^1, \tilde{\ell}_{n+1}^2)) = \Delta(\ell_n^1, \sigma_n^i, \sigma_n^o)(\ell_{n+1}^1) \cdot \text{un}(\ell_{n+1}^1)(\tilde{\ell}_{n+1}^2)$ (i.e., it is independent of $\tilde{\ell}_n^2$). Hence using the above equality and inductive hypothesis we have:

$$\begin{aligned} \Pr_{\mu}^{\alpha_H, \beta_H}(\mathcal{E}_{n+1}^1 \mid \mathcal{E}_{n+1}^2) &= \Pr_{\mu}^{\alpha_H, \beta_H}(\mathcal{E}_{1,2}(\rho_G^1, \rho_G^2) \mid \mathcal{E}_1(\rho_G^1)) \cdot \Pr_{\mu}^{\alpha_H, \beta_H}(\bar{\mathcal{E}}_{n+1}^1 \mid \bar{\mathcal{E}}_{n+1}^2) \\ &= \text{ObsSeq}(\rho_G^1)(\rho_G^2) \cdot \Pr_{\mu}^{\alpha_H, \beta_H}(\bar{\mathcal{E}}_{n+1}^1 \mid \bar{\mathcal{E}}_{n+1}^2) \quad \text{(By inductive hypothesis)} \\ &= \text{ObsSeq}(\rho_G^1)(\rho_G^2) \cdot \text{un}(\ell_{n+1}^1)(\ell_{n+1}^2) \quad \text{(By previous equality)} \\ &= \text{ObsSeq}(\rho_G^1 \sigma_n^i \sigma_n^o \ell_{n+1}^1)(\rho_G^2 \sigma_n^i \sigma_n^o \ell_{n+1}^2) \end{aligned}$$

The desired result follows. ■

We will now establish the equivalences of the probabilities of the cones.

Lemma 2. *For all finite prefixes ρ_G^1 in G , the following assertions hold:*

1. *For all strategies $\alpha_G, \beta_G, \beta_G^A$ (all-powerful), we have*

$$\Pr_{\ell_0}^{\alpha_G, \beta_G}(\text{Cone}(\rho_G^1)) = \Pr_{\mu}^{\hat{g}(\alpha_G), \hat{g}(\beta_G)}(\text{Cone}(h_1(\rho_G^1))); \quad \Pr_{\ell_0}^{\alpha_G, \beta_G^A}(\text{Cone}(\rho_G^1)) = \Pr_{\mu}^{\hat{g}(\alpha_G), \hat{g}(\beta_G^A)}(\text{Cone}(h_1(\rho_G^1))).$$

2. *For all strategies $\alpha_H, \beta_H, \beta_H^C$ (complete-observation), we have*

$$\Pr_{\ell_0}^{\hat{h}(\alpha_H), \hat{h}(\beta_H)}(\text{Cone}(\rho_G^1)) = \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1))); \quad \Pr_{\ell_0}^{\hat{h}(\alpha_H), \hat{h}(\beta_H^C)}(\text{Cone}(\rho_G^1)) = \Pr_{\mu}^{\alpha_H, \beta_H^C}(\text{Cone}(h_1(\rho_G^1))).$$

Proof. We will present the result for the first item, and the proof for second item is identical. Let us denote by $\alpha_H = \hat{g}(\alpha_G)$ and $\beta_H = \hat{g}(\beta_G)$. We will prove the result by induction on the length of the prefixes. The base case is as follows: let the length of the prefix ρ_G^1 be 1, with $\rho_G^1 = \ell_0$. We observe that $\Pr_{\ell_0}^{\alpha_G, \beta_G}(\text{Cone}(\ell_0)) = 1$, and $\Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\ell_0))) = 1$, and for all other cones of length 1 the probability is zero. This completes the base case.

We now consider the inductive case: by inductive hypothesis we assume that $\Pr_{\ell_0}^{\alpha_G, \beta_G}(\text{Cone}(\rho_G^1)) = \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1)))$; and show that

$$\Pr_{\ell_0}^{\alpha_G, \beta_G}(\text{Cone}(\rho_G^1 a_n b_n \ell_{n+1})) = \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1 a_n b_n \ell_{n+1}))).$$

Let ℓ_n be the last state of ρ_G^1 . We first consider the left-hand side (LHS):

$$\begin{aligned} & \Pr_{\ell_0}^{\alpha_G, \beta_G}(\text{Cone}(\rho_G^1 a_n b_n \ell_{n+1})) \\ &= \Pr_{\ell_0}^{\alpha_G, \beta_G}(\text{Cone}(\rho_G^1)) \cdot \left(\sum_{\rho' \in \text{ActMt}(\rho_G^1)} \text{ObsSeq}(\rho_G^1)(\rho') \cdot \alpha_G(\rho')(a_n) \cdot \beta_G(\rho_G^1 a_n)(b_n) \cdot \Delta(\ell_n, a_n, b_n)(\ell_{n+1}) \right) \\ &= \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1))) \cdot \left(\sum_{\rho' \in \text{ActMt}(\rho_G^1)} \text{ObsSeq}(\rho_G^1)(\rho') \cdot \alpha_G(\rho')(a_n) \cdot \beta_G(\rho_G^1 a_n)(b_n) \cdot \Delta(\ell_n, a_n, b_n)(\ell_{n+1}) \right) \\ &= \sum_{\rho' \in \text{ActMt}(\rho_G^1)} \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_{12}(\rho_G^1, \rho'))) \cdot \alpha_G(\rho')(a_n) \cdot \beta_G(\rho_G^1 a_n)(b_n) \cdot \Delta(\ell_n, a_n, b_n)(\ell_{n+1}) \end{aligned}$$

Above the first equality is by definition, the second equality by inductive hypothesis, and the last equality is obtained from Lemma 1 as follows: by Lemma 1 we have $\text{ObsSeq}(\rho_G^1)(\rho') = \Pr_{\mu}^{\alpha_H, \beta_H}(\mathcal{E}_{1,2}(\rho_G^1, \rho') \mid \mathcal{E}_1(\rho_G^1))$, and hence

$$\begin{aligned} & \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1))) \cdot \sum_{\rho' \in \text{ActMt}(\rho_G^1)} \text{ObsSeq}(\rho_G^1)(\rho') \\ &= \sum_{\rho' \in \text{ActMt}(\rho_G^1)} \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1))) \cdot \Pr_{\mu}^{\alpha_H, \beta_H}(\mathcal{E}_{1,2}(\rho_G^1, \rho') \mid \mathcal{E}_1(\rho_G^1)) \\ &= \sum_{\rho' \in \text{ActMt}(\rho_G^1)} \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_{12}(\rho_G^1, \rho'))). \end{aligned}$$

We now consider the right-hand side (RHS) $\Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_1(\rho_G^1 a_n b_n \ell_{n+1})))$ and the RHS can be expanded as: (below for brevity we write $\hat{\rho} = h_{12}(\rho_G^1, \rho')$)

$$\sum_{\rho' \in \text{ActMt}(\rho_G^1)} \sum_{\ell'_{n+1}} \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(\hat{\rho})) \cdot \alpha_H(\hat{\rho})(a_n) \cdot \beta_H(\hat{\rho} a_n)(b_n) \cdot \delta((\ell_n, \ell'_n, a_n), b_n)(\ell_{n+1}, \ell'_{n+1})$$

Since we have

$$\alpha_H(h_{12}(\rho_G^1, \rho'))(a_n) = \alpha_G(\rho')(a_n); \quad \text{and} \quad \beta_H(h_{12}(\rho_G^1, \rho') a_n)(b_n) = \beta_G(\rho_G^1 a_n)(b_n),$$

the above expression for RHS is equivalently described as:

$$\sum_{\rho' \in \text{ActMt}(\rho_G^1)} \sum_{\ell'_{n+1}} \Pr_{\mu}^{\alpha_H, \beta_H}(\text{Cone}(h_{12}(\rho_G^1, \rho'))) \cdot \alpha_G(\rho')(a_n) \cdot \beta_G(\rho_G^1 a_n)(b_n) \cdot \Delta(\ell_n, a_n, b_n)(\ell_{n+1}) \cdot \text{un}(\ell_{n+1})(\ell'_{n+1})$$

Since $\sum_{\ell'_{n+1}} \text{un}(\ell_{n+1})(\ell'_{n+1}) = 1$, it follows that LHS is equal to the RHS. The result for correspondence for all-powerful strategy β_G^A is essentially copy-paste of the above proof replacing appropriately β_G by β_G^A . This completes the proof and the desired result follows. \blacksquare

It follows that there is a sure, almost-sure, positive winning strategy in G for Parity(p_G) iff there is a corresponding one in H for Parity(p_H) and hence from Theorem 1 we obtain the following result.

Theorem 2. *The following assertions hold:*

1. (All-powerful Player 2). *The sure, almost-sure and positive winning for safety objectives; the sure and almost-sure winning for reachability objectives and Büchi objectives; the sure and positive winning for coBüchi objectives; and the sure winning for parity objectives can be solved in EXPTIME for games with probabilistic uncertainty with all-powerful strategies for Player 2. The positive winning for reachability objectives can be solved in PTIME.*

2. (Not all-powerful Player 2). *The sure, almost-sure winning for safety objectives; and the sure winning for parity objectives can be solved in EXPTIME; the almost-sure winning for reachability objectives and Büchi objectives; the positive winning for safety and coBüchi objectives can be solved in 2EXPTIME for games with probabilistic uncertainty without all-powerful strategies for Player 2. The positive winning for reachability objectives can be solved in EXPTIME.*

5 Reduction: POMDPs to Games with Probabilistic Uncertainty

In this section we present a reduction in the reverse direction and show that POMDPs with parity objectives can be reduced to games with probabilistic uncertainty and parity objectives. We first present the reduction and then show the correctness of the reduction by mapping prefixes, strategies, and establishing the equivalence of the probability measure.

Reduction: POMDPs to games with probabilistic uncertainty. Let $H = (S, A, \delta, \mathcal{O})$ be a POMDP with a parity objective ϕ , we construct the game of probabilistic uncertainty $G = (L, \Sigma_I, \Sigma_O, \Delta, \text{un})$ as follows:

- $L = S$;
- $\Sigma_I = A$;
- $\Sigma_O = \{\perp\}$;
- For $\ell \in L$ and $a \in \Sigma_I$ let $\Delta(\ell, a, \perp)(\ell') = \delta(\ell, a)(\ell')$, i.e., the transition function is same as the transition function of the POMDP. In other words, the state space is the same, the action choices of the POMDP corresponds to the input action choice, and the output action set is singleton, and the transition function mimics the transition function of the POMDP. Below we use the probabilistic uncertainty to capture the partial-observation of the POMDP.
- The uncertainty function is as follows: $\text{un}(\ell)(\ell') = \begin{cases} 0 & \text{if } \text{obs}(\ell) \neq \text{obs}(\ell') \\ \frac{1}{|\text{obs}(\ell)|} & \text{if } \text{obs}(\ell) = \text{obs}(\ell') \end{cases}$

The parity objective is the same as the original parity objective.

Mapping of prefixes. Given a prefix (or a finite history) $\rho_H = s_0 a_0 s_1 a_1 s_2 \dots s_n$ in H we construct a prefix in G as $\rho_G = s_0 a_0 \perp s_1 a_1 \perp s_2 \dots s_n$ by simply inserting the \perp actions. This construction defines a bijection $h : \text{Prefs}_H \rightarrow \text{Prefs}_G$ between prefixes. We can naturally extend the mapping to sets of prefixes. Let $\Psi \subseteq \text{Prefs}_H$, then $h'(\Psi) = \{h(\rho) \mid \rho \in \Psi\}$.

Lemma 3. *For prefixes ρ, ρ' in G the following assertion holds:*

$$\text{ObsSeq}(\rho)(\rho') = \begin{cases} \frac{1}{\prod_{i=1}^n |o_i|} & \text{If } \text{obs}(h^{-1}(\rho)) = \text{obs}(h^{-1}(\rho')) = o_1 a_1 o_2 \dots a_{n-1} o_n \\ 0 & \text{Otherwise} \end{cases}$$

Proof. We prove the result by induction on the length of prefixes. We will only consider ρ and ρ' that have the same length, as otherwise by definition the observation sequence probability is 0. We first consider the base case.

Base case. Let ℓ_0 be the initial state. Then $\rho = \ell_0$ and let $\rho' = \ell$ for some $\ell \in L$. Then:

$$\text{ObsSeq}(\ell_0, \ell) = \text{un}(\ell_0, \ell) = \frac{1}{|\text{obs}(\ell_0)|}$$

if ℓ_0 and ℓ have the same observation and 0 otherwise. This proves the base case.

Inductive step. We now consider prefixes of length $n + 1$, and by inductive hypothesis the result holds for prefixes of length n . Then

$$\text{ObsSeq}(\rho a_n \perp \ell_{n+1})(\rho' a_n \perp \ell'_{n+1}) = \text{ObsSeq}(\rho)(\rho') \cdot \text{un}(\ell_{n+1})(\ell'_{n+1}).$$

We now consider two cases to complete the proof.

- If $\text{obs}(h^{-1}(\rho a_n \perp \ell_{n+1})) \neq \text{obs}(h^{-1}(\rho' a_n \perp \ell'_{n+1}))$, then either $\text{obs}(h^{-1}(\rho)) \neq \text{obs}(h^{-1}(\rho'))$ or $\text{obs}(\ell_{n+1}) \neq \text{obs}(\ell'_{n+1})$. It follows that one of the factors ($\text{ObsSeq}(\rho)(\rho')$ or $\text{un}(\ell_{n+1})(\ell'_{n+1})$) is equal to 0 and hence:

$$\text{ObsSeq}(\rho a_n \perp \ell_{n+1})(\rho' a_n \perp \ell'_{n+1}) = 0$$

– Otherwise, we have $\text{obs}(h^{-1}(\rho a_n \perp \ell_{n+1})) = \text{obs}(h^{-1}(\rho' a_n \perp \ell'_{n+1})) = o_1 a_1 o_2 \dots a_{n-1} o_n a_n o_{n+1}$. Then:

$$\text{ObsSeq}(\rho a_n \perp \ell_{n+1})(\rho' a_n \perp \ell'_{n+1}) = \text{ObsSeq}(\rho)(\rho') \cdot \text{un}(\ell_{n+1})(\ell'_{n+1}) = \frac{1}{\prod_{i=1}^n |o_i|} \cdot \frac{1}{|o_{n+1}|} = \frac{1}{\prod_{i=1}^{n+1} |o_i|}$$

The desired result follows. \blacksquare

Mapping of strategies. We first present the mapping of strategies from H to G and then from G to H . Note that in the game G , there is no choice for Player 2, and hence we remove the Player 2 strategies in the descriptions below.

Mapping strategies from H to G . Let α_H be an observation-based Player-1 strategy in H and $\rho_G = s_0 a_0 \perp s_1 a_1 \perp s_2 \dots s_n$ be a prefix in G . We define a Player-1 strategy α_G in G as follows: $\alpha_G(\rho_G) = \alpha_H(h^{-1}(\rho_G))$.

Mapping strategies from G to H . Let α_G be a Player-1 strategy in G and $\rho_H = s_0 a_0 s_1 a_1 s_2 \dots s_n$ be a prefix in H with $o = o_0 a_0 o_1 a_1 o_2 \dots o_n$ as its observation sequence. Note that as Player 2 has only one strategy (always playing \perp) we omit it from discussion. Note that every $\rho \in \text{ActMt}(h(\rho_H))$ can have different actions with different probabilities enabled. We define a Player-1 strategy α_H in H as follows: for an action $a \in A$ we have

$$\alpha_H(\rho_H)(a) = \sum_{\rho' \in \text{ActMt}(h(\rho_H))} \text{ObsSeq}(h(\rho_H))(\rho') \cdot \alpha_G(\rho')(a).$$

We now show that the strategy α_H is an observation-based strategy for Player 1 in the POMDP.

Lemma 4. *The strategy α_H obtained from strategy α_G is an observation-based strategy for Player 1 in H .*

Proof. Let ρ_H and ρ'_H be two prefixes in H that match in observation sequence and we need to argue that α_H plays the same for both prefixes ρ_H and ρ'_H . Observe that since ρ_H and ρ'_H has the same observation sequence, we have $\text{ActMt}(h(\rho_H)) = \text{ActMt}(h(\rho'_H))$. Moreover it follows from Lemma 3 that $\text{ObsSeq}(h(\rho_H))$ only depends on the observation sequence of ρ_H and hence for all $\rho' \in \text{ActMt}(h(\rho_H)) = \text{ActMt}(h(\rho'_H))$ we have $\text{ObsSeq}(h(\rho_H))(\rho') = \text{ObsSeq}(h(\rho'_H))(\rho')$. It follows that for all actions $a \in A$ we have $\alpha_H(\rho_H)(a) = \alpha_H(\rho'_H)(a)$. It follows that α_H is observation based. \blacksquare

Correspondence of probabilities. In the following two lemmas we establish the correspondence of the probabilities for the mappings.

Lemma 5. *Let us consider the mapping of strategies from H to G . For all prefixes ρ_H in H we have*

$$\text{Pr}_\mu^{\alpha_H}(\text{Cone}(\rho_H)) = \text{Pr}_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H))).$$

Proof. The proof is based on induction on the length of the prefix ρ_H . We denote the last state of ρ_H by ℓ_n .

Base case. For prefixes of length 1 where $\rho_H = \ell_0$ we get $\text{Pr}_\mu^{\alpha_H}(\text{Cone}(\ell_0)) = 1$ and $\text{Pr}_{\ell_0}^{\alpha_G}(\text{Cone}(h(\ell_0))) = 1$. For all other prefixes both sides are equal to 0. Hence the base case follows.

Inductive step. By inductive hypothesis we assume the result for prefixes ρ_H of length n (i.e., we assume that $\text{Pr}_\mu^{\alpha_H}(\text{Cone}(\rho_H)) = \text{Pr}_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H)))$) and will show that

$$\text{Pr}_\mu^{\alpha_H}(\text{Cone}(\rho_H a_n \ell_{n+1})) = \text{Pr}_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H a_n \ell_{n+1}))).$$

First we expand the left hand side (LHS) and by definition we get that:

$$\text{Pr}_\mu^{\alpha_H}(\text{Cone}(\rho_H a_n \ell_{n+1})) = \text{Pr}_\mu^{\alpha_H}(\text{Cone}(\rho_H)) \cdot \alpha_H(\rho_H)(a_n) \cdot \delta(\ell_n, a_n)(\ell_{n+1}).$$

We now expand the right hand side (RHS) and get that:

$$\begin{aligned} \text{Pr}_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H a_n \ell_{n+1}))) &= \\ \text{Pr}_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H))) &\cdot \left(\sum_{\rho' \in \text{ActMt}(h(\rho_H))} \text{ObsSeq}(h(\rho_H))(\rho') \cdot \alpha_G(\rho')(a_n) \cdot \Delta(\ell_n, a_n, \perp)(\ell_{n+1}) \right) \end{aligned}$$

Using the inductive hypothesis, the definition of the game, and the mapping of strategies we get on the RHS:

$$\Pr_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H a_n \ell_{n+1}))) = \Pr_{\mu}^{\alpha_H}(\text{Cone}(\rho_H)) \cdot \left(\sum_{\rho' \in \text{ActMt}(h(\rho_H))} \text{ObsSeq}(h(\rho_H))(\rho') \cdot \alpha_H(h^{-1}(\rho'))(a_n) \cdot \delta(\ell_n, a_n)(\ell_{n+1}) \right)$$

For all ρ' that do not match the observation sequence of $h(\rho_H)$, we have $\text{ObsSeq}(h(\rho_H))(\rho') = 0$ (by Lemma 3), and as α_H is observation based for all $\rho' \in \text{ActMt}(\rho_H)$ that matches the observation sequence of $h(\rho_H)$, the strategy α_H plays the same. Let us denote by $\rho' \approx h(\rho_H)$ that ρ' matches the observation sequence of $h(\rho_H)$. Then we have

$$\begin{aligned} & \sum_{\rho' \in \text{ActMt}(h(\rho_H))} \text{ObsSeq}(h(\rho_H))(\rho') \cdot \alpha_H(h^{-1}(\rho'))(a_n) \\ &= \sum_{\rho' \in \text{ActMt}(h(\rho_H)), \rho' \approx h(\rho_H)} \text{ObsSeq}(h(\rho_H))(\rho') \cdot \alpha_H(h^{-1}(\rho'))(a_n) \\ &= \sum_{\rho' \in \text{ActMt}(h(\rho_H)), \rho' \approx h(\rho_H)} \text{ObsSeq}(h(\rho_H))(\rho') \cdot \alpha_H(\rho_H)(a_n) \\ &= \alpha_H(\rho_H)(a_n); \end{aligned}$$

where the first equality follows as for all sequences ρ' that do not match the observation sequence of $h(\rho_H)$ we have $\text{ObsSeq}(h(\rho_H))(\rho') = 0$; the second equality follows as for all $\rho' \approx h(\rho_H)$ we have $\alpha_H(h^{-1}(\rho'))(a_n) = \alpha_H(\rho_H)(a_n)$ (as α_H is observation based); and the last equality follows because as ObsSeq is a probability distribution we have $\sum_{\rho' \in \text{ActMt}(h(\rho_H)), \rho' \approx h(\rho_H)} \text{ObsSeq}(h(\rho_H))(\rho') = 1$. Hence we have

$$\Pr_{\ell_0}^{\alpha_G}(\text{Cone}(h(\rho_H a_n \ell_{n+1}))) = \Pr_{\mu}^{\alpha_H}(\text{Cone}(\rho_H)) \cdot \alpha_H(\rho_H)(a_n) \cdot \delta(\ell_n, a_n)(\ell_{n+1})$$

Thus we have that LHS and RHS coincide and this completes the proof. \blacksquare

Lemma 6. *Let us consider the mapping of strategies from G to H . For all prefixes ρ_G in G we have*

$$\Pr_{\mu}^{\alpha_H}(\text{Cone}(h^{-1}(\rho_G))) = \Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G))$$

Proof. The inductive proof is as follows and we will denote the last state of ρ_G as ℓ_n . The base case is similar to the base case of Lemma 5. We now present the inductive case.

Inductive step. By inductive hypothesis we assume the result for prefixes ρ_G of length n (i.e., we assume that $\Pr_{\mu}^{\alpha_H}(\text{Cone}(h^{-1}(\rho_G))) = \Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G))$) and will show that

$$\Pr_{\mu}^{\alpha_H}(\text{Cone}(h^{-1}(\rho_G a_n \ell_{n+1}))) = \Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G a_n \ell_{n+1})).$$

First we expand the right hand side (RHS) and by definition we get that:

$$\Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G a_n \ell_{n+1})) = \Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G)) \cdot \left(\sum_{\rho' \in \text{ActMt}(\rho_G)} \text{ObsSeq}(\rho_G)(\rho') \cdot \alpha_G(\rho')(a_n) \cdot \Delta(\ell_n, a_n, \perp)(\ell_{n+1}) \right)$$

As $\Delta(\ell_n, a_n, \perp)(\ell_{n+1})$ does not depend on ρ' we get:

$$\Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G a_n \ell_{n+1})) = \Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G)) \cdot \Delta(\ell_n, a_n, \perp)(\ell_{n+1}) \cdot \left(\sum_{\rho' \in \text{ActMt}(\rho_G)} \text{ObsSeq}(\rho_G)(\rho') \cdot \alpha_G(\rho')(a_n) \right)$$

We will now show that the expansion of the left hand side (LHS) also gives the same expression. Let $\rho_H = h^{-1}(\rho_G)$. By expanding the LHS we get:

$$\begin{aligned}
\Pr_{\mu}^{\alpha_H}(\text{Cone}(h^{-1}(\rho_G a_n \ell_{n+1}))) &= \Pr_{\mu}^{\alpha_H}(\text{Cone}(h^{-1}(\rho_G))) \cdot \alpha_H(h^{-1}(\rho_G))(a_n) \cdot \delta(\ell_n, a_n)(\ell_{n+1}) \\
&= \Pr_{\mu}^{\alpha_H}(\text{Cone}(\rho_H)) \cdot \alpha_H(\rho_H)(a_n) \cdot \delta(\ell_n, a_n)(\ell_{n+1}) \\
&= \Pr_{\mu}^{\alpha_H}(\text{Cone}(\rho_H)) \cdot \alpha_H(\rho_H)(a_n) \cdot \Delta(\ell_n, a_n, \perp)(\ell_{n+1}) \\
&= \Pr_{\ell_0}^{\alpha_G}(\text{Cone}(\rho_G)) \cdot \alpha_H(\rho_H)(a_n) \cdot \Delta(\ell_n, a_n, \perp)(\ell_{n+1});
\end{aligned}$$

where the first equality is by definition; the second equality is by simply re-writing $h^{-1}(\rho_G)$ as ρ_H ; the third equality is by the definition of Δ and δ ; and the final equality is the inductive hypothesis. By definition of α_H we have $\alpha_H(\rho_H)(a_n) = \left(\sum_{\rho' \in \text{ActMt}(\rho_G)} \text{ObsSeq}(\rho_G)(\rho') \cdot \alpha_G(\rho')(a_n) \right)$; and hence it follows that LHS and RHS coincide. Thus the desired result follows. \blacksquare

The previous two lemmas establish the equivalence of the probability measure and completes the reduction of POMDPs to games with probabilistic uncertainty. Hence the lower bounds for POMDPs also gives us the lower bound for games with probabilistic uncertainty. Hence Theorem 2, along with the reduction from POMDPs and Theorem 1 gives us the following result for games with probabilistic uncertainty (the results are also summarized in Table 1).

Theorem 3. *The following assertions hold:*

1. (All-powerful Player 2). *The sure, almost-sure and positive winning for safety objectives; the sure and almost-sure winning for reachability objectives and Büchi objectives; the sure and positive winning for coBüchi objectives; and the sure winning for parity objectives are all EXPTIME-complete for games with probabilistic uncertainty with all-powerful strategies for Player 2. The positive winning for reachability objectives is PTIME-complete.*
2. (Not all-powerful Player 2). *The sure, almost-sure winning for safety objectives; and the sure winning for parity objectives are all EXPTIME-complete; the almost-sure winning for reachability objectives and Büchi objectives; the positive winning for safety and coBüchi objectives can be solved in 2EXPTIME and is EXPTIME-hard for games with probabilistic uncertainty without all-powerful strategies for Player 2. The positive winning for reachability objectives can be solved in EXPTIME.*
3. (Undecidability results). *The positive winning problem for Büchi objectives, the almost-sure winning problem for coBüchi objectives, and the positive and almost-sure winning problem for parity objectives are undecidable for games with probabilistic uncertainty.*

	Sure		Almost		Positive	
	All-powerful	Not-all-powerful	All-powerful	Not-all-powerful	All-powerful	Not-all-powerful
Safety	EXP-complete	EXP-complete	EXP-complete	EXP-complete	EXP-complete	2EXP, EXP
Reachability	EXP-complete	EXP-complete	EXP-complete	2EXP, EXP	PTIME-complete	EXP, PTIME
Büchi	EXP-complete	EXP-complete	EXP-complete	2EXP, EXP	Undec.	Undec.
coBüchi	EXP-complete	EXP-complete	Undec.	Undec.	EXP-complete	2EXP, EXP
Parity	EXP-complete	EXP-complete	Undec.	Undec.	Undec.	Undec.

Table 1. Complexity of games with probabilistic uncertainty with parity objectives, where for each entry we present the upper and lower bound, or undecidability.

6 Conclusion

In this work we considered games with probabilistic uncertainty, which is natural for many problems, and has not been considered before. We present a reduction of such games to classical partial-observation games and a reduction

of POMDPs to games with probabilistic uncertainty. As a consequence we establish the precise decidability frontier for games with probabilistic uncertainty. Table 1 summarizes our results. For most problems we establish EXPTIME-complete bounds. For some decidable problems we establish 2EXPTIME upper bounds, and EXPTIME lower bounds, and establishing the precise complexity results are interesting open problems.

References

1. C. Baier, N. Bertrand, and M. Größer. On decision problems for probabilistic Büchi automata. In *FoSSaCS*, LNCS 4962, pages 287–301. Springer, 2008.
2. N. Bertrand, B. Genest, and H. Gimbert. Qualitative determinacy and decidability of stochastic games with signals. In *LICS*, pages 319–328. IEEE Computer Society, 2009.
3. D. Berwanger and L. Doyen. On the power of imperfect information. In *FSTTCS*, Dagstuhl Seminar Proceedings 08004. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), 2008.
4. P. Billingsley. *Probability and Measure*. Wiley-Interscience, 1995.
5. J.R. Büchi and L.H. Landweber. Solving sequential conditions by finite-state strategies. *Transactions of the AMS*, 138:295–311, 1969.
6. K. Chatterjee, L. Doyen, and T. A. Henzinger. Qualitative analysis of partially-observable markov decision processes. In *MFCS*, pages 258–269, 2010.
7. K. Chatterjee, L. Doyen, T. A. Henzinger, and J.-F. Raskin. Algorithms for omega-regular games of incomplete information. *Logical Methods in Computer Science*, 3(3:4), 2007.
8. A. Condon. The complexity of stochastic games. *Information and Computation*, 96(2):203–224, 1992.
9. C. Courcoubetis and M. Yannakakis. The complexity of probabilistic verification. *Journal of the ACM*, 42(4):857–907, 1995.
10. L. de Alfaro, T.A. Henzinger, and O. Kupferman. Concurrent reachability games. *TCS*, 386(3):188–217, 2007.
11. L. de Alfaro and R. Majumdar. Quantitative solution of omega-regular games. In *STOC'01*, pages 675–683. ACM Press, 2001.
12. J. Filar and K. Vrieze. *Competitive Markov Decision Processes*. Springer-Verlag, 1997.
13. O. Kupferman and M.Y. Vardi. μ -calculus synthesis. In *Proc. 25th International Symp. on Mathematical Foundations of Computer Science*, volume 1893 of *Lecture Notes in Computer Science*, pages 497–507. Springer-Verlag, 2000.
14. M.O. Rabin. *Automata on Infinite Objects and Church's Problem*. Number 13 in Conference Series in Mathematics. American Mathematical Society, 1969.
15. J. H. Reif. Universal games of incomplete information. In *STOC*, pages 288–308. ACM Press, 1979.
16. W. Thomas. Languages, automata, and logic. In *Handbook of Formal Languages*, volume 3, Beyond Words, chapter 7, pages 389–455. Springer, 1997.
17. M. Y. Vardi. Automatic verification of probabilistic concurrent finite-state systems. In *FOCS'85*, pages 327–338. IEEE Computer Society Press, 1985.