Interval Scheduling and Colorful Independent Sets*

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Abstract. Numerous applications in scheduling, such as resource allocation or steel manufacturing, can be modeled using the NP-hard INDEPENDENT SET problem (given an undirected graph and an integer k, find a set of at least k pairwise non-adjacent vertices). Here, one encounters special graph classes like 2-union graphs (edge-wise unions of two interval graphs) and strip graphs (edge-wise unions of an interval graph and a cluster graph), on which INDEPENDENT SET remains NP-hard but admits constant-ratio approximations in polynomial time.

We study the parameterized complexity of INDEPENDENT SET on 2-union graphs and on subclasses like strip graphs. Our investigations significantly benefit from a new structural "compactness" parameter of interval graphs and novel problem formulations using vertex-colored interval graphs. Our main contributions are:

- 1. We show a complexity dichotomy: restricted to graph classes closed under induced subgraphs and disjoint unions, INDEPENDENT SET is polynomial-time solvable if both input interval graphs are cluster graphs, and is NP-hard otherwise.
- 2. We chart the possibilities and limits of effective polynomial-time preprocessing (also known as kernelization).
- 3. We extend Halldórsson and Karlsson (2006)'s fixed-parameter algorithm for Independent Set on strip graphs parameterized by the structural parameter "maximum number of live jobs" to show that the problem (also known as Job Interval Selection) is fixed-parameter tractable with respect to the parameter k and generalize their algorithm from strip graphs to 2-union graphs. Preliminary experiments with random data indicate that Job Interval Selection with up to fifteen jobs and $5 \cdot 10^5$ intervals can be solved optimally in less than five minutes.

^{*}A preliminary version of this article appeared in the proceedings of the 23rd International Symposium on Algorithms and Computation (ISAAC 2012), volume 7676 in Lecture Notes in Computer Science, pp. 247–256, Springer, 2012. Besides providing full proof details, this revised and extended version improves running times, shows that Job Interval Selection is fixed-parameter tractable with respect to the standard parameter k, and introduces the parameter c-compactness. Moreover, it adds an experimental evaluation of the algorithms.

1 Introduction

Many fundamental scheduling problems can be modeled as finding maximum independent sets in generalizations of interval graphs (Kolen et al., 2007). Intuitively, finding a maximum independent set corresponds to scheduling a maximum number of jobs (represented by time intervals) on a limited set of machines in a given time frame.

In this context, we consider two popular generalizations of interval graphs, namely 2-union graphs (Bar-Yehuda et al., 2006) and strip graphs (Halldórsson and Karlsson, 2006): An undirected graph G = (V, E) is a 2-union graph if it is the edge-wise union of two interval graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set V, that is, $G = (V, E_1 \cup E_2)$, where an interval graph is a graph whose vertices one-to-one correspond to intervals on the real line and there is an edge between two vertices if and only if their intervals intersect. If one of the two interval graphs G_1 or G_2 is even a cluster graph, that is, if it consists of pairwise disjoint cliques, then G is called a strip graph.

Examples for solving scheduling problems using (weighted) INDEPENDENT SET on 2-union graphs include resource allocation scenarios (Bar-Yehuda et al., 2006) and coil coating in steel manufacturing (Höhn et al., 2011; Möhring, 2011). Formally, we are interested in the following problem:

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2-Union Independent Set
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Input: Two interval graphs $G_1 = (V, E_1), G_2 = (V, E_2)$, and a natural number k.

Question: Is there a size-k independent set in $G = (V, E_1 \cup E_2)$?

If G is a strip graph, then the problem is known as Job Interval Selection (Spieksma, 1999). We make two main conceptual contributions:

- 1. Since 2-UNION INDEPENDENT SET is NP-hard (Bar-Yehuda et al., 2006), there is little hope to find optimal solutions within polynomial time. Instead of following the route of approximation algorithms and heuristics (Spieksma, 1999; Bar-Yehuda et al., 2006; Höhn et al., 2011), we aim for solving the problem optimally using fixed-parameter algorithms (Downey and Fellows, 2013; Flum and Grohe, 2006; Niedermeier, 2006), a concept to date largely neglected in the field of scheduling problems (Marx, 2011; Mnich and Wiese, 2014).
- 2. In order to obtain our results, we provide "colorful reformulations" of 2-UNION INDEPENDENT SET and JOB INTERVAL SELECTION, providing characterizations of these problems in terms of vertex-colored interval graphs, thus replacing the conceptually more complicated 2-union and strip graphs.

1.1 Known Results

Results for 2-Union Independent Set. Checking whether a graph is a 2-union graph is NP-hard (Gyárfas and West, 1995; Jiang, 2013). Therefore, we require two separate interval graphs as input to 2-UNION INDEPENDENT SET.

To date, a number of polynomial-time approximation algorithms has been devised to solve 2-UNION INDEPENDENT SET. Bar-Yehuda et al. (2006) showed that vertex-weighted 2-UNION INDEPENDENT SET admits a polynomial-time ratio-4 approximation. For the special case of so-called $K_{1,5}$ -free graphs (which comprises the case that both input graphs are proper interval graphs), Bafna et al. (1996) provided a ratio-3.25 approximation.

In the context of applying 2-UNION INDEPENDENT SET to coil coating—a process in steel manufacturing—Höhn et al. (2011) showed NP-hardness of 2-UNION INDEPENDENT SET on so-called M-composite 2-union graphs (which arise in their application), and showed a dynamic programming based algorithm running in polynomial time for constant M, where the degree of the polynomial depends on M. They additionally provided experimental studies based on heuristics using mathematical programming.

Regarding parameterized complexity, Jiang (2010) proved that 2-UNION INDEPENDENT SET is W[1]-hard parameterized by the independent set size k, thus excluding any hope for fixed-parameter tractability with respect to k. Jiang's W[1]-hardness result holds even when both input graphs are proper interval graphs.

Results for Job Interval Selection. Job Interval Selection was introduced by Nakajima and Hakimi (1982) and was shown APX-hard by Spieksma (1999), who also provided a ratio-2 greedy approximation algorithm. Chuzhoy et al. (2006) improved this to a ratio-1.582 approximation algorithm. Halldórsson and Karlsson (2006) showed fixed-parameter tractability results for Job Interval Selection in terms of the structural parameter "maximum number of live jobs" and in terms of the parameter "total number of jobs". Moreover, they showed that recognizing strip graphs is NP-hard.

1.2 Our Results

We provide a refined computational complexity analysis for 2-UNION INDEPENDENT SET. Herein, our results mainly touch parameterized complexity.

We start by proving a complexity dichotomy that shows that all problem variants encountered in our work remain NP-hard: roughly speaking, we show that INDEPENDENT SET is polynomial-time solvable if the input is the edge-wise union of two cluster graphs, while it is NP-hard otherwise.

Results for Job Interval Selection. We complement known polynomial-time approximability results (Spieksma, 1999; Chuzhoy et al., 2006) for Job Interval Selection with parameterized complexity results and extend the tractability results by Halldórsson and Karlsson (2006) in several ways:

1. We generalize their fixed-parameter algorithm for Job Interval Selection parameterized by the maximum number of "live jobs" to 2-Union Independent Set. Moreover, for Job Interval Selection, we show that it can be turned into a fixed-parameter algorithm with respect to the parameter k ("number

of selected intervals"). Note that the latter appears to be impossible for 2-Union Independent Set, which is W[1]-hard for the parameter k (Jiang, 2010).

- 2. We prove the non-existence of polynomial-size problem kernels for Job Interval Selection with respect to k and structural parameters like the maximum clique size ω , thus lowering hopes for provably efficient and effective preprocessing.
- 3. We show that, if the input graph is the edge-wise union of a cluster graph and a *proper* interval graph, then Job Interval Selection admits a problem kernel comprising $4k^2\omega$ intervals that can be computed in linear time.

Results for 2-Union Independent Set. Since 2-Union Independent Set is W[1]-hard with respect to the parameter k (Jiang, 2010) and NP-hard even when natural graph parameters like "maximum clique size ω " or "maximum vertex degree Δ " are constants (which is implied by our complexity dichotomy), 2-Union Independent Set is unlikely to be fixed-parameter tractable for any of these parameters.

However, we identify a new natural interval graph parameter that highly influences the computational complexity of 2-UNION INDEPENDENT SET: we call an interval graph c-compact if its intervals are representable using at most c distinct start and end points. That is, c is the "number of numbers" required in an interval representation. Similar "number of numbers" parameters have previously been exploited to obtain fixed-parameter algorithms for problems unrelated to interval graphs (Fellows et al., 2012).

We use c_{\forall} to denote the minimum number such that *both* input interval graphs are c_{\forall} -compact and c_{\exists} to denote the minimum number such that at least *one* input interval graph is c_{\exists} -compact. We obtain the following results:

- 1. We give a simple polynomial-time data reduction rule for 2-UNION INDE-PENDENT SET. The analysis of its effectiveness naturally leads to the compactness parameter: the reduction rule yields a c_{\forall}^3 -vertex problem kernel. This improves to a $2c_{\forall}^2$ -vertex problem kernel if one of the input graphs is a proper interval graph.
- 2. The problem kernel with respect to c_{\forall} shows that 2-UNION INDEPENDENT SET is fixed-parameter tractable with respect to c_{\forall} . By generalizing Halldórsson and Karlsson (2006)'s fixed-parameter algorithm from Job Interval Selection to 2-Union Independent Set, we improve this to a time- $O(2^{c_{\exists}} \cdot n)$ fixed-parameter algorithm for the parameter $c_{\exists} \leq c_{\forall}$.

Table 1 summarizes our results. Experiments with random data indicate that, within less than five minutes, one can optimally solve JOB INTERVAL SELECTION with up to fifteen jobs and $5 \cdot 10^5$ intervals and 2-UNION INDEPENDENT SET with $c_{\exists} \leq 15$ and $5 \cdot 10^5$ intervals.

Table 1: Overview of parameterized complexity results for Job Interval Selection, where G_2 —one of the two input graphs—is a cluster graph, and 2-Union Independent Set, where G_2 is any interval graph. Results for various graph classes of G_1 —the other input graph—are shown. The complexity dichotomy in Theorem 1 shows that all these problem variants remain NP-hard.

Class of G_1	Job Interval Selection	2-Union Independent Set
interval	randomized FPT algorithm: $O(5.5^k \cdot n)$ time (Theorem 4)	FPT algorithm: $O(2^{c_{\exists}} \cdot n)$ time (Theorem 7)
	No polynomial-size kernel w. r. t. k and ω (Theorem 5)	problem kernel: c_{\forall}^3 vertices in $O(n \log^2 n)$ time (Theorem 8)
proper interval	problem kernel: $4k^2\omega$ vertices in $O(n)$ time (Theorem 6)	problem kernel: $2c_{\forall}^2$ vertices in $O(n \log^2 n)$ time (Theorem 8)

Organization of this Work. In Section 2, we introduce basic notation and the concepts of parameterized algorithmics.

Section 3 introduces the compactness parameter for interval graphs and some basic observations on compactness. In the remaining sections, we assume to work on c-compact representations of interval graphs such that c is minimum.

Section 4 presents our colored model of 2-UNION INDEPENDENT SET and JOB INTERVAL SELECTION and discusses pros and cons of the new model.

Section 5 presents a computational complexity dichotomy that has consequences both for Job Interval Selection and 2-Union Independent Set.

Section 6 presents our results specific to Job Interval Selection, whereas Section 7 contains the results for the more general 2-Union Independent Set. Finally, we present experimental results in Section 8 and conclude in Section 9.

2 Preliminaries

Throughout the work, we use the notation [c] as shorthand for the subset $\{1, 2, \ldots, c\}$ of natural numbers.

We consider undirected, finite graphs G = (V, E) with vertex set V(G) and edge set E(G). If not stated otherwise, we use n := |V| and m := |E|. Two vertices $v, w \in V$ are adjacent or neighbors if $\{v, w\} \in E$. The open neighborhood $N_G(v)$ of a vertex $v \in V$ is the set of vertices that are adjacent to v, the closed neighborhood is $N_G[v] := N_G(v) \cup \{v\}$. For a vertex set $U \subseteq V$, we define $N_G[U] := \bigcup_{v \in U} N_G[v]$. If the graph G is clear from context, we drop the subscript G. For a vertex set $V' \subseteq V$, the induced subgraph G[V'] is the graph obtained from G by deleting all vertices in $V \setminus V'$.

An independent set is a set of pairwise non-adjacent vertices. A matching is a set of pairwise disjoint edges. The chromatic index $\chi'(G)$ of G is the minimum

number of colors required in a *proper edge coloring*, that is, in a coloring of edges of G such that no pair of edges sharing a vertex has the same color.

A path in G from v_1 to v_ℓ is a sequence $(v_1, v_2, \ldots, v_\ell) \in V^\ell$ of vertices with $\{v_i, v_{i+1}\} \in E$ for $i \in [\ell-1]$. Its length is $\ell-1$. We denote a path on ℓ vertices by P_ℓ . Two vertices v and v are connected in v if there is a path from v to v in v to v in v to v in v connected component of v is a maximal set of pairwise connected vertices. If in each connected component of v, all its vertices are pairwise adjacent (that is, they form a v connected v to v contain a v contain a v as induced subgraph.

The disjoint union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \uplus G_2 = (V_1 \uplus V_2, E_1 \uplus E_2)$, where $V_1 \cap V_2 = \emptyset$. The edge-wise union of two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set is $G_1 \cup G_2 = (V, E_1 \cup E_2)$. A class of graphs \mathcal{C} is closed under induced subgraphs if $G = (V, E) \in \mathcal{C}$ implies $G[V'] \in \mathcal{C}$ for any $V' \subseteq V$. A class of graphs \mathcal{C} is closed under disjoint unions if $G_1, G_2 \in \mathcal{C}$ implies $G_1 \uplus G_2 \in \mathcal{C}$.

An interval graph is a graph whose vertices can be represented as closed intervals on the real line such that two vertices v and w are adjacent if and only if the intervals corresponding to v and w intersect. We denote the start point of the interval associated with v by v_s and its end point by v_e . A graph is a proper interval graph if it allows for an interval representation such that for no two intervals v and w it holds that $v \subseteq w$. Equivalently, proper interval graphs are precisely those interval graphs that do not contain a $K_{1,3}$ as induced subgraph (Brandstädt et al., 1999).

Fixed-Parameter Algorithms. The main idea in fixed-parameter algorithms is to accept exponential running time, which is seemingly inevitable in solving NP-hard problems, but to restrict it to one aspect of the problem, the *parameter*. More precisely, a problem Π is *fixed-parameter tractable (FPT)* with respect to a parameter k if there is an algorithm solving any instance of Π with size n in $f(k) \cdot \text{poly}(n)$ time for some computable function f (Downey and Fellows, 2013; Flum and Grohe, 2006; Niedermeier, 2006). Such an algorithm is potentially efficient for small values of k.

Problem Kernelization. One way of deriving fixed-parameter algorithms is problem kernelization (Guo and Niedermeier, 2007; Kratsch, 2014). As a formal approach of describing efficient data reduction that preserves optimal solutions, problem kernelization is a powerful tool for attacking NP-hard problems. A kernelization algorithm consists of polynomial-time executable data reduction rules that, applied to any instance x with parameter k, yield an equivalent instance x' with parameter k', such that both the size |x'| and k' are bounded by some functions g and g' in k, respectively. The function g is referred to as the size of the problem kernel (x',k'). Mostly, the focus lies on finding problem kernels of polynomial size.

3 Compact Interval Graphs

A parameter that we will see to highly influence the computational complexity of 2-UNION INDEPENDENT SET is the "compactness" of an interval graph, which corresponds to the number of distinct numbers required in its interval representation.

Definition 1. An interval representation is c-compact if the start and end point of each interval lies in [c]. Moreover, the intervals are required to be sorted by increasing start points.

An interval graph is *c-compact* if it admits a *c-*compact interval representation.

In this work, we state all running times under the assumption that a c-compact interval representation for minimum c is given. We show in the following that such a representation can be efficiently computed. To this end, we make a few observations.

Observation 1. Let G be an interval graph and c be the minimum integer such that G is c-compact. Then G has exactly c maximal cliques.

Proof. Let c' be the number of maximal cliques in G. We show c = c' by proving $c' \le c$ and $c \le c'$ independently.

First, it is easy to see that a c-compact interval graph has at most c maximal cliques: each interval end point v_e gives rise to at most one maximal clique, which consists of the intervals containing the point v_e . Hence, $c' \leq c$.

Second, the interval graph G allows for an ordering of its c' maximal cliques such that the cliques containing an arbitrary vertex occur consecutively in the ordering (Fulkerson and Gross, 1965). Hence, a c'-compact interval representation can be constructed in which each vertex v is represented by the interval $[v_s, v_e]$, where v_s is the number of the first maximal clique containing v and v_e is the number of the last maximal clique containing v. It follows that $c \leq c'$.

From Observation 1, it immediately follows that

Observation 2. An *n*-vertex interval graph is *n*-compact.

In the remainder of this article, we will assume to be given a c-compact representation for $minimum\ c$. This assumption is justified by the fact that such a c-compact representation is computable in $O(n\log n)$ time from an arbitrary interval representation or even in linear time from a graph given as adjacency list.

Observation 3. Any interval representation of an interval graph G can be converted into a c-compact representation for G in $O(n \log n)$ time such that

- i) at each position in [c], there is an interval start point and an interval end point, and
- ii) c is the minimum number such that G is c-compact.

Proof. We first sort all *event points* (start or end points of intervals) in increasing order in $O(n \log n)$ time. Then, in linear time, we iterate over all event points in increasing order and move each event point to the smallest possible integer position that maintains all pairwise intersections. It remains to show (i) and (ii).

(i) First observe that every interval start point v_s is also an end point for some interval: otherwise, we would have moved the event point v_e' (possibly, $v_e' = v_e$) that directly follows v_s to the position $v_e' - 1$, maintaining all pairwise intersections. It follows that G is c'-compact, where c' is the number of different end point positions.

Second, every interval end point v_e is also a start point for some interval: otherwise, we could have moved v_e to the position v_e-1 maintaining all pairwise intersections. It follows that each end point v_e gives rise to a distinct maximal clique, because the interval starting at v_e cannot be part of the maximal cliques raised by earlier end points.

(ii) From (i), it follows that, for any two positions i,j of event points, the set of intervals containing i and the set of intervals containing j are distinct and, therefore, i and j give rise to distinct maximal cliques in G. Thus, our algorithm computes a c'-compact representation of G with $c' \leq c$, where c is the number of maximal cliques in G. From Observation 1, it follows that c' is the minimum number such that G is c'-compact.

If the input graph is given in form of an adjacency list, we can compute a c-compact representation for minimum c in linear time.

Observation 4. Given an interval graph G as adjacency list, a c-compact representation for minimum c can be computed in O(n+m) time.

Proof. Using a linear-time algorithm by Corneil et al. (2009, Section 8), we obtain an n-compact interval representation of G. Using this, we can execute the algorithm in the proof of Observation 3 in linear time, since the list of sorted event points can be obtained in linear time using counting sort: the list has n elements and the sorting keys are integers not exceeding n.

4 Colorful Independent Sets

Many of our results significantly benefit from a novel but natural embedding of 2-Union Independent Set into a more general problem: Colorful Independent Set with Lists. We discuss this embedding in the following.

4.1 Colorful Independent Sets and Job Interval Selection

The first step in formalizing 2-UNION INDEPENDENT SET as COLORFUL IN-DEPENDENT SET WITH LISTS is an alternative formulation of the classical scheduling problem Job Interval Selection.

The task in Job Interval Selection is to execute a maximum number of jobs out of a given set, where each job has multiple possible execution intervals,

each job is executed at most once, and a machine can only execute one job at a time. We formally state this problem in terms of colored interval graphs, where the colors correspond to jobs and intervals of one color correspond to multiple possible execution times of the same job:

JOB INTERVAL SELECTION

Input: An interval graph G=(V,E), a coloring col: $V\to [\gamma]$, and a natural number k.

Question: Is there a size-k colorful independent set in G?

Here, *colorful* means that no two vertices of the independent set have the same color.

Note that the colored formulation of Job Interval Selection is indeed equivalent to the known formulation (Spieksma, 1999) as special case of 2-Union Independent Set, where one input interval graph is a cluster graph:

Input: An interval graph $G_1 = (V, E_1)$, a cluster graph $G_2 = (V, E_2)$, and a natural number k.

Question: Is there a size-k independent set in $G = (V, E_1 \cup E_2)$?

In this second formulation, the maximal cliques of G_2 correspond to jobs, and the intervals in G_1 that are part of the same maximal clique in G_2 correspond to multiple possible execution times of the same job. That is, the maximal cliques in G_2 one-to-one correspond to the colors in the colorful problem formulation.

In Section 6.2, we restate the fixed-parameter algorithms for Job Interval Selection by Halldórsson and Karlsson (2006) in terms of our colorful formulation. This formulation uncouples the algorithms from the geometric arguments originally used by Halldórsson and Karlsson (2006) and allows for a more combinatorial point of view. Exploiting this, we turn Halldórsson and Karlsson (2006)'s fixed-parameter algorithms for the total number of jobs (which translates to the number γ of colors in our formulation) into a fixed-parameter algorithm for the smaller parameter k—the number of jobs we want to execute.

4.2 From Strip Graphs to 2-Union Graphs

Our more combinatorially stated version of Halldórsson and Karlsson (2006)'s fixed-parameter algorithm easily applies to Colorful Independent Set with Lists, which is a canonical generalization of Job Interval Selection:

Colorful Independent Set with Lists

Input: An interval graph G = (V, E), a list-coloring col: $V \to 2^{[\gamma]}$, and a natural number k.

Question: Is there a size-k colorful independent set in G?

Here, *colorful* means that the intersection of the color sets of any two vertices in the independent set is empty.

We will later show that COLORFUL INDEPENDENT SET WITH LISTS is actually even more general than 2-UNION INDEPENDENT SET. The colored reformulation turned out to be the key in generalizing Halldórsson and Karlsson (2006)'s algorithm for Job Interval Selection to 2-Union Independent Set.

4.3 Advantages and Limitations of the Model

The colorful formulation of Job Interval Selection helps us to transform Halldórsson and Karlsson (2006)'s fixed-parameter algorithm for the parameter "number γ of colors" into a fixed-parameter algorithm for the parameter "size k of the sought colorful independent set", where $\gamma \leq k$. Moreover, the algorithm for the colorful formulation of Job Interval Selection easily generalizes to Colorful Independent Set with Lists and, as we will see, to 2-Union Independent Set.

The advantage of considering Colorful Independent Set with Lists instead of 2-Union Independent Set is that one can concentrate on a single given interval graph instead of two merged ones, thus making the numerous structural results on interval graphs applicable. Possibly, the colorful view on finding independent sets and scheduling might be useful in further studies.

Not always, however, the colorful viewpoint is superior to the geometric one. Herein, it is important to note that Colorful Independent Set with Lists is actually a more general problem than 2-Union Independent Set and it is cumbersome to formulate precisely 2-Union Independent Set in terms of Colorful Independent Set with Lists. Thus, when exploiting the specific combinatorial properties of 2-Union Independent Set, for example in the kernelization algorithm in Section 7, the colored model is not helpful.

Moreover, in the following Section 5, we prove hardness results for finding independent sets not only on 2-union and strip graphs. Hence, the colorful model is not exploited there.

5 A Complexity Dichotomy

In this section, we determine the computational complexity of INDEPENDENT SET on edge-wise unions of graphs in dependence of the allowed input graph classes. Formally, we define the considered problem as follows:

COMMON INDEPENDENT SET

Input: Two graphs $G_1 = (V, E_1), G_2 = (V, E_2)$, and a natural number k. **Question:** Is there a size-k independent set in $G = (V, E_1 \cup E_2)$?

Note that COMMON INDEPENDENT SET contains 2-UNION INDEPENDENT SET as special case since the only difference is that it does not restrict the two input graphs to be interval graphs.

If we assume that the input graphs G_1 and G_2 are members in a graph class that is closed under induced subgraphs and disjoint unions (as, for example, the widely studied chordal graphs and, in particular, interval graphs, cluster graphs, and forests (Brandstädt et al., 1999)), we can use the main result of this section to precisely state for which classes COMMON INDEPENDENT SET is NP-hard and for which it is polynomial-time solvable, thus giving a *complexity dichotomy* of the problem.

Now, we first precisely state the dichotomy theorem. Since it is quite technical, we immediately illustrate it by some examples in form of implications to the complexity of JOB INTERVAL SELECTION and 2-UNION INDEPENDENT SET. We conclude the section with the proof of the theorem.

Theorem 1. Let C_1 and C_2 be graph classes such that

- C_1 and C_2 are closed under disjoint unions and induced subgraphs, and
- C_1 and C_2 each contain at least one graph that has an edge.

Then, Common Independent Set restricted to input graphs $G_1 \in \mathcal{C}_1$ and $G_2 \in \mathcal{C}_2$

- i) is solvable in $O(n^{1.5})$ time if C_1 and C_2 only contain cluster graphs, and
- ii) NP-hard otherwise.

We illustrate Theorem 1 using some examples. First, choose $C_1 = C_2$ to be the class of interval graphs that have maximum degree 2 and maximum clique size 2. Since P_3 is an interval graph but not a cluster graph in C_1 , we obtain the following NP-hardness result for 2-UNION INDEPENDENT SET:

Corollary 1. 2-Union Independent Set is NP-hard even if both of the following hold:

- the maximum degree of each input interval graphs is 2,
- the maximum clique size of each input interval graph is 2.

Now, choose C_1 to be the class of disjoint unions of paths of length at most two and C_2 to be the class of cluster graphs of clique size at most two. Then, we have $P_3 \in C_1$, which is not a cluster graph and, since disjoint unions of paths are interval graphs, we obtain the result that 2-UNION INDEPENDENT SET is NP-hard even if one input interval graph is a cluster graph consisting of cliques of size two and if the other is a disjoint union of paths of length at most two. Transferring this to the colored model as described in Section 4, we obtain:

Corollary 2. Job Interval Selection is NP-hard even if the input interval graph is a disjoint union of paths of length at most two and contains each color at most two times.

It remains to prove Theorem 1. For the first part of the theorem, Bar-Yehuda et al. (2006) and Halldórsson and Karlsson (2006) mentioned that COMMON INDEPENDENT SET is polynomial-time solvable if both input graphs are cluster graphs. We prove here an explicit upper bound on the running time.

Lemma 1. COMMON INDEPENDENT SET is solvable in $O(n^{1.5})$ time if both input interval graphs are cluster graphs.

Proof. Let $G_1 = (V, E_1), G_2 = (V, E_2)$ be cluster graphs on n vertices, and let $G = (V, E_1 \cup E_2)$. Let C_i denote the set of connected components (cliques) of G_i for $i \in \{1, 2\}$. Define a bipartite graph $H = (C_1 \uplus C_2, E_H)$ with edge set E_H such that there is an edge $\{U_1, U_2\} \in E_H$ if and only if there is a vertex $v \in V$ that occurs in clique U_1 of G_1 and in clique U_2 of G_2 . We claim that G has an independent set of size k if and only if H has a matching of size k.

First, let I be an independent set of size k for G. We construct a matching M in H. To this end, for each $v \in I$, add an edge $\{U_1, U_2\}$ to M, where U_1 is the clique in G_1 that contains v and U_2 is the clique in G_2 that contains v. To verify that M is a matching in H, observe that each clique of G_1 or G_2 can contain only one vertex of I. Therefore, the edges in M are pairwise disjoint and |M| = |I| = k.

Second, let M be a matching of size k in H. We construct an independent set I for G with |I| = k as follows: for each edge $\{U_1, U_2\} \in M$, include an arbitrary vertex contained in both U_1 and U_2 in I. Since each clique of G_1 or G_2 is incident to at most one edge of M, we have chosen at most one vertex per clique of G_1 and G_2 , respectively. Hence, I is an independent set. Furthermore, this implies that we did not choose a vertex twice, so |I| = |M| = k. This completes the proof of the claim.

It remains to prove the running time. Note that we can verify in linear time that a graph is a cluster graph. Moreover, its connected components can be listed in linear time using depth first search. Hence, also the bipartite graph H is computable in linear time. Moreover, by construction, the graph H has at most n edges, and, therefore, we can compute a maximum matching in H in $O(n^{1.5})$ time using the algorithm of Hopcroft and Karp (Schrijver, 2003, Theorem 16.4).

We point out that Lemma 1 generalizes to finding a maximum-weight independent set if the vertices in the input cluster graphs have weights; to this end, we compute a weighted maximum matching in the auxiliary bipartite graph, where each edge is assigned the maximum weight of the vertices occurring in both clusters it connects.

To prove the second part of Theorem 1, we will employ a reduction from 3-SAT.

3-SAT

Input: A Boolean formula ϕ in conjunctive normal form with at most three variables per clause.

Question: Does ϕ have a satisfying assignment?

In fact, we will see two very similar reductions from 3-SAT to COMMON IN-DEPENDENT SET, where the second is an extension of the first. This has the following benefits. The first reduction is an adaption of a simple NP-hardness reduction by Garey et al. (1976) and, while not sufficient to show Theorem 1, it has properties that we will exploit to exclude polynomial-size problem kernels for Job Interval Selection in Section 6.3: **Lemma 2.** In polynomial time, a 3-SAT instance ϕ can be reduced to a COMMON INDEPENDENT SET instance (G_1, G_2, k) such that

- i) G_1 consists of pairwise disjoint paths of length at most two and
- ii) G_2 consists of k connected components, each of which is a triangle or an edge.

Moreover, k is proportional to the number of clauses in ϕ .

This first reduction, which we will use to prove Lemma 2, can then be modified to prove the following lemma, which we will exploit to show Theorem 1. Note that, in comparison to Lemma 2, the following lemma restricts G_2 to consist only of isolated edges and vertices. A polynomial-time many-one reduction that additionally ensures G_2 to have k connected components, like in Lemma 2, does not exist unless P = NP: in this case, COMMON INDEPENDENT SET is polynomial-time solvable using 2-SAT.

Lemma 3. In polynomial time, a 3-SAT instance ϕ can be reduced to a COMMON INDEPENDENT SET instance (G_1, G_2, k) such that

- i) G_1 consists of pairwise disjoint paths of length at most two,
- ii) G_2 consists only of isolated edges and vertices, and
- iii) $G_1 \cup G_2$ has chromatic index three.

Moreover, k is proportional to the number of clauses in ϕ .

Proof of Lemmas 2 and 3. We first show Lemma 2. To this end, we transform a 3-SAT formula ϕ into three graphs G'_1, G'_2 , and G'_3 . The "three graphs approach" will easily allow us to show that the edge-wise union $G_1 := G'_1 \cup G'_3$ consists of pairwise disjoint paths of length at most two and that $G_2 := G'_2$ consists of k pairwise disjoint triangles and edges.

We now give the details of the construction. Let ϕ be a formula in conjunctive normal form with the clauses C_1, \ldots, C_m , each of which contains at most three variables from the variable set $\{x_1, \ldots, x_n\}$. For a variable x_i , let m_i denote the number of clauses in ϕ that contain x_i . For each clause C_j in ϕ , create the gadget shown in Figure 1, where an edge labeled $\ell \in \{1, 2, 3\}$ belongs to G'_{ℓ} . We call the non-triangle vertices leaves and the non-triangle edges antennas. With each variable x_i in C_j , we associate a leaf vertex $C_j(x_i)$.

For each variable x_i , create a cycle gadget X_i (as illustrated in Figure 2), with $2m_i$ edges, alternatingly labeled 2 and 3, and $2m_i$ vertices, which we alternatingly call T-vertex and F-vertex.

For each variable x_i of a clause C_j , we merge the leaf vertex $C_j(x_i)$ with a vertex of the variable gadget X_i for the variable x_i as follows: if x_i appears non-negated in C_j , then we identify $C_j(x_i)$ with an F-vertex of X_i , otherwise, we identify $C_j(x_i)$ with a T-vertex of X_i . Since X_i has m_i pairwise disjoint edges with label 3, each of which has one F- and one T-vertex, we can realize all these

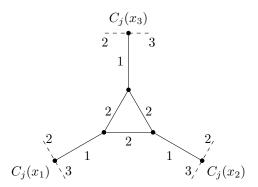


Figure 1: Solid edges form the gadget for a clause C_j containing the variables x_1 , x_2 , and x_3 . With each variable x_i , we associate a leaf vertex $C_j(x_i)$, which will be merged with either a T- or an F-vertex in the gadget for variable x_i (Figure 2), depending on whether C_j contains the variable x_i negated or not. The dashed edges represent edges of the variable gadgets. Edges labeled by a number ℓ belong to the graph G'_{ℓ} .

connections such that no edge with label 3 shares vertices with more than one antenna. The connections are illustrated by dashed lines in Figures 1 and 2. We set $G_1 := G'_1 \cup G'_3$ and $G_2 := G'_2$ and output the COMMON INDEPENDENT SET instance (G_1, G_2, k) , where $k := m + \sum_{i=1}^n m_i$.

It remains to show that the construction is correct and that G_1 and G_2 satisfy the required properties. Indeed, G_1 consists of pairwise disjoint paths of lengths at most two, since it only contains the edges labeled 1 or 3, of which each family forms a matching, and no edge with label 3 is ever connected to two edges labeled 1 and vice versa (only antennas are labeled 1). Moreover, G_2 consists of k isolated triangles and edges (labeled 2): it contains one isolated triangle for each of the m clauses and m_i isolated edges for each variable x_i . It remains to establish the correctness of the reduction by showing that ϕ is satisfiable if and only if $G_1 \cup G_2$ has an independent set of size k.

First, let I be an independent set for $G_1 \cup G_2$ that satisfies $|I| = m + \sum_{i=1}^n m_i = k$. Note that, for each variable x_i , the variable gadget of x_i is a cycle of $2m_i$ vertices. Clearly, I contains at most half of these vertices. Moreover, I contains at most one triangle vertex of each of our m clause gadgets. Hence, $|I| \leq m + \sum_{i=1}^n m_i$ implies that I contains a triangle vertex of each of the m clause gadgets. Equivalently,

- for each variable gadget X_i , either all T-vertices or all F-vertices are contained in I, and
- for each clause gadget, one of its leaf vertices is *not* contained in *I*.

Equivalently, in each clause C_j , we find at least one of the following situations:

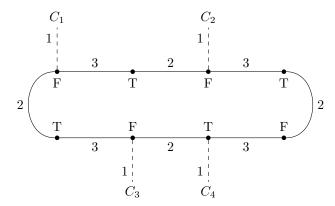


Figure 2: The solid edges form the gadget for a variable contained in the clauses C_1 , C_2 , and C_3 non-negated, but in C_4 negated. Dashed edges belong to the clause gadgets of the clauses C_1, \ldots, C_4 . Edges labeled by a number ℓ belong to the graph G'_{ℓ} .

- C_j contains a positive literal x_i (then $C_j(x_i)$ is an F-vertex in X_i) and I contains all T-vertices of X_i , or
- C_j contains a negative literal \bar{x}_i (then $C_j(x_i)$ is a T-vertex in X_i) and I contains all F-vertices of X_i .

Therefore, on the one hand, setting a variable x_i to true if and only if I contains the T-vertices of X_i yields a satisfying assignment for ϕ .

On the other hand, if we have a satisfying assignment for ϕ , putting into I all T-vertices of X_i if x_i is true and all F-vertices otherwise allows us to choose I so that it contains a triangle vertex of each clause gadgets and, thus, so that $|I| \geq k$.

We can now easily turn this reduction into a reduction that also proves Lemma 3. To this end, note that subdividing an edge of a graph twice increases the size of the graph's maximum independent set by exactly one. Thus, instead of using the clause gadget in Figure 1, we can use the gadget shown in Figure 3 and ask for an independent set of size $k := 10m + \sum_{i=1}^{n} m_i$: indeed, the gadget in Figure 3 is obtained from the simpler one in Figure 1 by subdividing each triangle edge twice and each antenna four times. Thus, we increase the maximum independent set size by $3 + 3 \cdot 2 = 9$ per clause gadget and, hence, ask for $k := 10m + \sum_{i=1}^{n} m_i$ instead of $m + \sum_{i=1}^{n} m_i$.

The benefit of replacing the gadget in Figure 1 by the gadget in Figure 3 is that G_2 now consists only of isolated edges and vertices instead of triangles. Moreover, the resulting graph $G_1 \cup G_2 = G'_1 \cup G'_2 \cup G'_3$ has chromatic index three, where the edge labels yield a proper edge coloring. Thus, using Figure 3, we proved Lemma 3.

Using Lemma 3, it is now easy to prove Theorem 1.

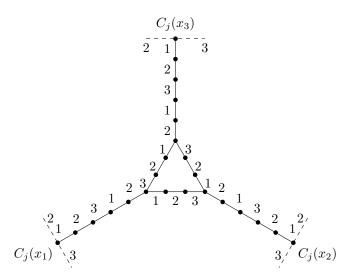


Figure 3: Advanced gadget for a clause C_j containing the variables x_1 , x_2 , and x_3 . With each variable x_i , we associate a leaf vertex $C_j(x_i)$, which is merged with either a T- or an F-vertex in the gadget for variable x_i (Figure 2), depending on whether C_j contains the variable x_i negated or not. The dashed edges represent edges of the variable gadgets. Edges labeled by a number ℓ belong to the graph G'_{ℓ} .

Proof of Theorem 1. Statement (i) immediately follows from Lemma 1. It remains to show (ii). To this end, observe that, without loss of generality, C_1 contains not only cluster graphs. Therefore, it contains a graph that has a P_3 as induced subgraph. Since C_1 is closed under induced subgraphs and disjoint unions, it follows that C_1 contains all graphs that consist of pairwise disjoint paths of length at most two. With the same argument and exploiting that C_2 contains at least one graph with an edge, we obtain that C_2 contains all graphs consisting of isolated vertices and edges. Hence, NP-hardness follows from Lemma 3.

Concluding this section, we derive further hardness results for Job Interval Selection from Lemma 2.

Corollary 3. Even when restricted to instances

- with an input graph that consists of disjoint paths of length at most two
- and that ask for a colorful independent set of size k with k being equal to the number γ of input colors,

Job Interval Selection remains

i) NP-hard, and

ii) cannot be solved in $2^{o(k)} \cdot n^{O(1)}$ time unless the Exponential Time Hypothesis fails.¹

Herein, (i) simply translates into our colored model the statement of Lemma 2 that COMMON INDEPENDENT SET remains NP-hard even if one input graph is a cluster graph with k connected components. Moreover, (ii) follows by combining a result of Impagliazzo et al. (2001) with the fact that k is proportional to the number of clauses in the input 3-SAT formula (Lemma 2).

6 Job Interval Selection

In this section, we investigate the parameterized complexity of Job Interval Selection. As warm-up for working with the colored model, Section 6.1 first gives a simple search tree algorithm that solves Job Interval Selection in linear time if the sought solution size k and the maximum number Γ of colors in any maximal clique of G are constant.

Section 6.2 then proceeds with a reformulation of the fixed-parameter algorithm of Halldórsson and Karlsson (2006) with respect to a structural parameter into our colored model, which makes it easy for us to generalize the algorithm to 2-Union Independent Set and also to show that the problem is linear-time solvable if only k is constant (as opposed to requiring both k and Γ being constant). However, the space requirements as well as the running time exponentially depend on k.

We conclude our findings for Job Interval Selection in Section 6.3 by showing that the problem has no polynomial-size problem kernel in general, but on proper interval graphs.

6.1 A Simple Search Tree Algorithm

As a warm-up for working with the colored formulation of JOB INTERVAL SELECTION, this section presents a simple search tree algorithm leading to the following theorem:

Theorem 2. Job Interval Selection is solvable in $O(\Gamma^k \cdot n)$ time, where Γ is the maximum number of colors occurring in any maximal clique.

Only for Γ < 6 the worst-case running time of Theorem 2 can compete with our generalizations of the dynamic program of Halldórsson and Karlsson (2006) in Section 6.2 (Theorem 4). However, as opposed to the dynamic programs presented in Section 6.2, the space requirements of the search tree algorithm are polynomial.

The first ingredient in our search tree algorithm is the following lemma, which shows that a search tree algorithm only has to consider the "first" intervals of the interval graph for inclusion into an optimal solution. This is illustrated in Figure 4.

¹The Exponential Time Hypothesis basically states that there is no $2^{o(n)}$ -time algorithm for n-variable 3-SAT (Impagliazzo et al., 2001; Lokshtanov et al., 2011).

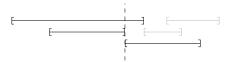


Figure 4: The set K of intervals that start no later than any interval in G ends are drawn black. The other intervals are drawn gray.

Lemma 4. Let K be the set of intervals that start no later than any interval in G ends. Then, there is a maximum colorful independent set that contains exactly one vertex of K.

Proof. Let I be a maximum colorful independent set for G with $I \cap K = \emptyset$ and let v^* be the interval in G that ends first. Obviously, $v^* \in K$ and any interval $v \in I$ intersecting v^* is in K. Hence, since $I \cap K = \emptyset$, I contains no interval intersecting v^* . It follows that I contains a vertex w such that $\operatorname{col}(w) = \operatorname{col}(v^*)$, otherwise $I \cup \{v^*\}$ would be a larger colorful independent set. Now, $I' = (I \setminus \{w\}) \cup \{v^*\}$ is a colorful independent set for G with |I'| = |I| and $v^* \in I' \cap K$.

Finally, note that I cannot contain more than one vertex of K since the intervals in K pairwise intersect.

The second ingredient in our search tree algorithm is the following lemma, which shows that knowing the color of the interval in K that is to be included in an optimal solution is sufficient to choose an optimal interval from K into a maximum colorful independent set.

Lemma 5. Let K be the set of intervals that start no later than any interval in G ends. Moreover, assume that there is a maximum colorful independent set containing an interval of color c from K.

Then, there is a maximum colorful independent set that contains the interval of color c from K that ends first.

Proof. Let I be a maximum colorful independent set, let $v \in K \cap I$ and let $\operatorname{col}(v) = c$. Moreover, let v^* be the interval in K with $\operatorname{col}(v^*) = c$ that ends first. By Lemma 4, I contains at most one interval of K. Then, since v intersects all intervals that intersect v^* , we know that $I' = (I \setminus \{v\}) \cup \{v^*\}$ is a colorful independent set with |I'| = |I|.

Using Lemma 4 and Lemma 5, it is easy to prove Theorem 2.

Proof of Theorem 2. The algorithm works as follows. First, find the set K of intervals that start no later than any interval in G ends. Let $C := \bigcup_{v \in K} \operatorname{col}(v)$ be the set of colors occurring in K. Note that these computations can be executed in O(n) time. Since the intervals in K form a maximal clique, it follows that $|C| \leq \Gamma$. By Lemma 4 and Lemma 5, it is now sufficient, for each color $c \in C$ and the first-ending interval v with $\operatorname{col}(v) = c$, to try choosing v for inclusion into the solution and to try recursively finding a colorful independent set of

size k-1 in the interval graph G without vertices having color c or intersecting v (that is, starting after v ends).

The recursion depth is bounded by k, each recursion step causes at most Γ new recursion steps, and each recursion step requires O(n) time, yielding a total running time of $O(\Gamma^k \cdot n)$.

6.2 Generalizations of Halldórsson and Karlsson (2006)'s Dynamic Program

In this section, we first present the dynamic program for Job Interval Selection by Halldórsson and Karlsson (2006) in terms of our colored model. Based on this presentation, we show modifications in order to lower its space requirements, we generalize it to Colorful Independent Set with Lists and, finally, transform it into a fixed-parameter algorithm with respect to the parameter k.

It is easy to see that the dynamic programs in this section can be straightforwardly generalized to the problem variant where each interval has assigned a weight and we search for a colorful independent set of maximum weight, rather than of maximum size.

Dynamic Program for Parameter "Number γ **of Colors".** Let (G,k) be an instance of Job Interval Selection, where G is given in c-compact representation for minimum c. For $i \in [c+1]$ and $C \subseteq [\gamma]$, we use T[i,C] to denote the size of a maximum colorful independent set in G that uses only intervals whose start point is at least i and whose color is in C. Obviously, for i = c+1 and any $C \subseteq [\gamma]$, we have T[i,C] = 0. Knowing T[i,C] for some $i \in [c+1]$ and all $C \subseteq [\gamma]$, we can easily compute T[i-1,C] for all $C \subseteq [\gamma]$, since there are only two cases:

- 1. There is a maximum independent set of intervals with start point at least i-1 and colors belonging to C that contains an interval v with $v_s=i-1$. Then, $T[i-1,C]=1+T[v_e+1,C\setminus\{\operatorname{col}(v)\}].$
 - 2. Otherwise, T[i-1,C] = T[i,C].

It follows that we can compute the size $T[1, [\gamma]]$ of a maximum colorful independent set in G using the recurrence

$$T[i-1,C]$$

$$= \max \begin{cases} T[i,C], \\ 1 + \max_{v \in V, v_s = i-1} T[v_e + 1, C \setminus \{\operatorname{col}(v)\}]. \\ \operatorname{col}(v) \in C \end{cases}$$
(DP- γ)

In this way, we obtain an alternative formulation of the dynamic program of Halldórsson and Karlsson (2006) using colored interval graphs instead of a geometric formulation. We can evaluate recurrence (DP- γ) in $O(2^{\gamma}n)$ time by

iterating over the intervals in G in order of decreasing start points and, for each interval, iterating over all subsets of $[\gamma]$. In this way, we first handle all intervals with start point c, then with c-1 and so on, so that we compute the table entries for decreasing start points $i \in [c+1]$. Herein, the c-compact representation not only ensures that the intervals are sorted by their start points, but also that, for each $i \in [c]$, some interval starts in i and, therefore, that the table entry T[i, C] indeed gets filled for all $C \subseteq [\gamma]$. This algorithm yields an alternative proof for a result by Halldórsson and Karlsson (2006):

Proposition 1 (Halldórsson and Karlsson (2006)). Job Interval Selection is solvable in $O(2^{\gamma} \cdot n)$ time.

Dynamic Program for Parameter "Maximum Number Q of Live Colors". Halldórsson and Karlsson (2006) improved recurrence (DP- γ) from using the parameter γ to the structural parameter $Q \leq \gamma$, which is defined as follows:

Definition 2. Let G be an interval graph given in c-compact representation and with vertex colors in $[\gamma]$. For each $i \in [c+1]$, let

 $L_i \subseteq [\gamma]$ be the set of colors that appear on intervals with start point at most i (note that $L_{c+1} = [\gamma]$), and

 $R_i \subseteq [\gamma]$ be the set of colors that appear on intervals with start point at least i (note that $R_{c+1} = \emptyset$).

Then $Q := \max_{i \in [c+1]} |L_i \cap R_i|$ is the maximum number of *live colors*. That is, a color c is *live* at a point i if there is an interval with color c that starts no later than i as well as an interval with color c that starts no earlier than i.

Using this definition, we first observe that, when searching for a maximum colorful independent set containing only intervals with start point at least i, it is safe to allow this independent set to contain all colors of $\bar{L}_i := [\gamma] \setminus L_i$: this is because an interval with start point before i cannot have a color in \bar{L}_i . Hence, we are only interested in the values T[i,C] for $i \in [c+1]$ and $\bar{L}_i \subseteq C \subseteq [\gamma]$. Second, a colorful independent set that only contains intervals with start point at least i only contains intervals of color R_i . Therefore, it is safe to allow only colors contained in R_i and we see that we are only interested in the values T[i,C] for $i \in [n+1]$ and $\bar{L}_i \subseteq C \subseteq R_i$. There are at most 2^Q such subsets, since for each C with $\bar{L}_i \subseteq C \subseteq R_i$, we have $C \setminus \bar{L}_i \subseteq L_i \cap R_i$.

Exploiting these observations in (DP- γ), we can compute T[i-1, C] for all C with $\bar{L}_{i-1} \subseteq C \subseteq R_{i-1}$ as

$$T[i-1, C]$$

$$= \max \begin{cases} T[i, (C \cup \bar{L}_i) \cap R_i], \\ 1 + \max_{v \in V, v_s = i-1} T[v_e + 1, (C \cup \bar{L}_{v_e+1}) \cap (R_{v_e+1} \setminus \{\text{col}(v)\})]. \\ \text{col}(v) \in C \end{cases}$$
(DP-Q)

As we have not changed the semantics of a table entry compared to $(DP-\gamma)$, the size of a maximum colorful independent set in G is, as before, $T[1, [\gamma]]$. Hence, the improved dynamic program of Halldórsson and Karlsson (2006) also works in our colored model:

Proposition 2 (Halldórsson and Karlsson (2006)). JOB INTERVAL SELECTION is solvable in $O(2^Q \cdot n)$ time, where Q is the maximum number of live colors as defined in Definition 2.

Improving the Space Complexity. Having stated the dynamic programs of Halldórsson and Karlsson (2006) in terms of our colored model, we now build upon these algorithms. Obviously, the dynamic programming table of recurrence (DP-Q) has $2^Q \cdot (c+1)$ entries. We improve it to $2^Q \cdot (\ell+2)$, where ℓ is the length of the longest interval in the input interval graph. That is, if Q and ℓ are constant, we can solve arbitrarily large input instances using a constant-size dynamic programming table. Note that, even if ℓ is not bounded by a constant, we have $\ell \leq c-1$, and therefore $2^Q \cdot (\ell+2) \leq 2^Q \cdot (c+1)$, since the input instance is given in a c-compact representation.

The improvement of space complexity is based on a simple observation: when computing T[i-1,C] in (DP-Q), there is a largest possible i'>i-1 and some color set C' for which we access T[i',C']. By definition of T, $i'=v_e+1$ for some interval v with start point $v_s=i-1$. We have $i'-1=v_e\leq v_s+\ell=i-1+\ell$, and, hence, $i'\leq i+\ell$. It follows that we only need $2^Q(\ell+2)$ table entries, since the entry T[i-1,C] does not need the value $T[i+\ell+1,C]$ and can therefore reuse the space previously occupied by $T[i+\ell+1,C]$. This we simply achieve by storing T[i,C] for $i\in [c+1]$ and $C\subseteq [\gamma]$ in a table $T'[i\bmod (\ell+2),C]$ that has only $\ell+2$ entries in the first coordinate. Having shrunken the dynamic programming table in this way, we obtain the following lemma:

Proposition 3. Job Interval Selection is solvable in $O(2^Q \cdot n)$ time and $O(2^Q \ell + \gamma c)$ space when the input graph is given in c-compact representation, ℓ is the maximum interval length, and Q is the maximum number of live colors as defined in Definition 2.

Herein, $O(2^{Q}\ell)$ space is used by the dynamic programming table and $O(\gamma c)$ space is used to hold the sets L_i and R_i from Definition 2, which we used to speed up the dynamic programming.

Generalization to Colorful Independent Set with Lists. We now generalize (DP-Q) to Colorful Independent Set with Lists. That is, vertices are now allowed to have multiple colors instead of just one and we search for a maximum independent set that is colorful in the sense that no pair of vertices may have common colors. The algorithm for Colorful Independent Set with Lists will allow us to solve 2-Union Independent Set in Section 7.1.

Due to the formulation of (DP-Q) in our colored model, the generalization to COLORFUL INDEPENDENT SET WITH LISTS turns out to be easy. For $i \in [c+1]$ and $C \subseteq [\gamma]$, we use T[i, C] to denote the size of a maximum colorful independent

set in G that uses only intervals with start point at least i and whose colors are a subset of C.

Completely analogously to Job Interval Selection, we can compute the size $T[1, [\gamma]]$ of a maximum colorful independent set by computing T[i-1, C] for each $i \in [c+1]$ and all color sets C with $\bar{L}_{i-1} \subseteq C \subseteq R_{i-1}$ as

$$T[i-1,C]$$

$$= \max \begin{cases} T[i,(C \cup \bar{L}_i) \cap R_i], \\ 1 + \max_{v \in V, v_s = i-1} T[v_e + 1,(C \cup \bar{L}_{v_e+1}) \cap (R_{v_e+1} \setminus \operatorname{col}(v))]. \\ \operatorname{col}(v) \subseteq C \end{cases}$$
(DP-Q*)

The improvement of the space complexity demonstrated for Job Interval Selection also works here. Hence, we can merge Propositions 1–3 into the following theorem:

Theorem 3. Given an interval graph with γ colors and maximum interval length ℓ in c-compact interval representation, COLORFUL INDEPENDENT SET WITH LISTS is solvable in $O(2^Q \cdot n)$ time and $O(2^Q \ell + \gamma c)$ space, where Q is the maximum number of live colors as defined in Definition 2.

Algorithm for Parameter "Solution Size k". We now improve recurrence (DP- γ) to a fixed-parameter algorithm for Job Interval Selection with respect to the parameter $k \leq \gamma$. Our first step is providing a randomized fixed-parameter algorithm for Job Interval Selection. The algorithm correctly answers if a no-instance of Job Interval Selection is given. In contrast, it rejects "yes"-instances with a given error probability ε . The randomized algorithm can be derandomized to show the following theorem:

Theorem 4. Job Interval Selection can be solved with error probability ε in $O(5.5^k \cdot |\ln \varepsilon| \cdot n)$ time and $O(2^k \cdot \ell)$ space. The algorithm can be derandomized to deterministically solve Job Interval Selection in $O(12.8^k \cdot \gamma n)$ time.

Comparing this theorem with the hardness result in Corollary 3 from Section 5, the running time of the derandomized algorithm is optimal up to factors in the base. However, in practical applications, the randomized algorithm is probably preferable over the derandomized one, since the error probability can be chosen very low without increasing the running time significantly.

To prove Theorem 4, we use the color-coding technique by Alon et al. (1995) to reduce the number γ of colors in the given instance to k. After that, recurrence (DP- γ) can be evaluated in $O(2^k \cdot n)$ time. Depending on whether we reduce the number of colors randomly or deterministically, this method will yield the first or the second running time.

Proof of Theorem 4. Let $(G, \operatorname{col}, k)$ be an instance of Job Interval Selection. In a first step, we assign each color in $[\gamma]$ a color in [k] uniformly at random. Let $\delta \colon [\gamma] \to [k]$ denote this recoloring and let $(G, \operatorname{col}', k)$ denote the resulting instance with $\operatorname{col}'(v) = \delta(\operatorname{col}(v))$ for all vertices v. Note that, in general, δ is not

injective. Then, we use $(DP-\gamma)$ to compute a size-k colorful independent set in the resulting instance. Since the resulting instance has only k colors, this works in $O(2^k \cdot n)$ time.

We now first analyze the probability that a colorful independent set for $(G, \operatorname{col}, k)$ is also a colorful independent set for $(G, \operatorname{col}', k)$ and vice versa. Then, we analyze how often we have to repeat the procedure of recoloring and computing recurrence $(\operatorname{DP}-\gamma)$ in order to achieve the low error probability ε .

First, assume that the recolored instance $(G, \operatorname{col}', k)$ is a "yes"-instance. Then, there is a colorful independent set I with $|I| \geq k$. The set I is a colorful independent set also for the original instance $(G, \operatorname{col}, k)$, since each color in col is mapped to only one color in col' . It follows that $(G, \operatorname{col}, k)$ is a "yes"-instance.

Now, assume that the original instance (G,col,k) is a "yes"-instance. We analyze the probability of the recolored instance $(G,\operatorname{col}',k)$ being a "yes"-instance. Let I be a colorful independent set for (G,col,k) . The set I is a colorful independent set for $(G,\operatorname{col}',k)$ if the vertices in I have pairwise distinct colors with respect to col' . Since the vertices in I have pairwise distinct colors with respect to col and we assign each color in $[\gamma]$ a color in [k] uniformly at random, the colors of the vertices of I with respect to col' are also chosen uniformly at random and independently from each other. Thus, the probability of I being colorful with respect to col' is $p:=k!/k^k$: out of k^k possible ways of coloring the k vertices in I with k colors, there are k! ways of doing so in a colorful manner. Hence, the probability of $(G,\operatorname{col}',k)$ also being a "yes"-instance, is $p:=k!/k^k$.

In order to lower the error probability of not finding a colorful independent set if it exists to ε , we repeat the process of recoloring and running recurrence (DP- γ) $t(\varepsilon)$ times. That is, we want

$$(1-p)^{t(\varepsilon)} \le \varepsilon.$$

Exploiting that $1 + x \le e^x$ holds for all $x \in \mathbb{R}$, the above inequality is satisfied by any number $t(\varepsilon)$ of recoloring trials that satisfies

$$e^{-p \cdot t(\varepsilon)} \le \varepsilon.$$

Taking the logarithm on both sides and rearranging terms,

$$t(\varepsilon) \ge \ln \varepsilon \cdot \frac{1}{-p} = |\ln \varepsilon| \cdot \frac{k^k}{k!}.$$

Using Stirling's lower bound for the factorial, one obtains $k^k/k! \in O(e^k)$. To conclude the proof, it is now enough to put together the observations that each run of recurrence $(\mathbf{DP}-\gamma)$ with k colors takes $O(2^k \cdot n)$ time and that we have to repeat it only $t(\varepsilon) \in O(|\ln \varepsilon| \cdot e^k)$ times to get an error probability of ε . Thus, the overall procedure takes $O(|\ln \varepsilon| \cdot (2e)^k \cdot n)$ time.

We now derandomize the presented algorithm: instead of repeatedly choosing random recolorings $\delta \colon [\gamma] \to [k]$, we deterministically enumerate the recolorings

according to a k-color coding scheme (Chen et al., 2007): a k-color coding scheme \mathcal{F} is a set of recolorings such that, for each subset $C \subseteq [\gamma]$ with |C| = k, there is a recoloring $\delta \in \mathcal{F}$ such that the colors in C will be mapped to pairwise distinct colors by δ . That is, whatever colors a colorful independent set I of size k in G might have, there is one recoloring in \mathcal{F} such that I is colorful after recoloring. Thus, the dynamic program (DP- γ) will find it.

A k-color coding scheme \mathcal{F} can be computed in $O(6.4^k \cdot \gamma)$ time (Chen et al., 2007). Moreover, it consists of $O(6.4^k \cdot \gamma)$ colorings. That is, in $O(6.4^k \gamma \cdot 2^k n)$ time, we can run (DP- γ) for each coloring in \mathcal{F} , thus proving (ii).

Many algorithms that are based on the color-coding techniques can be sped up using algebraic techniques (Koutis and Williams, 2009). It would be interesting to see whether they can also be used to speed up the running time of Theorem 4 (at least in the asymptotic sense).

Finally, note that the color-coding technique as used in Theorem 4 for Job Interval Selection could be applied to Colorful Independent Set with Lists in the same way. However, the result will not be a fixed-parameter algorithm with respect to the parameter "solution size k", but with respect to the total number of colors found in the lists of the solution vertices. This number could potentially be much larger than k and even n, thus making such a fixed-parameter algorithm not particularly attractive for Colorful Independent Set with Lists.

6.3 Polynomial-Time Preprocessing

In this section, we first show that efficient and effective data reduction in form of polynomial-size problem kernels is most likely unfeasible for Job Interval Selection. Then, we show that it becomes feasible when we restrict the colored input graph to be a *proper* interval graph.

Non-Existence of Polynomial-Sized Problem Kernels. We show that Job Interval Selection is unlikely to admit problem kernels of polynomial size with respect to various parameters. To this end, we employ the "cross composition" technique introduced by Bodlaender et al. (2014). A cross composition is a polynomial-time algorithm that, given t instances x_i with $0 \le i < t$ of an NP-hard starting problem A, outputs an instance (y,k) of a parameterized problem B such that $k \in \text{poly}(\max_{0 \le i < t} |x_i| + \log t)$ and (y,k) is a "yes"-instance for B if and only if there is some $0 \le i < t$ with x_i being a "yes"-instance for A. A theorem by Bodlaender et al. (2014) now states that if a problem B admits such a cross composition, then there is no polynomial-size problem kernel for B unless the polynomial hierarchy collapses to the third level, which is widely disbelieved.

In the following, we present a cross composition for JOB INTERVAL SELECTION parameterized by the combination of the size ω of a maximum clique and the number γ of colors, yielding the following theorem:

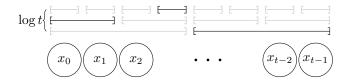


Figure 5: Schematic view of the cross composition for JOB INTERVAL SELECTION. Circles at the bottom represent the t input instances. Bars at the top represent the auxiliary intervals spanning over the input instances. Here, each of the $\log t$ rows stands for a new color. A solution (black intervals) for the instance must select one interval in each row, thereby selecting one of the t input instances (x_2 in this example).

Theorem 5. Unless the polynomial hierarchy collapses, JOB INTERVAL SELECTION does not admit a polynomial-size problem kernel with respect to the combined parameter "number γ of colors" and "maximum clique size ω ".

In particular, there are no polynomial-size problem kernels for the combined parameters (ω, k) or (ω, Q) , where Q is the number of "live colors" (Definition 2).

The second part of the theorem follows from the first part since both k and Q are at most γ .

Proof. We present a cross composition from the NP-hard starting problem Job Interval Selection with the further restriction that the sought solution size k equals the number of colors γ . We saw in Corollary 3 in Section 5 that this restriction remains NP-hard. The framework of Bodlaender et al. (2014) allows us to force all of the t input instances x_i to have the same value for k and, thus, each instance uses the same color set [k]. We assume, without loss of generality, that t is a power of two (otherwise, we add some "no"-instances to the list of input instances). The steps of the cross composition are as follows (see Figure 5):

- 1. Place the start and end points of the n intervals of each input instance x_i into the integer range $[i \cdot n, (i+1) \cdot n-1]$.
- 2. Introduce $\log t$ extra colors $k+1, k+2, \ldots, k+\log t$; the resulting instance then asks for an independent set of size $k + \log t$.
- 3. For each $1 \leq i \leq \log t$, introduce 2^i auxiliary intervals $v_0, v_1, \ldots, v_{2^i-1}$ with color k+i such that the auxiliary interval v_j spans exactly over the instances x_ℓ with

$$j \cdot \frac{t}{2^i} \le \ell \le (j+1) \cdot \frac{t}{2^i} - 1.$$

To show that this construction is indeed a cross composition for the parameters "number γ of colors" and "maximum clique size ω ", observe that $\gamma, \omega \leq \max_i |x_i| + \log t$ and it remains to prove that the constructed instance $(G, k + \log t)$ is a "yes"-instance if and only if one of the input instances is a "yes"-instance.

First, if the constructed graph G has a colorful independent set I of size $k+\log t$, then I contains an interval of each color. In particular, I contains an auxiliary interval of each of the colors k+1 to $k+\log t$. We show that all of the k non-auxiliary intervals of I are from the same input instance. To this end, note that, for each $1 \le i \le \log t$, each auxiliary interval v_j of color k+i spans over exactly

$$(j+1)\cdot\frac{t}{2^i}-j\cdot\frac{t}{2^i}=\frac{t}{2^i}$$
 instances.

Since instances spanned by auxiliary intervals of I are disjoint, the $\log t$ auxiliary intervals in I span exactly

$$\sum_{i=1}^{\log t} \frac{t}{2^i} = t - 1 \quad \text{instances.}$$

Hence, exactly one input instance is not spanned, implying that all non-auxiliary intervals in I are from this very instance.

Second, let x_{ℓ} be a "yes"-instance, that is, there is an independent set of size k in x_{ℓ} that contains the colors [k]. We extend this to a colorful independent set of size $k + \log t$ for G. To this end, it is sufficient to add the intervals from a $(\log t)$ -separating colorful independent set, where a colorful independent set I is i-separating for some integer i if

- no interval in I spans x_{ℓ} ,
- I has size i and contains all colors $\{k+1,\ldots,k+i\}$, and
- there is a single interval of color k + i that is not in I and covers all instances not spanned by the intervals in I.

Obviously, there is a 1-separating colorful independent set, since the intervals with color k+1 separate the input instances in exactly two halves. To complete the proof, it remains to extend this 1-separating independent set to be $(\log t)$ -separating. To this end, we use induction.

Assume that I is an i-separating colorful independent set for some $1 \le i < \log t$. We show how to extend it to be (i+1)-separating. The auxiliary intervals in I span exactly

$$\sum_{i=1}^{i} \frac{t}{2^{j}} = t - \frac{t}{2^{i}} \quad \text{instances.}$$

That is, $t/2^i$ instances are not spanned by I but by a single interval of color k+i. Since each interval with color k+i+1 spans $t/2^{i+1}$ instances and is contained in an interval of color k+i, there are precisely two intervals of color k+i+1 that span the instances not spanned by I. Since they are disjoint, one of them does not span x_ℓ , add this interval to I.

Polynomial-Size Problem Kernel on Proper Interval Graphs. We restrict Job Interval Selection to proper interval graphs, which remains NP-hard, as shown in Section 5. Here, the negative result of Theorem 5 collapses: the following polynomial-time data reduction routine produces a problem kernel containing $4k^2 \cdot \omega$ intervals, where ω is the maximum clique size in the input graph.

Reduction Rule 1. For a graph G and a color c, let G[c] denote the subgraph of G induced by the intervals of color c.

If G[c] has an independent set of size at least 2k-1, then remove all intervals of color c and decrease k by one.

In order to show that a reduction rule is *correct*, one has to show that the output instance is a "yes"-instance if and only if the input instance is.

Lemma 6. Reduction Rule 1 is correct and can be applied exhaustively in O(n) time.

Proof. Let (G', k-1) denote the instance produced by Reduction Rule 1 from an instance (G, k) by removing all intervals of a color c from G. Clearly, a colorful independent set I for G is also a colorful independent set for G' if we remove the interval with color c from I. Hence, if (G, k) is a "yes"-instance, then so is (G', k-1).

In the following, let (G', k-1) be a "yes"-instance with solution I and let I_c denote an independent set of size 2k-1 in G[c]. If $|I| \geq k$, then (G, k) is a "yes"-instance. Otherwise, $|I| \leq k-1$. Furthermore, I does not contain an interval with color c. Consider an interval $u \in I$. If $N_G(u) \cap I_c$ contains at least three intervals x, y, z, then $G[\{u, x, y, z\}]$ is a $K_{1,3}$, contradicting G being a proper interval graph. Therefore, each interval in I overlaps at most two intervals in I_c . Hence, the intervals in I overlap at most $2|I| \leq 2(k-1) < |I_c|$ intervals, implying that $I_c \setminus N[I] \neq \emptyset$. Thus, there is an interval in I_c that can be added to I, thereby obtaining a solution for (G, k).

It remains to argue the claimed running time. To this end, note that a maximum independent set in G[c] can be computed in $O(n_c)$ time with $n_c := |V(G[c])|$, since G[c] is an ordinary (that is, monochromatic) proper interval graph. Hence, computing maximum independent sets for all colors can be done in O(n) time in total. Since applying the rule for one color does not affect other colors, this application is exhaustive, that is, Reduction Rule 1 is not applicable to the resulting instance.

In order to prove the problem kernel bound, we further need the following trivial "data reduction rule" that returns a "yes"-instance if we can greedily find an optimal solution.

Reduction Rule 2. Let I be a maximal colorful independent set of G. If $|I| \ge k$, then return a small trivial "yes"-instance.

Lemma 7. Reduction Rule 2 is correct can be applied in O(n) time.

Proof. The correctness of Reduction Rule 2 is obvious. It remains to prove the running time.

A maximal colorful independent set of G can be found by greedily picking the first-ending valid interval v into an independent set I and deleting all intervals that overlap v. Herein, we can keep a size- γ array whose i-th entry is 1 if color i is already used. Using this array, we can check in constant time whether an interval is valid for inclusion in I. Moreover, since invalid vertices do not become valid again, the whole procedure can be executed in O(n) time, given that the intervals are sorted.

Given these two data reduction rules, we can now prove the following theorem.

Theorem 6. JOB INTERVAL SELECTION on proper interval graphs admits a problem kernel with at most $4k^2 \cdot \omega$ intervals that is computable in O(n) time. Herein, ω is the maximum clique size of the input graph.

Proof. To show the problem kernel bound, consider an instance (G,k) of Job Interval Selection that is reduced with respect to Reduction Rule 1 and to which Reduction Rule 2 has been applied. It follows that there is a maximal colorful independent set I of G with |I| < k. Since G is a proper interval graph, the neighborhood of each vertex v can be partitioned into two cliques: one consisting of intervals containing v_s , one consisting of intervals containing v_e . Thus, each vertex in I has at most $2\omega - 1$ neighbors and, hence, we can bound $|N[I]| \le 2k\omega$.

Now, let $X:=V(G)\setminus N[I]$ and let G':=G[X]. Then, since I is maximal, all intervals in X have a color that appears in I, of which there are at most k-1. For each color c of these, let G'[c] denote the subgraph of G' that is induced by all intervals of color c in X and let I_c denote a maximum independent set of G'[c]. Since G is reduced with respect to Reduction Rule 1, $|I_c| \leq 2(k-1)$. Again, since G'[c] is a proper interval graph, each interval $u \in I_c$ has at most $2\omega - 1$ neighbors in G'[c]. Thus, the total number of intervals in G'[c] is at most $4(k-1)(\omega-1)$. Since G' contains at most k-1 colors, we can bound $|V(G')| \leq 4(k-1)^2(\omega-1)$, implying a bound of $|V(G)| + |N[I]| \leq 4(k-1)^2(\omega-1) + 2k\omega \leq 4k^2\omega$ for the number of intervals in G.

The running time bound follows from Lemma 6 and Lemma 7. \Box

7 2-Union Independent Set

In Section 6, we studied the Job Interval Selection problem, which is equivalent to 2-Union Independent Set where one of the two input interval graphs is a cluster graph. In this section, we investigate the parameterized complexity 2-Union Independent Set.

Corollary 1 has already shown that 2-UNION INDEPENDENT SET is NP-hard even if the maximum clique size of both input interval graphs and the maximum vertex degree are at most two. Moreover, we already know that 2-UNION INDEPENDENT SET is W[1]-hard with respect to the parameter k (Jiang, 2010).

Hence, with respect to these three parameters, 2-UNION INDEPENDENT SET is unlikely to be fixed-parameter tractable.

In contrast, this section shows how the compactness of the input interval graphs affects the computational complexity of 2-UNION INDEPENDENT SET. To this end, as before, let c_{\forall} be the minimum number such that both input interval graphs are c_{\forall} -compact and let c_{\exists} be the minimum number such that at least one of both input interval graphs is c_{\exists} -compact (see Definition 1).

First, in Section 7.1, we show a fixed-parameter algorithm with respect to the parameter c_{\exists} . The algorithm is an adaption of our algorithm for COLORFUL INDEPENDENT SET WITH LISTS (Theorem 3) to 2-UNION INDEPENDENT SET. In the analysis of its complexity, the parameter c_{\exists} naturally arises as complexity measure.

Second, in Section 7.2, we show a simple polynomial-time data reduction rule for 2-Union Independent Set. Again, in the analysis of its effectiveness, the parameter c_{\forall} naturally arises as complexity measure.

Since in both applications, compactness-related parameters arose quite naturally, we suspect that the parameter may be useful in the development in fixed-parameter algorithms for other NP-hard problems on interval graphs.

7.1 A Dynamic Program for 2-Union Independent Set

We describe an algorithm that solves 2-Union Independent Set in $O(2^{c_3} \cdot n)$ time. To this end, we reformulate 2-Union Independent Set as a special case of Colorful Independent Set with Lists and then solve the resulting instance using the dynamic program (DP- Q^*) from Section 6.2 (Theorem 3).

An instance of 2-Union Independent Set can be solved by an algorithm for Colorful Independent Set with Lists as follows: without loss of generality, assume that of the input interval graphs G_2 is c_{\exists} -compact. We interpret each number in $[c_{\exists}]$ as a color and give the input graph G_1 as input to Colorful Independent Set with Lists such that each vertex v of G_1 gets the colors corresponding to the numbers contained in the interval that represents v in G_2 . Then a solution for Colorful Independent Set with Lists is a solution for 2-Union Independent Set and vice versa:

- Two vertices v and w may be together in a solution of 2-UNION INDE-PENDENT SET if and only if their intervals neither intersect in G_1 nor in G_2 .
- Two vertices v and w may be together in a solution of COLORFUL IN-DEPENDENT SET WITH LISTS if and only if neither their intervals in G_1 intersect nor their color lists intersect (which are precisely their intervals in G_2).

We stated earlier that Colorful Independent Set with Lists is a more general problem than 2-Union Independent Set. This now becomes clear: whereas Colorful Independent Set with Lists allows arbitrary color lists,

the instances generated from 2-UNION INDEPENDENT SET only use intervals of natural numbers as color lists.

To execute the transformation from 2-Union Independent Set to Colorful Independent Set with Lists, we just take each interval of G_2 and add the numbers that it contains to the color list of the corresponding vertex in G_1 . Since each interval in G_2 contains at most c_{\exists} numbers, the transformation from 2-Union Independent Set to Colorful Independent Set with Lists is executable in $O(c_{\exists} \cdot n)$ time. The resulting Colorful Independent Set with Lists with Lists instance has c_{\exists} colors and, by Theorem 3, is solvable in additionally $O(2^{c_{\exists}} \cdot n)$ time.

Theorem 7. 2-Union Independent Set is solvable in $O(2^{c_{\exists}} \cdot n)$ time when at least one input interval graph is c_{\exists} -compact.

7.2 Polynomial-Time Preprocessing

We provide polynomial-time data reduction for 2-UNION INDEPENDENT SET. It will turn out that the presented data reduction rule yields a polynomial-size problem kernel for the parameter c_{\forall} , where both input interval graphs G_1 , G_2 are c_{\forall} -compact.

The intuition behind the data reduction rule is simple: assume that we have a vertex that is represented by the interval v in the first input interval graph G_1 and by v' in the second input interval graph G_2 . Moreover, assume that there is another vertex represented by the intervals u in G_1 and u' in G_2 . Then, if $v \subseteq u$ and $v' \subseteq u'$, we would never choose the vertex represented by u and u' into a maximum independent set, as it "blocks" a superset of vertices for inclusion into a maximum independent set compared to the vertex represented by v and v'. Hence, we delete the intervals v and v'.

To lead this intuitive idea to a problem kernel, we introduce the concept of the *signature* of a vertex, give a reduction rule that bounds the number of vertices having a given signature, and finally bound the number of signatures in a 2-union graph.

Definition 3. Let (G_1, G_2, k) denote an instance of 2-UNION INDEPENDENT SET and let v be a vertex of G_1 and G_2 . The *signature* $\operatorname{sig}(v)$ of v is a four-dimensional vector $(-v_s, v_e, -v'_s, v'_e)$, where v_s and v_e are v's start and end points in G_1 , and v'_s and v'_e are its start and end points in G_2 .

Reduction Rule 3. Let (G_1, G_2, k) denote an instance of 2-Union Independent Set. For each pair of vertices u, v of G_1 and G_2 such that $sig(v) \leq sig(u)$ (component-wise), delete u from G_1 and G_2 .

Lemma 8. Reduction Rule 3 is correct and can be applied in $O(n \log^2 n)$ time.

Proof. Let (G_1, G_2, k) be an instance of 2-UNION INDEPENDENT SET and let u, v be vertices of G_1 and G_2 such that $\operatorname{sig}(v) \leq \operatorname{sig}(u)$. Observe that this implies $v_s \geq u_s, v_e \leq u_s, v_s' \geq u_s'$, and $v_e' \leq u_e'$. Hence, $N_{G_1}[v] \subseteq N_{G_1}[u]$ and $N_{G_2}[v] \subseteq N_{G_1}[u]$

 $N_{G_2}[u]$ and, therefore, $N_{G_1 \cup G_2}[v] \subseteq N_{G_1 \cup G_2}[u]$. Hence, instead of choosing u into an independent set, we can always choose v. Therefore, it is safe to delete u.

Regarding the running time, Kung et al. (1975) showed that the set of maxima of n vectors in d dimensions can be computed using $O(n \log^{d-2} n)$ comparisons, directly implying the stated running time.

Theorem 8. Let (G_1, G_2, k) be an instance of 2-Union Independent Set such that G_1 and G_2 are c_{\forall} -compact.

Then, a problem kernel with c_{\forall}^3 vertices can be constructed in $O(n \log^2 n)$ time. The size reduces to $2c_{\forall}^2$ vertices if one of the input graphs is proper interval.

Proof. Let (G_1, G_2, k) be an instance of 2-UNION INDEPENDENT SET. We assume that G_1 and G_2 have been preprocessed according to Observation 3, that is, at each position of the interval representations of G_1 and G_2 , there is an interval start point as well as an interval end point. The problem kernel (G_1^*, G_2^*, k) is then obtained from (G_1, G_2, k) by applying Reduction Rule 3 to (G_1, G_2, k) . By definition of G_1^* and G_2^* , the graphs G_1^* and G_2^* contain at most one vertex of each signature. Hence, it is sufficient to show that there are at most c_{\forall}^3 different signatures corresponding to vertices in the new instance (G_1^*, G_2^*, k) .

Consider the set $S_{i,j}$ of all signatures $s = (-v_s, v_e, -v_s', v_e')$ with $v_s = i$ and $v_s' = j$ such that v remains in G_1^* and G_2^* . If $|S_{i,j}| > c_{\forall}$, then we find $s_1, s_2 \in S_{i,j}$ such that s_1 and s_2 agree in the second or fourth coordinate, since there are at most c_{\forall} possible values for each of them. Since then s_1 and s_2 agree in three coordinates, it follows that either $s_1 \leq s_2$ or $s_2 \leq s_1$, contradicting the assumption that Reduction Rule 3 has been applied to (G_1^*, G_2^*, k) . Obviously, there are at most c_{\forall}^2 sets of signatures $S_{i,j}$ and, thus, there are at most c_{\forall}^3 signatures in total.

In the following, we show that the described instance has at most $2c_{\forall}^2$ vertices if G_1 is a proper interval graph. To this end, we will use two relations R_1, R_2 between pairs $(i,j) \in [c_{\forall}] \times [c_{\forall}]$ and signatures corresponding to vertices of G_1^* . We then show that each signature is the image under one of R_1 and R_2 and that both relations are in fact functions, that is, they map each pair to at most one signature. This proves that there are at most $2 \cdot |[c_{\forall}] \times [c_{\forall}]| = 2c_{\forall}^2$ signatures corresponding to vertices in G_1^* . Since G_1^* contains at most one vertex per signature, the theorem will follow.

We define the relations R_1 and R_2 as follows: for a pair (i, j), the relation R_1 associates (i, j) with all signatures $s = \text{sig}(v) = (-v_s, v_e, -v'_s, v'_e)$ of $\mathcal{S}_{i,j}$ that minimize v'_e . For all signatures $s = \text{sig}(w) = (-w_s, w_e, -w'_s, w'_e)$ with w remaining in G_1^* and that are *not* images under R_1 , the relation R_2 associates (w_e, w'_s) with s. By definition, every signature is the image of some pair under either R_1 or R_2 .

Observe that R_1 is a function: since G_1^* and G_2^* are reduced with respect to Reduction Rule 3, for each pair $(i,j) \in [c_{\forall}] \times [c_{\forall}]$, there is at most one signature $s = (-v_s, v_e, -v_s', v_e') \in \mathcal{S}_{i,j}$ that minimizes v_e' .

It remains to show that R_2 is also a function. Towards a contradiction, assume that R_2 maps some pair to two signatures. Then, there are distinct vertices v and w in G_1^* with signatures $s_1 := \text{sig}(v) = (-v_s, v_e, -v_s', v_e')$ and

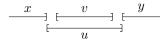


Figure 6: The constellation $u_s = x_e < v_s$ and $v_e < y_s = u_e$ that induces a $K_{1,3}$ in G_1 .

 $s_2 := \operatorname{sig}(w) = (-w_s, w_e, -w'_s, w'_e)$ such that $(v_e, v'_s) = (w_e, w'_s)$. Since the vertices v and w are in G_1^* , we know that $v_s \neq w_s$, since otherwise $s_1 \leq s_2$ or $s_2 \leq s_1$. By symmetry, let $w_s < v_s$. Since $s_2 \in \mathcal{S}_{w_s, w'_s}$, and, therefore, \mathcal{S}_{w_s, w'_s} is nonempty, there is a signature $s_3 := R_1(w_s, w'_s) = \operatorname{sig}(u) = (-u_s, u_e, -u'_s, u'_e)$. Since s_2 is an image under R_2 , it is not an image under R_1 and, thus, we have $s_2 \neq s_3$. The definition of R_1 implies $u_s = w_s$, $u'_s = w'_s$, and $u'_e \leq w'_e$. Therefore, we have $u_e > w_e$ since, otherwise, $s_3 \leq s_2$.

Since $u_s = w_s < v_s$ and $v_e = w_e < u_e$, we now have a constellation $u_s < v_s \le v_e < u_e$ of intervals that contradicts G_1 being a proper interval graph: since G_1 has been preprocessed according to Observation 3, the start point u_s is also the end point x_e of some interval x and the end point u_e is also the start point y_s of some interval y. Note that $x_e = u_s < v_s \le v_e < u_e = y_s$ implies that x, y, v, and u are pairwise distinct. This, as depicted in Figure 6, implies that G_1 contains a $K_{1,3}$ as induced subgraph, which contradicts G_1 being a proper interval graph. The $K_{1,3}$ consists of the central vertex u and the leaves v, x, y. \square

We can generalize Theorem 8 for the problem of finding an independent set of weight at least k: we only have to keep that vertex for each signature in the graph that has the highest weight. Since there are at most c_{\forall}^4 different signatures, we obtain a problem kernel with c_{\forall}^4 vertices for the weighted variant of 2-UNION INDEPENDENT SET.

8 Experimental Evaluation

In this section, we aim for giving a proof of concept by demonstrating to which extent instances of Colorful Independent Set with Lists are solvable within an acceptable time frame of five minutes. Herein, we chose Colorful Independent Set with Lists (see Section 4.2 for the definition) since it is the most general problem studied in our work and algorithms for it also solve 2-Union Independent Set and Job Interval Selection.

We implemented the dynamic programming algorithm (DP- Q^*) from Section 6.2 that solves Colorful Independent Set with Lists in $O(2^Q \cdot n)$ time and $O(2^Q \ell + \gamma c)$ space (Theorem 3), where γ is the number of colors, c is the compactness of the input interval graph, ℓ is the maximum length of an interval, and Q is the structural parameter "maximum number of live colors" (Definition 2). We applied the implemented algorithm to randomly generated instances.

Note that we abstained from implementing our data reduction rules (Section 6.3 and 7.2), since they do not apply to the most general form of COLORFUL INDEPENDENT SET WITH LISTS, which we aim to experiment with.

Implementation Details. The implementation of the algorithm is based on recurrence (DP- Q^*) from Section 6.2, but allows the vertices to have weights and finds a colorful independent set of maximum weight. The source code uses about 700 lines of C++ and is freely available.² The experiments were run on a computer with a 3.6 GHz Intel Xeon processor and 64 GiB RAM under Linux 3.2.0, where the source code has been compiled using the GNU C++ compiler in version 4.7.2 and using the highest optimization level (-O3).

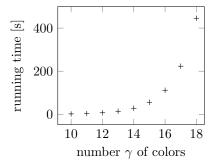
Data. In order to test the influence of various parameters on the running time and memory usage of the algorithm, we evaluated the algorithm on artificial, randomly generated data. To generate random interval graphs, we use a model that is strongly inspired by Scheinerman (1988). However, while Scheinerman (1988) chooses integer interval endpoints uniformly at random without repetitions from [2n], we choose integer interval endpoints uniformly at random from [c], where c is a maximum compactness chosen in advance. It then remains to assign colors and weights to the vertices.

In detail, to generate a random interval graph, we fix a maximum compactness c, a maximum number γ of colors, and a number n of intervals to generate. We then randomly generate n intervals: for each interval v, we choose a start point v_s and an end point v_e uniformly at random from [c]. Then, we add each color in $[\gamma]$ to the color list of v with probability 1/2 and uniformly at random assign v a weight from 1 to 10.

To interpret the experimental results, it is important to make some structural observations about the data generated by this random process.

- 1. The maximum interval length ℓ is at most c-1. Moreover, with a growing number n of generated intervals, the probability $(1-1/c^2)^n$ of not generating an interval that indeed has length c-1 approaches zero. That is, we expect the chosen parameter $c \approx \ell + 1$ to roughly linearly influence the memory usage of the algorithm (Theorem 3).
- 2. The maximum number of live colors Q is at most the number γ of colors. However, since every interval contains each color with equal probability, with increasing number n of intervals we will have $Q \approx \gamma$. Hence, we expect γ to exponentially influence the running time and memory usage of the algorithm (Theorem 3).
- 3. The sizes of the generated vertex color lists follow a binomial distribution. The expected color list size is $\gamma/2$.

²http://fpt.akt.tu-berlin.de/cis/



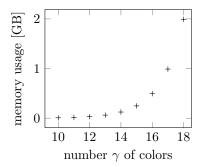


Figure 7: Dependence of running time and space requirements of the dynamic program (DP- Q^* , Section 6.2) for COLORFUL INDEPENDENT SET WITH LISTS on the number γ of colors in the input interval graph, each having 10^5 intervals and being 10^3 -compact.

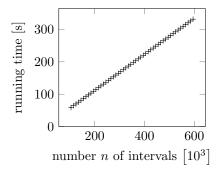
Experimental Results. We generated three data sets by varying each time one of the parameters $\{n, \gamma, c\}$ and keeping the other two constant. We applied our algorithm to find a maximum colorful independent set in each of the graphs.

For the first data set, we let the number γ of colors vary between 10 and 18 and fixed $n=10^5$ and $c=10^3$. Figure 7 clearly exhibits the exponential dependence of running time and memory usage on the number γ of colors, which both roughly double when increasing the number of colors by one. We see that, in this setup, we can solve COLORFUL INDEPENDENT SET WITH LISTS within a time frame of five minutes for $\gamma \leq 17$.

For the second data set, we let the number n of intervals vary between 10^5 and $6 \cdot 10^5$. We again fixed $c = 10^3$. We chose $\gamma = 15$ as number of colors. As expected, Figure 8 shows a roughly linear dependence of the running time on the number n of intervals. Moreover, the memory usage is almost constantly about 250 MB with a slight increase, since we left the compactness c constant and with increasing number n of intervals, the maximum interval length ℓ approaches the maximum compactness c.

For the third data set, we finally let the compactness c vary between 10^2 and 10^3 . We again fixed $\gamma=15$ and $n=10^5$. Figure 9 shows the linear dependence of the memory usage on the compactness $c\approx\ell+1$. In contrast, the running time remains roughly constant with increasing c. The observed local minima of the running time are exactly at those values of c where c is a power of two. In this case, we observed that the time spent per table look-up decreases. We suspect that this has technical reasons.

Summary. The running time and memory usage of the algorithm on randomly generated data very reliably behave as predicted by Theorem 3 and most likely scale to larger data. We have seen that on moderate values of $\gamma \leq 15$, the algorithm can solve instances with up to $5.5 \cdot 10^5$ intervals in a time frame of about five minutes. However, in application data, like for example from the steel



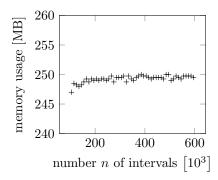
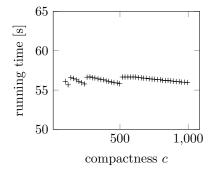


Figure 8: Dependence of running time and space requirements of the dynamic program (DP- Q^* , Section 6.2) for COLORFUL INDEPENDENT SET WITH LISTS on the number n of intervals in the input interval graph, each being colored with subsets of $\{1, \ldots, 15\}$ and being 10^3 -compact.



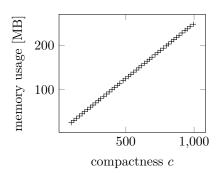


Figure 9: Dependence of running time and space requirements of the dynamic program (DP- Q^* , Section 6.2) for COLORFUL INDEPENDENT SET WITH LISTS on the compactness c of the input interval graph, each having 10^5 intervals colored using subsets of $\{1, \ldots, 15\}$.

manufacturing application of Höhn et al. (2011), the number of colors can be much higher. To efficiently solve such instances with our algorithm, it is crucial that these instances have a low maximum number Q of "live colors", that is, these instances must be more structured than our randomly generated interval graphs.

9 Conclusion

We charted the complexity landscape of INDEPENDENT SET on subclasses of 2-union graphs, which are of relevance for applications in scheduling, and which generalize interval graphs. Our focus was on determining the complexity of finding exact solutions, whereas, so far, approximation algorithms have been much better researched in the literature (Bafna et al., 1996; Spieksma, 1999; Bar-Yehuda et al., 2006; Chuzhoy et al., 2006).

Besides hardness results from our complexity dichotomy, we provided first results on effective polynomial-time preprocessing (kernelization) in this context. We also developed encouraging algorithmic results and evaluated them experimentally, which might find use in practical applications.

For future work, it would be interesting to determine whether 2-UNION INDEPENDENT SET is fixed-parameter tractable with respect to the "M-compositeness" parameter that is small in the steel manufacturing application considered by Höhn et al. (2011). Moreover, it seems worthwhile trying to speed up our randomized algorithm for JOB INTERVAL SELECTION (Theorem 4) using the algebraic techniques described by Koutis and Williams (2009).

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